Geometric Moment Zeta Functions

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July 7, 2003

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1 Introduction

Given a family of algebraic varieties over finite fields, we introduce a sequence of higher moment zeta functions, called the geometric moment zeta functions, which measure the distribution of the closed points along the fibres of the family of varieties. As the moment parameter approaches to certain p-adic limit, one gets the p-adic limiting moment zeta function, which includes Dwork's p-adic unit root zeta function as a special case. We discuss the related results and open problems on these new functions.

2 Zeta functions over finite fields

Let $\mathbb{F}_q$ be the finite field of q elements of characteristic p. Let X be a scheme of finite type over $\mathbb{F}_q$ and let $|X|$ denote the set of closed points of X. If $x \in |X|$ is a closed point, the residue field of X at x is a finite extension field of $\mathbb{F}_q$, whose degree of extension is denoted by $\deg(x)$. The zeta function $Z(X,T)$ of X is defined by

$$Z(X,T) = \prod_{x\in|X|} (1 - T^{\deg(x)})^{-1} \in 1 + T\mathbb{Z}[[T]]. \quad (1)$$

An alternative definition for $Z(X,T)$ is in terms of counting rational points of X over various finite extensions of $\mathbb{F}_q$. Let $\bar{\mathbb{F}}_q$ denote a fixed algebraic closure of $\mathbb{F}_q$. The Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ is topologically generated by the q-th power Frobenius map

$$\text{Frob} : a \to a^q, \ a \in \bar{\mathbb{F}}_q.$$

For each positive integer k, $\bar{\mathbb{F}}_q$ contains a unique subfield of q^k elements, denoted by $\mathbb{F}_{q^k}$, given explicitly by

$$\mathbb{F}_{q^k} = \text{Fix}(\text{Frob}^k) = \{a \in \bar{\mathbb{F}}_q | a^{q^k} = a\}.$$

We have

$$\bar{\mathbb{F}}_q = \bigcup_{k=1}^{\infty} \mathbb{F}_{q^k}.$$

Let $X(\bar{\mathbb{F}}_q)$ be the set of geometric points of X. For each positive integer k, let $X(\mathbb{F}_{q^k})$ denote the set of $\mathbb{F}_{q^k}$-rational points. Clearly,

$$X(\mathbb{F}_{q^k}) = \mathrm{Fix}(\mathrm{Frob}^k | X(\bar{\mathbb{F}}_q)).$$

The alternative definition of $Z(X,T)$ is

$$Z(X,T) = \exp(\sum_{k=1}^{\infty} \frac{T^k}{k} \# X(\mathbb{F}_{q^k})), \tag{2}$$

where $\# X(\mathbb{F}_{q^k})$ denotes the number of $\mathbb{F}_{q^k}$-rational points of X.

The rationality of $Z(X,T)$ was conjectured by Weil (1949 [34]) and first proved by Dwork [7] using p-adic analysis. That is, we have

Theorem 2.1 *(Dwork, 1960) Let X be a scheme of finite type over $\mathbb{F}_q$. Then, $Z(X,T) \in \mathbb{Q}(T)$.*

Write

$$Z(X,T) = \frac{\prod_{i=1}^{s}(1-\alpha_i T)}{\prod_{j=1}^{r}(1-\beta_j T)}$$

in reduced form, where the α_i's and β_j's are algebraic integers. Taking logarithmic derivative, one obtains an explicit formula for rational point counting:

$$\# X(\mathbb{F}_{q^k}) = \sum_{j=1}^{r} \beta_j^k - \sum_{i=1}^{s} \alpha_i^k, \ k = 1, 2, \cdots.$$

The total degree $r+s$ of $Z(X,T)$ can be explicitly bounded, see Bombieri [2]. In the case of small characteristic p, a polynomial time p-adic algorithm is recently obtained in [21], see also [30] for an exposition of the algorithmic issues on zeta functions.

The Riemann hypothesis for $Z(X,T)$ was conjectured by Weil (1949) and proved by Deligne [5] in the general form.

Theorem 2.2 *(Deligne, 1980). Let X be a scheme of finite type over $\mathbb{F}_q$ of dimension n. Then,*

$$|\alpha_i| = q^{u_i/2}, \ |\beta_j| = q^{v_j/2}, \ u_i \in \mathbb{Z} \cap [0, 2n], \quad v_j \in \mathbb{Z} \cap [0, 2n], \tag{3}$$

where $\mathbb{Z} \cap [0, 2n]$ denotes the set of integers in the interval $[0, 2n]$. Furthermore, each α_i (resp. each β_j) and its Galois conjugates over $\mathbb{Q}$ have the same complex absolute value. That is, the α_i and β_j are Weil q-integers.

As the α_i and β_j are algebraic integers, one can also ask for their ℓ-adic absolute values if we fix an embedding of $\bar{\mathbb{Q}}$ to $\bar{\mathbb{Q}}_\ell$ of ℓ-adic numbers, where ℓ is a prime number. If $\ell \neq p$, the α_i and the β_j are eigenvalues of ℓ-adic representations and hence they are ℓ-adic units:

$$|\alpha_i|_\ell = |\beta_j|_\ell = 1.$$

For the remaining prime p, it is easy to prove

$$|\alpha_i|_p = q^{-r_i}, \ |\beta_j|_p = q^{-s_j}, \ r_i \in \mathbb{Q} \cap [0, 2n], \ s_j \in \mathbb{Q} \cap [0, 2n],$$

where we have normalized the p-adic absolute value by $|q|_p = q^{-1}$. Deligne's integrality theorem [6] implies the following improved information:

$$r_i \in \mathbb{Q} \cap [0, n], \ s_j \in \mathbb{Q} \cap [0, n].$$

Note that each α_i (resp. each β_j) and its Galois conjugates over $\mathbb{Q}$ may have different p-adic absolute values. The rational number r_i (resp. s_j) is called the slope of α_i (resp. β_j). Its denominator can be greater than 2, but can be effectively bounded depending on X. The nature of the slopes r_i and s_j can be viewed as the p-adic Riemann hypothesis for $Z(X,T)$. It is a very interesting subject with many open problems. A classical example is the Stickelberger theorem on Gauss sum. We shall not discuss it further here, see [31] for a systematic introduction to this slope problem.

