Lectures on Zeta Functions over Finite Fields

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1. Introduction

These are the notes from the summer school in Göttingen sponsored by NATO Advanced Study Institute on Higher-Dimensional Geometry over Finite Fields that took place in 2007. The aim was to give a short introduction on zeta functions over finite fields, focusing on moment zeta functions and zeta functions of affine toric hypersurfaces. Along the way, both concrete examples and open problems are presented to illustrate the general theory. For simplicity, we have kept the original lecture style of the notes. It is a pleasure to thank Phong Le for taking the notes and for his help in typing up the notes.

2. Zeta Functions over Finite Fields

Definitions and Examples

Let p be a prime, $q = p^a$ and $\mathbb{F}_q$ be the finite field of q elements. For the affine line $\mathbb{A}^1$, we have $\mathbb{A}^1(\mathbb{F}_q) = \mathbb{F}_q$ and $\#\mathbb{A}^1(\mathbb{F}_q) = q$.

Fix an algebraic closure $\overline{\mathbb{F}_q}$. $\mathrm{Frob}_q : \overline{\mathbb{F}_q} \mapsto \overline{\mathbb{F}_q}$, defined by $\mathrm{Frob}_q(x) = x^q$. For $k \in \mathbb{Z}_{>0}$,

$$\mathbb{F}_{q^k} = \mathrm{Fix}\left(\mathrm{Frob}_q^k|\overline{\mathbb{F}_q}\right), \ \ \mathbb{A}^1(\overline{\mathbb{F}_q}) = \overline{\mathbb{F}_q} = \bigcup_{k=1}^{\infty} \mathbb{F}_{q^k}.$$

Given a geometric point $x \in \overline{\mathbb{F}_q}$, the orbit $\{x, x^q, \ldots, x^{q^{\deg(x)-1}}\}$ of x under Frob_q is called the closed point of $\mathbb{A}^1$ containing x. The length of the orbit is called the degree of the closed point. We may correspond this uniquely to the monic irreducible polynomial $(t-x)(t-x^q)\ldots(t-x^{q^{\deg(x)-1}})$. Let $|\mathbb{A}^1|$ denote the set of closed points of $\mathbb{A}^1$ over $\mathbb{F}_q$. Similarly, let $|\mathbb{A}^1|_k$ denote the set of closed points of $\mathbb{A}^1$ of degree k. Hence

$$|\mathbb{A}^1| = \bigsqcup_{k=1}^{\infty} |\mathbb{A}^1|_k.$$

Example 2.1 *The zeta function of $\mathbb{A}^1$ over $\mathbb{F}_q$ is*

$$\begin{aligned} Z(\mathbb{A}^1, T) &= \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \#\mathbb{A}^1(\mathbb{F}_{q^k})\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} q^k\right) \\ &= \frac{1}{1-qT} \in \mathbb{Q}(T). \end{aligned}$$

The reciprocal pole is a Weil q-number. There is also a product decomposition

$$Z(\mathbb{A}^1, T) = \prod_{k=1}^{\infty} \frac{1}{(1-T^k)^{\#|\mathbb{A}^1|_k}}.$$

More generally, let X be quasi-projective over $\mathbb{F}_q$, or a scheme of finite type over $\mathbb{F}_q$. By birational equivalence and induction, one can often (but not always) assume that X is a hypersurface $\{f(x_1, \ldots, x_n) = 0 | x_i \in \overline{\mathbb{F}_q}\}$. Consider the Frobenius action on $X(\overline{\mathbb{F}_q})$. Let $|X|$ be the set of all closed points of X and $|X|_k$ be the set of closed points on X of degree k. As in the previous case, we have

$$X(\overline{\mathbb{F}_q}) = \bigsqcup_{k=1}^{\infty} X(\mathbb{F}_{q^k}), \quad |X| = \bigsqcup_{k=1}^{\infty} |X|_k.$$

Definition 2.2 *The zeta functions of X is*

$$Z(X, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \#X(\mathbb{F}_{q^k})\right)$$

$$= \prod_{k=1}^{\infty} \frac{1}{(1-T^k)^{\#|X|_k}} \in 1 + T\mathbb{Z}[[T]].$$

Question 2.3 *What does $Z(X, T)$ look like?*

The answer was proposed by André Weil in his celebrated Weil conjectures. More precisely, Dwork [7] proved that $Z(X, T)$ is a rational function. Deligne [6] proved that the reciprocal zeros and poles of $Z(X, T)$ are Weil $q-$numbers.