Let ℓ be a prime different from p. In terms of the ℓ-adic cohomology with compact support, one has the Grothendieck ℓ-adic trace formula [13]:

$$Z(X,T) = \prod_{i=0}^{2\dim(X)} \det(I - \mathrm{frob}T | H^i_c(X \otimes \bar{\mathbf{F}}_q, \mathbf{Q}_\ell))^{(-1)^{i-1}},$$

where frob $=$ Frob^{-1} is the geometric Frobenius map. The conjectural independence on ℓ of the ℓ-adic Betti numbers is not known in general. A weak evidence has recently been obtained by Katz [18] who gave a uniform (independent of ℓ) explicit upper bound for the ℓ-adic Betti numbers with compact support.

Theorem 2.3 *(Katz[18]) Let X be the affine variety defined by the vanishing of m polynomials $\{f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\}$ each having degree at most d. Then, for every prime number $\ell \neq p$, we have*

$$\sum_{i \geq 0} \dim_{\mathbf{Q}_\ell} H^i_c(X \otimes \bar{\mathbf{F}}_q, \mathbf{Q}_\ell) \leq 2^{m+2}(3+md)^{n+1}.$$

As an immediate consequence of the rationality for $Z(X,T)$, to be used later, we have

Corollary 2.4 *For each positive integer k, we have*

$$\begin{aligned} Z(X \otimes \mathbb{F}_{q^k}, T) &= \exp\left(\sum_{d=1}^{\infty} \frac{T^d}{d} \#X(\mathbb{F}_{q^{kd}})\right) \\ &= \frac{\prod_{i=1}^{s}(1-\alpha_i^k T)}{\prod_{j=1}^{r}(1-\beta_j^k T)}. \end{aligned} \tag{4}$$

3 Moment zeta functions

Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. We may view f as a family of algebraic varieties over finite fields parametrized by Y. If $y \in |Y|$, the fibre $X_y = f^{-1}(y)$ is an algebraic variety defined over the finite residue field $\mathbb{F}_{q^{\deg(y)}}$ of y. By the results in the previous section, we know that each zeta function $Z(X_y, T)$ is a rational function and satisfies the Riemann hypothesis. We would like to understand

Question 3.1 *How $Z(X_y, T)$ varies when y varies in $|Y|$? How the zeros and poles of $Z(X_y, T)$ vary when y varies in $|Y|$?*

To make this variation question more precise, a classical procedure is to understand all the high moments of this family of rational functions. The k-th moment of each rational function $Z(X_y, T)$ is obtained by raising each reciprocal zeros (resp. reciprocal poles) to its k-th power. By Corollary 2.4, this is simply given by

$$Z(k, X_y, T) = Z(X_y \otimes \mathbb{F}_{q^{\deg(y)k}}, T).$$

Thus, we have

Definition 3.2 *For each positive integer k, the k-th moment zeta function of the family f is defined to be the product*

$$\begin{aligned} Z(k, f, T) &= \prod_{y \in |Y|} Z(k, X_y, T^{\deg(y)}) \\ &= \prod_{y \in |Y|} Z(X_y \otimes \mathbb{F}_{q^{\deg(y)k}}, T^{\deg(y)}) \in 1 + T\mathbb{Z}[[T]]. \end{aligned} \tag{5}$$

The k-th moment zeta function can be similarly defined for a morphism between arithmetic schemes over $\mathbb{Z}$, see [32]. To emphasize that our ground field is the finite field $\mathbb{F}_q$, we can call the moment zeta function of this paper as the geometric moment zeta function.

The k-th moment zeta function $Z(k, f, T)$ is a special case of a more general partial zeta function introduced in [29], see [33] and [15] for further work in this direction. Arithmetically, the k-th moment zeta function can be viewed as the zeta function associated to a partial rational point counting problem. For a positive integer d, let

$$M_d(k, f) = \#\{x \in X(\mathbb{F}_{q^{dk}}) | f(x) \in Y(\mathbb{F}_{q^d})\}.$$

Then, it is easy to check that

$$Z(k, f, T) = \exp(\sum_{d=1}^{\infty} \frac{T^d}{d} M_d(k, f)).$$

According to an observation of Faltings (see [29]), the number $M_d(k, f)$ is the number of fixed points of a certain twisted Frobenius. Let $f^{\otimes k}$ be the k-th fibre product of X over Y. Let σ be the right shifting map on the coordinates of $f^{\otimes k}$. That is, for a point $(x^{(1)}, \cdots, x^{(k)}) \in f^{\otimes k}$,

$$\sigma(x^{(1)}, \cdots, x^{(k)}) = (x^{(k)}, x^{(1)}, \cdots, x^{(k-1)}).$$

The map σ is an automorphism of $f^{\otimes k}$ of order k. It commutes with the Frobenius map. One checks that

$$M_d(k, f) = \#\mathrm{Fix}(\sigma \circ \mathrm{Frob}^d | f^{\otimes k}(\bar{\mathbb{F}}_q)).$$

This formula together with the general ℓ-adic trace formula implies that $Z(k, f, T)$ is nearly rational in the sense of [29], as noted by Faltings.

The sequence of moment zeta functions $Z(k, f, T)$ $(k = 1, 2, \cdots,)$ contains critical information about the variation of the family $Z(X_y, T)$ parametrized by y. From a different point of view, $Z(k, f, T)$ captures the distribution of the closed points (or geometric points) of X along the fibres of f.

Our standard questions are then the rationality and Riemann hypothesis for these moment zeta functions. The rationality implies the existence of good structural distribution law. The Riemann hypothesis gives sharp information about the distribution law. As in the previous section, we have the following general result which is a special case from [29] on partial zeta functions.

Theorem 3.3 *(Wan[29]) Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. For each positive integer k, the k-th moment zeta function $Z(k, f, T)$ is a rational function in $\mathbb{Q}(T)$, whose reciprocal zeros and reciprocal poles are Weil q-integers.*

Proof. We include a proof here since it is very simple. Let ℓ be a prime number different from p. Let $\mathcal{F}_i = R^i f_! \mathbf{Q}_\ell$ be the relative ℓ-adic cohomology with compact support, which is a constructible ℓ-adic sheaf on Y. The fibre $\mathcal{F}_{i,y}$ at y is the ℓ-adic cohomology group $H^i_c(X_y \otimes \bar{\mathbb{F}}_q, \mathbf{Q}_\ell)$ with compact support. Applying the Grothendieck trace formula fibre by fibre, we have

$$Z(k, X_y, T) = \prod_{i \geq 0} \det(I - F_y^k T | \mathcal{F}_{i,y})^{(-1)^{i-1}},$$

where F_y denotes the geometric Frobenius map at y. Let $\mathcal{F}_i^k$ denote the k-th Adams operation of $\mathcal{F}_i$. It is a vitual sheaf. For example, one can write

$$\mathcal{F}_i^k = \sum_{j \geq 0} (-1)^{j-1} j \mathrm{Sym}^{k-j} \mathcal{F}_i \otimes \wedge^j \mathcal{F}_i.$$

Then,

$$Z(k, X_y, T) = \prod_{i,j \geq 0} \det(I - F_y T | \mathrm{Sym}^{k-j} \mathcal{F}_{i,y} \otimes \wedge^j \mathcal{F}_{i,y})^{(-1)^{i+j} j}.$$

This and the product definition of $Z(k, f, T)$ give

$$Z(k, f, T) = \prod_{i,j \geq 0} L(\mathrm{Sym}^{k-j} \mathcal{F}_i \otimes \wedge^j \mathcal{F}_i, T)^{(-1)^{i+j-1} j},$$

where the L on the right side denotes the L-function of the given sheaf. The theorem then follows from Grothendieck's general rationality theorem and Deligne's general theorem on Riemann hypothesis.

The above theorem gives only qualitative information about the nature of the moment zeta function. To be useful in applications, one needs to have a good control on the weights of the zeros (resp. poles) and the total number of zeros (resp. poles). Both questions are far from being well understood.

Let $D(k,f)$ denote the total degree of the rational function $Z(k,f,T)$. The first question to ask is to give a good estimate for $D(k,f)$. A crude explicit upper bound for $D(k,f)$ can be derived from the bound in Fu-Wan [14] for more general partial zeta functions. We now state this explicit bound in our current case of moment zeta functions.

Without loss of generality, we may assume that $X \hookrightarrow \mathbf{A}^{n+n'}$ is defined by:

$$\begin{cases} f_1(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+n'})=0, \\ \quad\vdots \\ f_m(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+n'})=0, \\ f_{m+1}(x_{n+1},\cdots,x_{n+n'})=0, \\ \quad\vdots \\ f_{m+m'}(x_{n+1},\cdots,x_{n+n'})=0, \end{cases}$$

$Y \hookrightarrow \mathbf{A}^{n'}$ is defined by the last m' equations and the map f is given by the projection

$$f:(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+n'})\in X \longrightarrow (x_{n+1},\cdots,x_{n+n'})\in Y,$$

where the polynomials f_i have coefficients in $\mathbf{F}_q$ with degrees at most d. Then, we have

Theorem 3.4 *(Fu-Wan [14]) In terms of the above notations, the total degree of $Z(k,f,T)$ is bounded by*

$$D(k,f)\le 2^{mk+m'+2}(3+(mk+m')d)^{nk+n'+1}.$$

Proof. Let X^k denote the k-fold product of X, embedded in $\mathbf{A}^{(n+n')k}$ whose coordinates are denoted by

$$x=(x^{(1)},\cdots,x^{(k)})=(x_{ij}),\ 1\le i\le (n+n'),\ 1\le j\le k,$$

where $x^{(j)}$ is the column vector $(x_{1j},\cdots,x_{(n+n')j})$. Recall that σ is the cyclic shift

$$\sigma(x^{(1)},\cdots,x^{(k)})=(x^{(k)},x^{(1)},\cdots,x^{(k-1)}).$$

Let $f^{\otimes k}$ be the algebraic subset of X^k cut out by the additional linear equations

$$x_{i1}=x_{i2}=\cdots=x_{ik},\ n+1\le i\le n+n'.$$

Thus, $f^{\otimes k}$ can be embedded in the smaller affine space $\mathbf{A}^{nk+n'}$ with $(mk+m')$ defining equations. Geometrically, $f^{\otimes k}$ is the k-fold fibre product of X over Y:

$$f^{\otimes k} = X \times_Y X \times \cdots \times_Y X.$$

As noted before,

$$M_d(k,f) = \#\mathrm{Fix}(\sigma \circ \mathrm{Frob}^d | f^{\otimes k}(\bar{\mathbf{F}}_q))$$

and

$$Z(k,f,T) = \exp(\sum_{d=1}^{\infty} \frac{\#\mathrm{Fix}(\sigma \circ \mathrm{Frob}^d | f^{\otimes k}(\bar{\mathbf{F}}_q))}{d}).$$