Moment Zeta Functions

Let $f : X \mapsto Y/\mathbb{F}_q$. One has

$$X(\overline{\mathbb{F}_q}) = \bigsqcup_{y \in Y(\overline{\mathbb{F}_q})} f^{-1}(y)(\overline{\mathbb{F}_q}).$$

Similarly

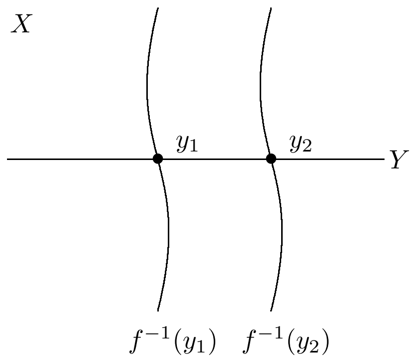

Figure 1. $f^{-1}(y)$

$$X(\mathbb{F}_q) = \bigsqcup_{y \in Y(\mathbb{F}_q)} f^{-1}(y)(\mathbb{F}_q).$$

From this we get

$$\#X(\mathbb{F}_{q^k}) = \sum_{y \in Y(\mathbb{F}_{q^k})} \#f^{-1}(y)(\mathbb{F}_{q^k})$$

for $k = 1, 2, 3, \ldots$. This number is known as the first moment of f over $\mathbb{F}_{q^k}$.

Definition 2.4 *For $d \in \mathbb{Z}_{>0}$, the d-th moment of f over $\mathbb{F}_{q^k}$ is*

$$M_d(f \otimes \mathbb{F}_{q^k}) = \sum_{y \in Y(\mathbb{F}_{q^k})} \#f^{-1}(y)(\mathbb{F}_{q^{dk}})$$

$k = 1, 2, 3, \ldots$

Definition 2.5 *The d-th moment zeta function of f over $\mathbb{F}_q$ is*

$$\begin{aligned} Z_d(f, T) &= \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} M_d(f \otimes \mathbb{F}_{q^k})\right) \\ &= \prod_{y \in |Y|} Z\left(f^{-1}(y) \otimes_{\mathbb{F}_{q^{\deg(y)}}} \mathbb{F}_{q^{d \times \deg(y)}}, T^{\deg(y)}\right) \in 1 + T\mathbb{Z}[[T]]. \end{aligned}$$

Geometrically $M_d(f \otimes \mathbb{F}_{q^k})$ can be thought of as certain point counting along the fibres of f. Note that $M_d(f, k)$ will increase as d increases. Figure 2 illustrates this. The sequence of moment zeta functions $Z_d(f, T)$ measures the arithmetic variation of rational points along the fibres of f. It naturally arises from the study of Dwork's unit root conjecture [28].

Question 2.6

1. *For a given f, what is $Z_d(f, T)$?*
2. *How does $Z_d(f, T)$ vary with d?*

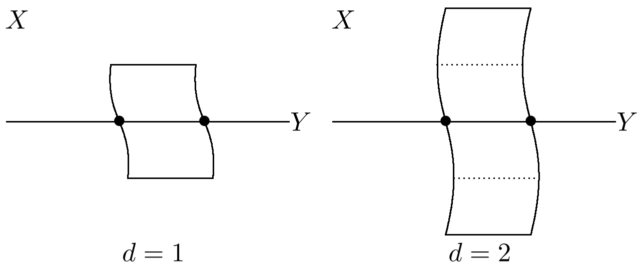

Figure 2. $f^{-1}(y)(\mathbb{F}_{q^d})$

As d increases the area where we count points will also increase.

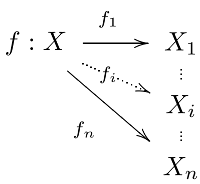

Figure 3. $f : X \mapsto X_1 \times \ldots \times X_n$

Partial Zeta Functions

Assume $f : X \mapsto X_1 \times \ldots \times X_n$ defined by $x \mapsto (f_1(x), \ldots, f_n(x))$ is an embedding. There are many ways to satisfy this property. For example the addition of the identity function $f_n : X \mapsto X$ will assure f is an embedding.

Let $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$. For $k = 1, 2, 3, \ldots$, let

$$M_{d_1,\ldots,d_n}(f \otimes \mathbb{F}_{q^k}) :=$$
$$\#\{x \in X(\overline{\mathbb{F}_q}) | f_1(x) \in X_1(\mathbb{F}_{q^{d_1 k}}), \ldots, f_n(x) \in X_n(\mathbb{F}_{q^{d_n k}})\} < \infty$$

Definition 2.7 *Define the partial zeta function of f over $\mathbb{F}_q$ to be*

$$Z_{d_1,\ldots,d_n}(f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} M_{d_1,\ldots,d_n}(f \otimes \mathbb{F}_{q^k})\right).$$

The partial zeta function measures the distribution of rational points of X independently along the fibres of the n-tuple of morphisms $(f_1, \cdots, f_n)$.

Example 2.8 *If $f_1 : X \mapsto X_1$ and $f_2 = \mathrm{Id} : X \mapsto X$, then $Z_{1,d}(f, T) = Z_d(f_1, T)$.*

Thus, partial zeta functions are generalizations of moment zeta functions.