One can then use the general ℓ-adic fixed point formula to conclude

$$D(k,f) \leq \sum_{i \geq 0} \dim_{\mathbf{Q}_\ell} H_c^i(f^{\otimes k} \otimes \bar{\mathbf{F}}_q, \mathbf{Q}_\ell).$$

Let

$$d = \max_{1 \leq i \leq n+n'} \deg(f_i).$$

As noted above, the algebraic set $f^{\otimes k}$ can be defined by $(mk+m')$ equations in $(nk+n')$ variables, each having degree at most d. By Katz's estimate for ℓ-adic Betti numbers, the desired bound for $D(k,f)$ follows. The theorem is proved.

The above total degree bound however grows exponentially in k. We expect that the true size of $D(k,f)$ is much smaller. In fact, a special case of a result in Fu-Wan [15] says that the total degree $D(k,f)$ is bounded by a polynomial function in k. That is, we have

Theorem 3.5 *(Fu-Wan[15]) Let $f: X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. There are two positive constants $c_1(f)$ and $c_2(f)$ such that for every positive integer k, we have the polynomial bound*

$$D(k,f) \leq c_1(f) k^{c_2(f)}.$$

The power constant $c_2(f)$ is explicit. But the coefficient $c_1(f)$ is not effective yet. For example, in the special case that f is the universal family of elliptic curves over $\mathbb{F}_q$, the total degree $D(k,f)$ is bounded by a linear polynomial in k, since the dimension for modular forms of weight k grows linearly in k. No non-trivial lower bound for $D(k,f)$ is known even in this elliptic family case.

We believe that the coefficient $c_1(f)$ can also be made to be effective using the representation theoretic approach in the proof of Theorem 3.3 together with delicate p-adic arguments on Newton polygons. The above theorem is important in deriving good archimedian estimate of $M_d(k,f)$ for large k, which is in turn crucial in the statistical study of rational points along the fibres of f, see Katz [19] and also [15] for some examples in this direction.

Another very interesting question is to understand the slopes of the zeros and poles of $Z(k,f,T)$, corresponding to the p-adic Riemann hypothesis for

$Z(k, f, T)$. This is in general quite difficult, already so in the special case of the unversal family of elliptic curves where the slopes reflect crucial arithmetic information about modular forms [23] and where the total degree $D(k, f)$ is already unknown (except for an upper bound linear in k).

We can further ask how the sequence $Z(k, f, T)$ varies when the integer k varies. This is treated in next section. It is related to Dwork's conjecture.

4 Limiting moment zeta functions

Let $f : X \longrightarrow Y$ be a morphism of schemes of finite types over $\mathbb{F}_q$. For each positive integer k, we have the k-th moment zeta function $Z(k, f, T)$ which is a rational function satisfying the Riemann hypothesis. We would like to understand how the moment zeta function $Z(k, f, T)$ varies as the integer k varies arithmetically.

To be more specific, we fix a prime number ℓ (which may be equal to the characteristic p). We let k vary ℓ-adically and want to understand how $Z(k, f, T)$ varies ℓ-adically. An initial property would be possible ℓ-adic continuity.

Let $K_{f,\ell} = \mathbb{Q}_\ell(\alpha_i(y), \beta_j(y))$, the extension field of $\mathbb{Q}_\ell$ obtained by adjoining all ℓ-adic *unit* zeros $\alpha_i(y)$ and all ℓ-adic *unit* poles $\beta_j(y)$ of $Z(X_y, T)$ for all $y \in |Y|$. The uniform upper bound [2] for the total degree of $Z(X_y, T)$ together with a standard algebraic number theory argument implies

Proposition 4.1 *The extension $K_{f,\ell}$ over $\mathbb{Q}_\ell$ is finite.*

Denote the uniformizer of $K_{f,\ell}$ by π. Let $d_{f,\ell}$ be the unramified degree of $K_{f,\ell}$ over $\mathbb{Q}_\ell$. Let

$$D_{f,\ell} = \ell^{d_{f,\ell}} - 1. \tag{6}$$

This is simply the order of the multiplicative group of the residue field of $K_{f,\ell}$.

Then, for each ℓ-adic unit α in $K_{f,\ell}$ (such as those ℓ-adic unit $\alpha_i(y)$ and those ℓ-adic unit $\beta_j(y)$), the power $\alpha^{D_{f,\ell}}$ is an ℓ-adic 1-unit in $K_{f,\ell}$. In particular, we have the limiting formula

$$\lim_{m\to\infty} \alpha^{D_{f,\ell}\ell^m} = 1.$$

For any two integers k_1 and k_2 satisfying $k_1 \equiv k_2 \pmod{D_{f,\ell}\ell^{m-1}}$ for some positive integer m, we have the congruences

$$\alpha^{k_1} \equiv \alpha^{k_2} \pmod{\pi^m},$$

and

$$Z(k_1, X_y, T) \equiv Z(k_2, X_y, T) \pmod{\pi^m}.$$

This ℓ-adic continuity result and the Euler product definition of $Z(k, f, T)$ show that the limit in the following definition exists as an ℓ-adic formal power series.