Question 2.9

1. *What is* $Z_{d_1,\ldots,d_n}(f,T)$*?*
2. *How does* $Z_{d_1,\ldots,d_n}(f,T)$ *vary as* $\{d_1,\ldots,d_n\}$ *varies?*

We have

Theorem 2.10 ([26]) *The partial zeta function* $Z_{d_1,\ldots,d_n}(f,T)$ *is a rational function. Furthermore, its reciprocal zeros and poles are Weil* q*-numbers.*

3. General Properties of $Z(f,T)$.

Trace Formula

By Grothendieck [14], $Z(X,T)$ can be expressed in terms of l-adic cohomology. More precisely, let $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Then,

Theorem 3.1 *There are finite dimensional vector spaces* $H_c^i(X)$ *with invertible linear action by* Frob_q *such that*

$$Z(X,T) = \prod_{i=0}^{2\dim(X)} \det(I - \mathrm{Frob}_q^{-1}T|H_c^i(X))^{(-1)^{i-1}},$$

where

$$H_c^i(X) = \begin{cases} H_c^i(\overline{X},\mathbb{Q}_l), & l \neq p, \textit{prime} \\ H_{rig,c}(X,\mathbb{Q}_p), & l = p. \end{cases}$$

This is used to show that $Z(X,T) \in \mathbb{Q}(T)$. One should note:

1. $Z(X,T)$ is independent of the choice of l.
2. $\det(I - \mathrm{Frob}_q^{-1}T|H_c^i(X))$ may depend on the choice of l due to possible cancellation. The conjectural independence on l is still open in general.

Riemann Hypothesis

Fix an embedding of $\overline{\mathbb{Q}_l} \hookrightarrow \mathbb{C}$. Let $b_i = \dim H_c^i(X)$. Consider the factorization

$$\det(I - \mathrm{Frob}_q^{-1}T|H_c^i(X)) = \prod_{j=1}^{b_i}(1-\alpha_{ij}T), \alpha_{ij} \in \mathbb{C}.$$

The α_{ij}'s are Weil q-numbers, that is,

1. The α_{ij}'s are algebraic integers over $\mathbb{Q}$.
2. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $|\alpha_{ij}| = |\sigma(\alpha_{ij})| = \sqrt{q}^{\omega_{ij}}$ for some integer ω_{ij}, called the weight of α_{ij} with $0 \leq \omega_{ij} \leq i$,$\forall j = 1, \ldots b_i$.

The $l \neq p$ case was proved by Deligne [6] and the $l = p$ case by Kedlaya [19].

Slopes (p-adic Riemann Hypothesis)

Consider an embedding $\overline{\mathbb{Q}_l} \hookrightarrow \mathbb{C}_p$. Then what is the $\mathrm{ord}_q(\alpha_{ij}) \in \mathbb{Q}_{\geq 0}$? This is referred to as the slope of α_{ij}.

By Riemann Hypothesis,

$$\alpha_{ij}\overline{\alpha_{ij}} = q^{\omega_{ij}},$$

$$0 \leq \mathrm{ord}_q(\alpha_{ij}) \leq \mathrm{ord}_q(\alpha_{ij}\overline{\alpha_{ij}}) = \omega_{ij} \leq i,$$

Further, Deligne's integrality theorem implies that

$$i - \dim(X) \leq \mathrm{ord}_q(\alpha_{ij}).$$

Question 3.2 *Given $X/\mathbb{F}_q$, the following questions arise:*

1. *What is $b_{i,l} := b_i$?*
2. *What is ω_{ij}?*
3. *What is the slope $\mathrm{ord}_q(\alpha_{ij})$?*

Example 3.3 *If X is a smooth projective variety over $\mathbb{F}_q$, then:*

1. *$H^i_c(X)$ is pure of weight i, i.e. $\omega_{ij} = i$ for $1 \leq j \leq b_i$. Thus $b_{i,l}$ is independent of l.*
2. *The q-adic Newton polygon (NP) of $\det(I - \mathrm{Frob}_q^{-1}T|H^i_c(X)) \in \mathbb{Z}[[T]]$ lies above the Hodge polygon of $H^i_c(X)$. This was conjectured by Katz [17] and proven by Mazur [20] and Ogus [2]. We will discuss this more later.*

4. Moment Zeta Functions

Let $f : X \to Y/\mathbb{F}_q$. For $d \in \mathbb{Z}_{>0}$, recall the d-th moment of $f \otimes \mathbb{F}_{q^k}$ is

$$M_d(f \otimes \mathbb{F}_{q^k}) = \sum_{y \in Y(\mathbb{F}_{q^k})} \#f^{-1}(y)(\mathbb{F}_{q^{dk}}).$$

Question 4.1

1. *How does $M_d(f \otimes \mathbb{F}_{q^k})$ vary as k varies?*
2. *How does $M_d(f \otimes \mathbb{F}_{q^k})$ vary with d?*
3. *How does $M_d(f \otimes \mathbb{F}_{q^k})$ vary with both d and k?*

Definition 4.2 *Define the d-th moment zeta function of f to be*

$$Z_d(f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} M_d(f \otimes \mathbb{F}_{q^k})\right).$$

Observe for $d = 1$ we have $Z_1(f,T) = Z(X,T)$. Recall that $Z_d(f,T) \in \mathbb{Q}(T)$ and its reciprocal zeros and poles are Weil q-numbers. This follows from the following more precise cohomological formula.