Definition 4.2 *Given a morphism $f : X \longrightarrow Y$ over $\mathbb{F}_q$, a prime number ℓ, an integer k, we define the limiting moment zeta function by*

$$Z_\ell(k, f, T) = \lim_{m\to\infty} Z(k + D_{f,\ell}\ell^m, f, T) \in 1 + T\mathbb{Z}_\ell[[T]]. \tag{7}$$

Note that $Z_\ell(k, f, T)$ is very different from $Z(k, f, T)$ in general. Let

$$M_{d,\ell}(k, f) = \lim_{m\to\infty} \#\{x \in X(\mathbb{F}_{q^{d(k+D_{f,\ell}\ell^m)}}) | f(x) \in Y(\mathbb{F}_{q^d})\}.$$

This limit exists as an ℓ-adic integer. It can be viewed as the "infinite" number of $\mathbb{F}_{q^d}$-rational points on a certain infinite dimensional variety defined over $\mathbb{F}_q$. The additive definition of the limiting moment zeta function can be written as

$$Z_\ell(k, f, T) = \exp(\sum_{d=1}^{\infty} \frac{T^d}{d} M_{d,\ell}(k, f)).$$

The series $Z_\ell(k, f, T)$ with ℓ-adic integral coefficients is clearly ℓ-adic analytic in the open unit disk $|T|_\ell < 1$. It can be viewed in certain sense as the zeta function of a certain infinite dimensional variety over $\mathbb{F}_q$ counted in certain direction. Our first fundamental question is to ask if the limiting moment zeta function $Z_\ell(k, f, T)$ is an ℓ-adic meromorphic function on the whole ℓ-adic plane $|T|_\ell < \infty$.

If $Z_\ell(k, f, T)$ is indeed ℓ-adic meromorphic everywhere, then there are ℓ-adic numbers α_i and β_j approaching to zero such that

$$Z_\ell(k, f, T) = \frac{\prod_{i=1}^{\infty}(1 - \alpha_i T)}{\prod_{j=1}^{\infty}(1 - \beta_j T)}.$$

In terms of the ℓ-adic sequence $M_{d,\ell}(d, f)$ parametrized by d, this means that for each integer $d > 0$, we have the formula

$$M_{d,\ell}(k, f) = \sum_{j=1}^{\infty} \beta_j^d - \sum_{i=1}^{\infty} \alpha_i^d.$$

Conversely, the existence of such a formula is equivalent to the ℓ-adic meromorphic continuation of $Z_\ell(k, f, T)$ to the whole ℓ-adic plane.

If $\ell \neq p$, the $\alpha_i(y), \beta_j(y)$ are always ℓ-adic units. In this case, we deduce that if k is a positive integer, then

$$Z_\ell(k, f, T) = Z(k, f, T),$$

which is a rational function in $\mathbb{Q}(T)$ whose reciprocal zeros and reciprocal poles are Weil q-integers, by the results in the previous section. If k is negative, it can be proved in a similar way using contragradient representations.

If $\ell = p$, the situation is more complicated. Assume that k is a positive integer. Then, $Z_p(k, f, T)$ is exactly the k-th power unit root zeta function of Dwork attached to the family f. Denote this k-th power unit root zeta function by $Z_{p-\mathrm{unit}}(k, f, T)$. This is a p-adic power series, whose coefficients are not contained in a fixed number field and hence not a rational function any more in general. Dwork [9] conjectured that $Z_{p-\mathrm{unit}}(k, f, T)$ is p-adic meromorphic in the whole p-adic plane $|T|_p < \infty$. This was proved recently in [25][26][27]. Note that here we only consider the slope zero (unit root) part. Similar results hold for higher slopes, see [28] for a simple introduction.

In summary, our first question has a positive answer. That is, we have

Theorem 4.3 *(Wan [26][27]). Let ℓ be a prime. Let k be an integer. Then, the limiting moment zeta function $Z_\ell(k, f, T)$ is ℓ-adic meromorphic everywhere. If $\ell \neq p$, it is rational over $\mathbb{Q}$, whose reciprocal zeros and reciprocal poles are Weil q-integers.*

The integer domain of the variable k in the function $Z_\ell(k, f, T)$ can be extended to a larger ℓ-adic domain as follows. Write $k = k_1 + D_{f,\ell}k_2$. Then, it is easy to check that the formal power series $Z_\ell(k_1 + D_{f,\ell}k_2, f, T)$ is ℓ-adically continuous in k_2. This continuity implies that we can define

Definition 4.4 *Let k_1 be a fixed integer. Let k_2 be any ℓ-adic integer. We define*

$$Z_\ell(k_1, k_2, f, T) = \lim_{m\to\infty} Z_\ell(k_1 + D_{f,\ell}k_2(m), f, T),$$

where $k_2(m)$ is any sequence of strictly increasing positive integers which converges ℓ-adically to k_2.

This is a well defined ℓ-adic power series for any integer $k_1 \in \mathbb{Z}$ and any ℓ-adic integer $k_2 \in \mathbb{Z}_\ell$. It is independent of the choice of the sequence $k_2(m)$ converging ℓ-adically to k_2.

The function $Z_\ell(k_1, k_2, f, T)$ in the two ℓ-adic variables (k_2, T) should be viewed as the ℓ-adic zeta function attached to the morphism f. It is then natural to ask

Question 4.5 *For $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_\ell$, is $Z_\ell(k_1, k_2, f, T)$ an ℓ-adic meromorphic function on the closed unit disk $|T|_\ell \leq 1$? or even on the whole ℓ-adic plane $|T|_\ell < \infty$?*

In general, the ℓ-adic meromorphic continuation of $Z_\ell(k_1, k_2, f, T)$ to the closed unit disk $|T|_\ell \leq 1$ is already not clear, even for the universal family of elliptic curves if $\ell \neq p$. It would be premature to conjecture the ℓ-adic meromorphic continuation to the entire ℓ-adic plane $|T|_\ell < \infty$. For this reason, we simply state the above problem as a question instead of a conjecture as the answer could be negative in general (there seems to have a little negative feeling in the general case). It is however interesting to find out when the answer is positive.

The previous theorem shows that the answer is positive if $k_2 \in \mathbb{Z}$. Another positive result in this direction is the following theorem which says that the answer is also positive for some special $k_2 \in \mathbb{Z}_\ell \cap \mathbb{Q}$.

For each integer $n > m$ for some fixed large integer m depending on $D_{f,\ell}$, the integer $(\ell^{(n+1)!-n!} - 1)$ is clearly divisible by $D_{f,\ell}$. Thus, for a positive integer d, we can write the formal symbal $d\ell^\infty$ (the zero element in $\mathbb{Z}_\ell$) in the form

$$d\ell^\infty = (d + \sum_{n=0}^{m} d\ell^{n!}(\ell^{(n+1)!-n!} - 1)) + \sum_{n=m+1}^{\infty} d\ell^{n!}(\ell^{(n+1)!-n!} - 1) = k_1 + D_{f,\ell}k_2.$$

Write

$$k_1 = d + \sum_{n=0}^{m} d\ell^{n!}(\ell^{(n+1)!-n!} - 1) = d\ell^{(m+1)!} \in \mathbb{Z},$$

$$k_2 = \sum_{n=m+1}^{\infty} d\ell^{n!}\frac{(\ell^{(n+1)!-n!} - 1)}{D_{f,\ell}} = \frac{-d\ell^{(m+1)!}}{D_{f,\ell}} \in \mathbb{Z}_\ell \cap \mathbb{Q}.$$

Theorem 4.6 *(Lenstra-Wan [22]) Let ℓ be a prime. Let d be a positive integer. Let k_1 and k_2 be defined as above for some large integer m. Then, $Z_\ell(k_1, k_2, f, T) \in \mathbb{Q}_\ell(T)$, an ℓ-adic rational function whose reciprocal zeros and reciprocal poles are Weil q-integers.*

This theorem can be proved quickly using the Grothendieck-Deligne results and Brauer's virtual lifting theorem for modulo ℓ representations.

Let us now look at an important special case.

Example 4.7 Let $n \geq 2$ be an integer. Let f be the family of Calabi-Yau projective hypersurfaces over $\mathbb{F}_q$ defined by the equation

$$X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1} + yX_0X_1 \cdots X_n = 0,$$

parametrized by the affine line $y \in \mathbb{A}^1$. In the case $n = 2$, this is a family of elliptic curves. In the case $n = 3$, this is a family of $K - 3$ surfaces. In the case $n = 4$, this is a family of Calabi-Yau quintic hypersurfaces.

If $\ell \neq p$, the zeta function $Z_\ell(k_1, k_2, f, T)$ is in general not known to be ℓ-adic meromorphic on the closed unit disk $|T|_\ell \leq 1$ if $k_2 \in \mathbb{Z}_\ell$, even in the special case that $n = 2$. If in addition, either $k_2 \in \mathbb{Z}$ or $\{k_1, k_2\}$ is defined as in the above theorem, then $Z_\ell(k_1, k_2, f, T)$ is rational.

If $\ell = p$, then there is only one (or none) non-trivial p-adic unit root for the zeta function $Z(X_y, T)$ and one can show that $D_{f,p} = p - 1$. Our result implies that the limiting moment zeta function $Z_p(k_1, k_2, f, T)$ is always p-adic meromorphic everywhere for all $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_p$. This follows from the rank one case of Dwork's conjecture as given in [27]. The cases $n = 2, 3$ with $k_2 \in \mathbb{Z}$ had been proved previously by Dwork (1971[8], 1973[9]). Such p-adic meromorphic continuation should be related to deep p-adic properties of the mirror map.

5 Moment L-functions

To give a further and more precise exposition, we need to work with the languages of ℓ-adic representations and ℓ-adic ètale cohomology.

Let Y be a geometrically connected smooth affine scheme of finite type over $\mathbb{F}_q$ with function field $\mathbb{F}_q(Y)$. Let $\pi_1(Y)$ be the arithmetic fundamental group of Y. It is the absolute Galois group of $\mathbb{F}_q(Y)$ modulo the closed subgroup generated by the inertial subgroups at closed points of Y. Let ℓ be a prime number. Let

$$\psi_\ell : \pi_1(Y) \longrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell) \tag{8}$$

be a continuous ℓ-adic representation. Equivalently, ψ_ℓ defines a lisse ℓ-adic ètale sheaf on Y. The L-function of ψ_ℓ is defined in a standard manner as follow:

$$L(\psi_\ell, T) = \prod_{y\in|Y|} \frac{1}{\det(I - \psi_\ell(\mathrm{Frob}_y)T^{\deg(y)})} \in 1 + T\mathbb{Z}_\ell[[T]], \tag{9}$$

where Frob_y denotes the Frobenius conjugacy class of $\pi_1(Y)$ at y. If $\psi_\ell = 1$ is the trivial representation, then $L(\psi_\ell, T) = Z(Y, T)$ is rational. More generally, if ψ_ℓ is of finite order, then $L(\psi_\ell, T)$ is also rational. This follows from the following general result.