Theorem 4.3 *Let $l \neq p$. Let $\mathfrak{F}^i = R^i f_! \mathbb{Q}_l$ be the i-th relative l-adic cohomology with compact support. Let $\sigma_{d,j,i} = Sym^{d-j}\mathfrak{F}^i \otimes \bigwedge^j \mathfrak{F}^i$. Then $Z_d(f,T) =$*

$$\prod_{i=0}^{2\dim(X/Y)} \prod_{j=0}^{d} \prod_{k=0}^{2\dim(Y)} \det\left(I - \mathrm{Frob}_q^{-1} T | H_c^k(\overline{Y}, \sigma_{d,j,i})\right)^{(-1)^{i+j+k-1}(j-1)}$$

Proof For an l-adic sheaf $\mathfrak{F}$ on Y, let $L(\mathfrak{F},T)$ denote the L-function of $\mathfrak{F}$. The trace formula in [14] applies to the L-function $L(\mathfrak{F},T)$:

$$L(\mathfrak{F},T) = \prod_{i=0}^{2\dim(Y)} \det(I - \mathrm{Frob}_q^{-1} T | H_c^i(\bar{Y}, \mathfrak{F}))^{(-1)^{i-1}}.$$

The d-th Adams operation of a sheaf $\mathfrak{F}$ can be written as the virtual sheaf [23]

$$[\mathfrak{F}]^d = \sum_{j \geq 0} (-1)^j (j-1) \left[Sym^{d-j}\mathfrak{F} \otimes \bigwedge^j \mathfrak{F} \right].$$

It follows that

$$\begin{aligned}
Z_d(f,T) &= \prod_{y\in|Y|} Z\left(f^{-1}(y) \otimes_{\mathbb{F}_{q^{\deg(y)}}} \mathbb{F}_{q^{\deg(y)d}}, T^{\deg(y)}\right) \\
&= \prod_{y\in|Y|} \prod_{i\geq 0} \det\left(I - (\mathrm{Frob}_{q^{\deg(y)}}^{-1})^d T^{\deg(y)} | \mathfrak{F}_y^i\right)^{(-1)^{i-1}} \\
&= \prod_{i\geq 0} \prod_{y\in|Y|} \det\left(I - T^{\deg(y)} (\mathrm{Frob}_{q^{\deg(y)}}^{-1}) | [\mathfrak{F}_y^i]^d\right)^{(-1)^{i-1}} \\
&= \prod_{i\geq 0} L([\mathfrak{F}^i]^d / Y, T)^{(-1)^i} \\
&= \prod_{i\geq 0} \prod_{j\geq 0} L\left(\sigma_{d,j,i}, T\right)^{(-1)^{i+j}(j-1)} \\
&= \prod_k \prod_{i\geq 0} \prod_{j\geq 0} \det\left(I - T\mathrm{Frob}_q^{-1} | H_c^k(\overline{Y}, \sigma_{d,j,i}, T)\right)^{(-1)^{i+j+k-1}(j-1)}.
\end{aligned}$$

□

To use this formula, one needs to know:

1. The total degree of $Z_d(f,T)$: number of zeros + number of poles.
2. The high weight trivial factor which gives the main term.
3. The vanishing of nontrivial high weight term which gives a good error bound.

Note:

1. There is an explicit upper bound for the total degree of $Z_d(f,T)$, which grows exponentially in d, see [9].
2. There exists a total degree bound of the form $c_1 d^{c_2}$ which is a polynomial in d. However, the constant c_1 is not yet known to be effective if $\dim Y \geq 2$, see [9].

Question 4.4 *How do we make c_1 effective?*

Example: Artin-Schreier hypersurfaces

Let

$$g(x_1,\dots,x_n,y_1,\dots,y_{n'}) \in \mathbb{F}_q[x_1,\dots,x_n,y_1,\dots,y_{n'}].$$

We may also rewrite this as $g = g_m + g_{m-1} + \ldots + g_0$, where g_i is the homogeneous part of degree i and $g_m \neq 0$.

Consider:

$$\begin{array}{l} X : \{x_0^p - x_0 = g(x_1,\dots,x_n,y_1,\dots,y_{n'})\} \hookrightarrow \mathbb{A}^{n+n'+1} \\ Y : \mathbb{A}^{n'} \\ f :\ X \mapsto Y, (x_0,x_1,\dots,x_n,y_1,\dots,y_{n'}) \mapsto (y_1,\dots,y_{n'}) \end{array}$$

One may then ask:

$$M_d(f) = \#\{x_i \in \mathbb{F}_{q^d}, y_i \in \mathbb{F}_q | x_0^p - x_0 = g(x,y)\} =?$$

Ideally for nice g, one hopes:

$$M_d(f) = q^{dn+n'} + O(q^{(dn+n')/2})$$

Theorem 4.5 (Deligne, [5]) *Assume that g is a Deligne polynomial of degree m, i.e., the leading form g_m is a smooth projective hypersurface in $\mathbb{P}^{n+n'}$ and $p \nmid m$. Then*

$$|M_1(f) - q^{n+n'}| \leq (p-1)(m-1)^{n+n'} q^{\frac{n+n'}{2}}.$$

For $d > 1$, a similar estimate can be obtained in some cases.