Theorem 5.1 *(Grothendieck[13]). Let ψ_ℓ be a continuous ℓ-adic representation of $\pi_1(Y)$ as above. If $\ell \neq p$, then $L(\psi_\ell, T)$ is a rational function over $\mathbb{Q}_\ell$.*

In the remaining case that $\ell = p$, the situation is more complicated and the L-function is not rational in general. Katz [16] conjectured that $L(\psi_p, T)$ is p-adic meromorphic everywhere. This turned out to be false in general [24]. Dwork's original conjecture [9] says that the L-function $L(\psi_p, T)$ is p-adic meromorphic if the p-adic representation ψ_p is geometric in some sense. We now briefly recall the definition of geometric representations of $\pi_1(Y)$.

Classically for the case $\ell \neq p$, the representation ψ_ℓ is called geometric if it comes from the relative ℓ-adic ètale cohomology of a morphism $f : X \to Y$ over $\mathbb{F}_q$. If $\ell \neq p$, the geometric Langlands conjecture as proved by Lafforgue [20] shows that up to a constant twist, every irreducible ℓ-adic representation of $\pi_1(Y)$ is geometric. So, for $\ell \neq p$, geometric ℓ-adic representations essentially give rise to all ℓ-adic representations.

On the other hand, if $\ell = p$, the situation is quite a bit different. To define geometric p-adic representations, one can start with the relative p-adic ètale cohomology of a morphism $f : X \to Y$ over $\mathbb{F}_q$. This definition is a bit narrow. It is well known that p-adic representations of $\pi_1(Y)$ corresponds exactly to unit root F-crystals on Y, see [16]. From F-crystal point of view, what arises from geometry is the relative crystalline cohomology or the more general relative rigid cohomology [1]. These give rise to overconvergent F-crystals, whose unit (slope zero) part is exactly the relative p-adic ètale cohomology, see [10] and [11]. For an F-crystal M on Y, shrinking Y if necessary and by Katz's isogeny theorem [17], we may assume that the F-crystal M is ordinary, which then has a Newton-Hodge decomposition. Each pure slope piece (after twisting so that it becomes slope zero) of M then gives a p-adic representation of $\pi_1(Y)$, which is no longer overconvergent in general. Thus, we say that a p-adic representation ψ_p of $\pi_1(Y)$ is geometric in the restricted sense if it arises from some pure slope part of an ordinary overconvergent F-crystal on Y. This is not the most general definition since it does not form a tensor category yet. The general definition is then the tensor category generated the restricted geometric p-adic representations and their contragradient representations, see [26]. Actually, in [26], we used the larger ambient category of nuclear overconvergent σ-modules instead of the much smaller ambient category of overconvergent F-crystals. Thus, the definition given here for geometric p-adic representations is more restricted than what is treated in [26].

There are a lot more non-geometric highly transcendental p-adic representations because of very wild ramifications. It is not clear how to characterize the geometric p-adic representations of $\pi_1(Y)$. This can be viewed as the truly p-adic geometric Langlands conjecture, which has not been formulated yet! This problem is transcendental in nature. The easier p-adic analogue of the geometric Langlands conjecture is the compatibility between the category of overconvergent F-crystals and the category of lisse ℓ-adic sheaves, where $\ell \neq p$. This problem is algebraic in nature and seems within reach in view of the recent progresses in this direction.

As a solution to Dwork's conjecture, we have

Theorem 5.2 *(Wan [26][27]). Let ψ_p be a continuous p-adic representation of $\pi_1(Y)$. If ψ_p is geometric, then $L(\psi_p, T)$ is p-adic meromorphic.*

From now on, we assume that ψ_ℓ is a geometric ℓ-adic representation, where ℓ may be equal to p. From the above results, we know that the L-function $L(\psi_\ell, T)$ is ℓ-adic meromorphic everywhere (in fact, rational if $\ell \neq p$). For an integer k, we can define the k-th moment L-function of ψ_ℓ by

$$L(\psi_\ell^k, T) = \prod_{y \in |Y|} \frac{1}{\det(I - \psi_\ell(\mathrm{Frob}_y)^k T^{\deg(y)})} \in 1 + T\mathbb{Z}_\ell[[T]]. \qquad (10)$$

This L-function is the k-th moment of the Euler factors of $L(\psi_\ell, T)$. It is also the L-function of the k-th Adams operation ψ_ℓ^k, which is a virtual ℓ-adic representation. Similarly, we have

Theorem 5.3 *(Wan [27][29]). Let ψ_ℓ be a geometric ℓ-adic representation of $\pi_1(Y)$. Then, for each integer k, $L(\psi_\ell^k, T)$ is ℓ-adic meromorphic everywhere (rational if $\ell \neq p$).*

A further question is to understand the ℓ-adic variation of $L(\psi_\ell^k, T)$ as the integer k varies ℓ-adically. Just as in the zeta function case, this leads to a suitable limiting moment L-function.