Assume $f^{-1}(y)$ is a Deligne polynomial of degree m for all $y \in \mathbb{A}^{n'}(\mathbb{F}_q)$. Then, applying Deligne's estimate fibre by fibre, one deduces

$$\#f^{-1}(y)(\mathbb{F}_{q^d}) = q^{dn} + E_y(d),$$

$$|E_y(d)| \leq (p-1)(m-1)^n q^{\frac{dn}{2}},$$

where $E_y(d)$ is some error term. From this, we get

$$\begin{aligned} M_d(f) &= \textstyle\sum_{y\in\mathbb{A}^{n'}(\mathbb{F}_q)} \#f^{-1}(y)(\mathbb{F}_{q^d}) \\ &= q^{dn+n'} + \textstyle\sum_{y\in\mathbb{A}^{n'}(\mathbb{F}_q)} E_y(d) \end{aligned}$$

Thus, we get the “trivial" estimate:

$$|M_d(f) - q^{dn+n'}| \leq (p-1)(m-1)^n q^{\frac{dn}{2}+n'}$$

Ideally, one would hope to replace n' by $n'/2$ in the above error bound.

If one applies the Katz type estimate via monodromy calculation as in [18], one gets $\sqrt{q}$ savings in good cases, i.e., with error term $O(q^{\frac{dn}{2}+n'-\frac{1}{2}})$. This is still far from the expected error bound $O(q^{\frac{dn+n'}{2}})$ if $n' \geq 2$.

Definition 4.6 *The d-th fibered sum of g is*

$$\bigoplus_Y^d g = g(x_{11}, \ldots, x_{1n}, y_1, \ldots, y_{n'}) + \ldots + g(x_{d1}, \ldots, x_{dn}, y_1, \ldots, y_{n'}).$$

Observe the y_i values remain the same while the x_{ij} values vary.

Theorem 4.7 (Fu-Wan, [9]) *Assume $\bigoplus_Y^d g$ is a Deligne polynomial of degree m. Then*

1. $|M_d(f) - q^{dn+n'}| \leq (p-1)(m-1)^{dn+n'} q^{\frac{dn+n'}{2}}$
2. $|M_d(f) - q^{dn+n'}| \leq c(p, n, n') d^{3(m+1)^n - 1} q^{\frac{dn+n'}{2}}$

The constant c is not known to be effective if $n' \geq 2$.

If p does not divide d, then $\bigoplus_Y^d g$ is a Deligne polynomial for a generic g of degree m. Thus, the assumption is satisfied for many g if p does not divide d. However, if $p \mid d$, there are no such g.

Question 4.8 *If $p|d$, what would be the best estimate $M_d(f)$?*

Example: Toric Calabi-Yau hypersurfaces

This geometric example is studied in a joint work with A. Rojas-Leon [21]. Let $n \geq 2$. We consider

$$X : \{x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n} - y = 0\} \hookrightarrow \mathbb{G}_m^n \times \mathbb{A}^1,$$

$$Y = \mathbb{A}^1,$$

$$f : (x_1, \cdots, x_n, y) \longrightarrow y.$$

For $y \neq (n+1)\zeta$, with $\zeta^{n+1} = 1$, we have

$$f^{-1}(y) : x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n} - y = 0$$

is an affine Calabi-Yau hypersurface in $\mathbb{G}_m^n$.

For $n = 2$, we have an elliptic curve. For $n = 3$, we have a K3 surface. For $n = 4$, we have a Calabi-Yau 3-fold. Recall

$$M_d(f) = \sum_{y \in \mathbb{F}_q} \# f^{-1}(y)(\mathbb{F}_{q^d}).$$

For $d = 1$, we have $M_1(f) = \# X(\mathbb{F}_q) = (q-1)^n$. For every $y \in \mathbb{F}_q$, we have

$$\#f^{-1}(y)(\mathbb{F}_{q^d}) = \frac{(q^d-1)^n - (-1)^n}{q^d} + E_y(d),$$

where $E_y(d)$ is some error term with $|E_y(d)| \leq nq^{d(n-1)/2}$. Thus,

$$M_d(f) = q\frac{(q^d-1)^n - (-1)^n}{q^d} + \sum_{y \in \mathbb{F}_q} E_y(d).$$

From this, we obtain the “trivial" estimate

$$|M_d(f) - \frac{(q^d-1)^n - (-1)^n}{q^{d-1}}| \leq nq^{d(n-1)/2+1}.$$

Theorem 4.9 (Rojas-Leon and Wan, [21]) *If $p \nmid (n+1)$, then*

1. $|M_d(f) - \left(\frac{(q^d-1)^n-(-1)^n}{q^{d-1}} + \frac{1}{2}(1+(-1)^d)q^{d(n-1)/2+1}\right) \leq Dq^{d(n-1)/2+\frac{1}{2}}$ *where D is an explicit constant depending only on n and d.*
2. *The purity decomposition of $Z_d(f,T)$ is determined.*

Question 4.10 *How do $M_d(f)$ and $Z_d(f,T)$ vary with d?*

5. Zeta Functions of Fibres

We continue with the previous example.

Example 5.1 *For $y \in \mathbb{F}_q$, let*

$$f^{-1}(y) = x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n} - y = 0 \hookrightarrow \mathbb{G}_m^n.$$

This is singular when $y \in \{(n+1)\zeta | \zeta^{n+1} = 1\}$. This family forms the mirror family of

$$\{x_0^{n+1} + \ldots + x_n^{n+1} - yx_0 \ldots x_n = 0\}.$$

Let $p \nmid (n+1), y \in \mathbb{F}_q \setminus \{(n+1)\zeta | \zeta^{n+1} = 1\}$. Then

$$Z(f^{-1}(y)/\mathbb{F}_q, T) = Z\left(\left\{\frac{(q^k-1)^n - (-1)^n}{q^k}\right\}_{k=1}^{\infty}, T\right) P_y(T)^{(-1)^n},$$

where $P_y(T) \in 1 + T\mathbb{Z}[T]$ of degree n, pure of weight $(n-1)$. Write

$$P_y(T) = (1-\alpha_1(y)T)\ldots(1-\alpha_n(y)T), \;\; |\alpha_i(y)| = \sqrt{q^{n-1}}.$$

Then we get the following:

Corollary 5.2

$$|\#f^{-1}(y)(\mathbb{F}_q) - \frac{(q-1)^n - (-1)^n}{q}| \leq n\sqrt{q^{n-1}}.$$

The star decomposition in [22][27] implies

Theorem 5.3 *There is a nonzero polynomial $H_p(y) \in \mathbb{F}_p[y]$ such that if $H_p(y) \neq 0$ for some $y \in \mathbb{F}_q$, then $\mathrm{ord}_q(\alpha_i(y)) = i-1$ for $1 \leq i \leq n$.*

Equivalently, this family of polynomials $f^{-1}(y)$ is generically ordinary. An alternative proof can be found in Yu [31].

Moment Zeta Functions

For $d > 0$, recall

$$M_d(f) = \sum_{y \in \mathbb{F}_q} \# f^{-1}(y)(\mathbb{F}_{q^d}),$$

$$M_d(f \otimes \mathbb{F}_{q^k}) = \sum_{y \in \mathbb{F}_{q^k}} \# f^{-1}(y)(\mathbb{F}_{q^{dk}}), k = 1, 2, 3, \ldots,$$

$$Z_d(f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} M_d(f \otimes \mathbb{F}_{q^k}) \right) \in \mathbb{Q}(T).$$

Let

$$S_d(T) = \prod_{k=0}^{[\frac{n-2}{2}]} \frac{1 - q^{dk}T}{1 - q^{dk+1}T} \prod_{i=0}^{n-1} (1 - q^{di+1}T)^{(-1)^{i+1}\binom{n}{i+1}}.$$

Theorem 5.4 (Rojas-Leon and Wan, [21]) *Assume that $(n+1)$ divides $(q-1)$. Then, the d-th moment zeta function for the above one parameter toric CY family f has the following factorization*

$$Z_d(f, T)^{(-1)^{n-1}} = P_d(T) \left(\frac{Q_d(T)}{P(d, T)} \right)^{n+1} A_d(T) S_d(T).$$

We now explain each of the above factors. First, $P_d(T)$ is the non-trivial factor which has the form

$$P_d(T) = \prod_{a+b=d, 0 \leq b \leq n} P_{a,b}(T)^{(-1)^{b-1}(b-1)},$$

and each $P_{a,b}(T)$ is a polynomial in $1 + T\mathbb{Z}[T]$, pure of weight $d(n-1)+1$, whose degree r is given explicitly and which satisfies the functional equation

$$P_{a,b}(T) = \pm T^r q^{(d(n-1)+1)r/2} P_{a,b}(1/q^{d(n-1)+1}T).$$

Second, $P(d,T) \in 1+T\mathbb{Z}[T]$ is the d-th Adams operation of the "non-trivial" factor in the zeta function of a singular fibre X_t, where $t=(n+1)\zeta_{n+1}$ and $\zeta_{n+1}^{n+1}=1$. It is a polynomial of degree $(n-1)$ whose weights are completely determined. Third, the quasi-trivial factor $Q_d(T)$ coming from a finite singularity has the form

$$Q_d(T) = \prod_{a+b=d, 0\leq b\leq n} Q_{a,b}(T)^{(-1)^{b-1}(b-1)},$$

where $Q_{a,b}(T)$ is a polynomial whose degree $D_{n,a,b}$ and the weights of its roots are given. Finally, the trivial factor $A_d(T)$ is given by:

$A_d(T) = (1-q^{\frac{d(n-1)}{2}}T)(1-q^{\frac{d(n-1)}{2}+1}T)(1-q^{\frac{d(n-2)}{2}+1}T)$ *if n and d are even.*