Let $D = D_{\psi_\ell}$ be the order of the image of the residue representation $\bar{\psi}_\ell$. Let k_1 and k_2 be integers. As in the previous section, it is easy to see that the power series $L(\psi_\ell^{k_1+Dk_2}, T)$ is ℓ-adically continuous in k_2. This continuity shows that $L(\psi_\ell^{k_1+Dk_2}, T)$ is a well defined ℓ-adic power series in $\mathbb{Z}_\ell[[T]]$ for all $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_\ell$. This power series is clearly ℓ-adic analytic in the open unit disk $|T|_\ell < 1$. It can be viewed as the ℓ-adic L-function attached to the representation ψ_ℓ. We can ask if this limiting moment L-function is ℓ-adic meromorphic on the closed unit disk or even everywhere. This is unknown if the rank of ψ_ℓ is greater than 1. But, if the rank of ψ_ℓ is 1, it is true.

Theorem 5.4 *(Wan [27]) Let ψ_ℓ be a rank one geometric ℓ-adic representation of $\pi_1(Y)$, where ℓ may be equal to p. For all integers k_1 and ℓ-adic integers $k_2 \in \mathbb{Z}_\ell$, the limiting moment L-function $L(\psi_\ell^{k_1+Dk_2}, T)$ is ℓ-adic meromorphic everywhere (rational if $\ell \neq p$).*

This result is easy if $\ell \neq p$ since ψ_ℓ is then geometrically of finite order by geometric class field theory. If $\ell = p$, the result is non-trivial and follows from our work on the rank one case [27] of Dwork's conjecture. For higher rank geometric representations, just as in the previous section on limiting moment zeta functions, $L(\psi_\ell^{k_1+Dk_2}, T)$ is known to be ℓ-adic meromorphic if either k_2 is an integer (Wan [26]) or $\{k_1, k_2\}$ arises from Theorem 4.6 (Lenstra-Wan [22]).

In the case that $\ell = p$, the above rank one theorem and its proof [27] can be pushed further as first observed by Coleman in his private notes on the author's proof of the Dwork conjecture. The full form of this refinement, which we now describe, has been carried out by Grosse-Klönne [12].

Let ψ_p be a continuous rank one geometric p-adic representation

$$\psi_p : \pi_1(Y) \longrightarrow \mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^*.$$

Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let W_p be the set of continuous p-adic homomorphisms from $\mathbb{Z}_p^*$ to $\mathbb{C}_p^*$. It is called the weight space. It is a rigid analytic space. It can be identified with a finite disjoint union of open unit disks. For $\chi \in W_p$, the composition $\chi \circ \psi_p$ is a continuous group homomorphism from $\pi_1(Y)$ to $\mathbb{C}_p^*$. For example, for each $k \in \mathbb{Z}$, the map $a \to a^k$ is a continuous group homomorphism from $\mathbb{Z}_p^*$ to $\mathbb{C}_p^*$. One can define an L-function as above by

$$L(\chi \circ \psi_p, T) = \prod_{y \in |Y|} \frac{1}{1 - (\chi \circ \psi_p)(\mathrm{Frob}_y) T^{\deg(y)}} \in 1 + T\mathbb{C}_p[[T]]. \tag{11}$$

In this way, $L(\chi \circ \psi_p, T)$ becomes a function in the two variables (χ, T). In the special case that χ is the k-th power map, $L(\chi \circ \psi_p, T)$ becomes the k-th moment L-function $L(\psi_p^k, T)$, which is just $L(\psi_p^{\otimes k}, T)$ since ψ_p has rank one.

Theorem 5.5 *(Grosse-Klönne [12]) Let ψ_p be a rank one geometric p-adic representation of $\pi_1(Y)$. Then, $L(\chi \circ \psi_p, T)$ is a two variable meromorphic function in the domain $(\chi, T) \in W_p \times \mathbb{C}_p$.*

In the special case that ψ_p comes from the universal family of elliptic curves over $\mathbb{F}_p$, this result was proved by Coleman [3] and the zero locus of the rigid meromorphic function $L(\chi \circ \psi_p, T)$ is closely related to the eigencurve [4]. The special value $L(\chi \circ \psi_p, 1)$, which is a meromorphic function in $\chi \in W_p$, is related to Iwasawa theory and p-adic L-functions.

To conclude this paper, we briefly discuss the distribution of the zeros of the p-adic meromorphic function $L(\psi_p^k, T)$ for geometric ψ_p and $k \in \mathbb{Z}$. For a positive real number t, let $N(\psi_p^k, t)$ denote the number of reciprocal zeros (or poles) of slopes at most t. The order of the meromorphic function $L(\psi_p^k, T)$ is defined to be the upper limit

$$\mu(\psi_p^k) = \lim_{t\to\infty} \sup \frac{\log(N(\psi_p^k, t) + 1)}{\log t}.$$

It is clear that

$$0 \leq \mu(\psi_p^k) \leq \infty.$$

Question 5.6 *Let ψ_p be a geometric p-adic representation. Is the order $\mu(\psi_p^k)$ finite for $k \in \mathbb{Z}$?*

We do know

Theorem 5.7 *(Wan[27]) Let ψ_p be a geometric p-adic representation. If ψ_p has rank one, then the order $\mu(\psi_p^k)$ is finite and uniformly bounded for all $k \in \mathbb{Z}$.*

If ψ_p has rank greater than one, we do not know if the order $\mu(\psi_p^k)$ is finite.

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