$A_d(T) = (1-q^{\frac{d(n-2)}{2}+1}T)$ *if n is even and d is odd.*

$A_d(T) = (1-q^{\frac{d(n-1)}{2}}T)$ *if n and d are odd.*

$A_d(T) = (1-q^{\frac{d(n-1)}{2}+1}T)^{-1}$ *if n is odd and d is even.*

Corollary 5.5 *Let $n=2$ and $f:\{x_1+x_2+\frac{1}{x_1x_2}-y=0\}\mapsto y$ with $p\nmid 3$. Then,*

$$Z_d(f,T)^{-1} = A_d(T)\frac{R_d(T)}{R_{d-2}(qT)},$$

where $A_d(T)$ is a trivial factor and $R_d(T)\in 1+T\mathbb{Z}[T]$ is a non-trivial factor which is pure of weight $d+1$ and degree $2(d-1)$.

For all $d\leq 1$, $R_d(T)=1$. $R_2(T)$ is a polynomial of degree 2 and weight 3. This suggests that $R_2(T)$ comes from a rigid Calabi-Yau variety. In general, $R_d(T)$ is of weight $d+1$ and degree $2(d-1)$.

As always, we may ask what are the slopes of $R_d(T)$?

The above one parameter family of Calabi-Yau hypersurfaces is the only higher dimensional example for which the moment zeta functions are determined so far. It shows that the calculation of the moment zeta function can be quite complicated in general. A related example is the one parameter family of higher dimensional Kloosterman sums, see [10][11] for the L-function of higher symmetric power which gives the main piece of the moment zeta function.

l-adic Moment Zeta Function ($l\neq p$)

Fix a prime $l\neq p$. Given $\alpha\in\mathbb{Z}_l^*$ and $d_1\equiv d_2 \bmod (l-1)l^{k-1}$ for some k, then $\alpha^{d_1}\equiv\alpha^{d_2} \bmod l^k$.

By rationality of $Z(f^{-1}(y),T)$ it follows that

$$\#f^{-1}(y)(\mathbb{F}_{q^d}) = \sum_i \alpha_i(y)^d - \sum_j \beta_j(y)^d$$

for some l-adic algebraic integers $\alpha_i(y)$ and $\beta_j(y)$. Consider

$$M_d(f) = \sum_{y\in Y(\mathbb{F}_q)} \#f^{-1}(y)(\mathbb{F}_{q^d}).$$

This can be rewritten as

$$= \sum_{y \in Y(\mathbb{F}_q)} \left(\sum_i \alpha_i(y)^d - \sum_j \beta_j(y)^d \right).$$

We may take some $D_l(f) \in \mathbb{Z}_{>0}$ such that if $d_1 \equiv d_2 \bmod D_l(f)l^{k-1}$ then

1. $M_{d_1}(f) \equiv M_{d_2}(f) \bmod l^k$.
2. $Z_{d_1}(f,T) \equiv Z_{d_2}(f,T) \bmod l^k \in 1 + T\mathbb{Z}[[T]]$.

Definition 5.6 *The l-adic weight space is defined to be*

$$W_l(f) = (\mathbb{Z}/D_l(f)\mathbb{Z}) \times \mathbb{Z}_l.$$

Let $s = (s_1, s_2) \in W_l(f)$. Take a sequence of $d_i \in \mathbb{Z}_{>0}$ such that

1. $d_i \to \infty$ *in* $\mathbb{C}$,
2. $d_i \equiv s_1 \bmod D_l(f)$,
3. $d_i \to s_2 \in \mathbb{Z}_l$.

With this we may define the l-adic moment zeta function

$$\zeta_s(f,T) = \lim_{i \to \infty} Z_{d_i}(f,T) \in 1 + T\mathbb{Z}_l[[T]].$$

This function is analytic in the l-adic open unit disk $|T|_l < 1$.

Question 5.7 *Is $\zeta_s(f,T)$ analytic on $|T|_l \leq 1$? What about in $|T|_l < \infty$?*

Embed $\mathbb{Z}$ in $W_l(f)$ in the following way:

$$\mathbb{Z} \hookrightarrow W_l(f),$$

$$d \mapsto (d,d).$$

Proposition 5.8 *If $d \in \mathbb{Z}_{>0} \hookrightarrow W_l(f)$, then $\zeta_d(f,T) = Z_d(f,T) \in \mathbb{Q}(T)$.*

Question 5.9 *What if $s \in W_l(f) \setminus \mathbb{Z}$? This is open even when f is a non-trivial family of elliptic curves over $\mathbb{F}_p$.*

p-adic Moment Zeta Functions ($l = p$)

As in the l-adic case, one has a p-adic continuous result.

If $d_1 \equiv d_2 \bmod D_p(f)p^{k-1}, d_1 \geq d_2 \geq c_f k$ for some k and sufficiently large constant c_f, then

$$M_{d_1}(f) \equiv M_{d_2}(f) \bmod p^k.$$

Also, define in the same way as above

$$\zeta_{s,p}(f,T) = \lim_{i\to\infty} Z_{d_i}(f,T) \in 1 + T\mathbb{Z}_p[[T]].$$

As before consider the embedding:

$$\mathbb{Z} \hookrightarrow W_p(f),$$

$$d \mapsto (d,d).$$

The following result was conjectured by Dwork [8].

Theorem 5.10 (Wan, [23][24][25]) *If $s = d \in \mathbb{Z} \hookrightarrow W_p(f)$, then $\zeta_{d,p}(f,T)$ is p-adic meromorphic in $|T|_p < \infty$.*

Furthermore, we have

Theorem 5.11 ([25]) *Assume the p-rank ≤ 1. Then for each $s \in W_p(f)$, $\zeta_{s,p}(f,T)$ is p-adic meromorphic in $|T|_p < \infty$.*

This can be extended a little further as suggested by Coleman.

Theorem 5.12 (Grosse-Klönne, [13]) *Assume the p-rank ≤ 1. For $s = (s_1, s_2)$ with $s_1 \in \mathbb{Z}/D_p(f)$ and $s_2 \in \mathbb{Z}_p/p^{\epsilon}$ (small denominator), then $\zeta_{s,p}(f,T)$ is p-adic meromorphic in $|T|_p < \infty$.*

Question 5.13 *In the case $s \in W_p(f) - \mathbb{Z}$ and p-rank > 1, it is unknown if $\zeta_{s,p}(f,T)$ is p-adic meromorphic, even on the closed unit disk $|T|_p \leq 1$.*

6. Moment Zeta Functions over $\mathbb{Z}$

Consider

$$f : X \mapsto Y/\mathbb{Z}[\frac{1}{N}].$$

The d-th moment zeta function of f is:

$$\zeta_d(f,s) = \prod_{p \nmid N} Z_d(f \otimes \mathbb{F}_p, p^{-s}).$$

Is this $\mathbb{C}$-meromorphic in $s \in \mathbb{C}$? Is $\zeta_d(f,s)$ or its special values p-adic continuous in some sense? If so, its p-adic limit $\zeta_s(f)(s \in \mathbb{Z}_p)$ is a p-adic zeta function of f.

Example 6.1 *Consider the map*

$$f : \{x_1 + x_2 + \frac{1}{x_1 x_2} - y = 0\} \mapsto y.$$

Then

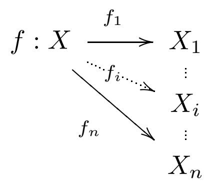

Figure 4. $f : X \mapsto X_1 \times \ldots \times X_n$

$$Z_d(f \otimes \mathbb{F}_p, T)^{-1} = A_d(T) \frac{R_d(f \otimes \mathbb{F}_p, T)}{R_{d-2}(f \otimes \mathbb{F}_p, pT)}$$

where $A_d(T)$ is a trivial factor and R_d is a non-trivial factor of degree $2(d-1)$ and weight $d+1$.

$$R_d(T) \leftrightarrow f^{\otimes d} = \{x_{11} + x_{12} + \frac{1}{x_{11}x_{12}} = \ldots = x_{d1} + x_{d2} + \frac{1}{x_{d1}x_{d2}}\}$$

Example 6.2 *For $d = 2$, we have*

$$x_1 + x_2 + \frac{1}{x_1 x_2} = y_1 + y_2 + \frac{1}{y_1 y_2}.$$

As Matthias Schuett observed during the workshop, $R_2(T) \leftrightarrow$ the unique new form of weight 4 and level 9.

Conjecture 6.3 $\prod_p R_d\left(f \otimes \mathbb{F}_p, p^{-s}\right)$ *is meromorphic in $s \in \mathbb{C}$ for all d.*

This conjecture is known to be true if $d \leq 2$. It should be realistic to prove the conjecture for all positive integers d.

7. l-adic Partial Zeta Functions

We now consider the system of maps where $X \mapsto X_1 \times \ldots \times X_n$ is an embedding (See Figure 4).

This allows us to define the partial zeta function

$$Z_{d_1,\ldots,d_n}(f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \#\{x \in X(\overline{\mathbb{F}_q}) | f_i(x) \in X_i(\mathbb{F}_{q^{d_i k}})\}\right) \in \mathbb{Q}(T).$$

Question 7.1 *Is there any p-adic or l-adic continuity result as $\{d_1, \ldots, d_n\}$ varies p-adically or l-adically?*

Example 7.2 *Consider the surface and three projection maps:*

$$f : x_1 + x_2 + \frac{1}{x_2 x_2} - x_3 = \quad 0 \begin{array}{l} \xrightarrow{f_1} x_1 \\ \xrightarrow{f_2} x_2 \\ \xrightarrow{f_3} x_3 \end{array}$$

Thus

$$M_{d_1,d_2,d_3}(f) = \#\{(x_1,x_2,x_3)|x_1 + x_2 + \frac{1}{x_1 x_2} - x_3 = 0, x_i \in \mathbb{F}_{q^{d_i}}, i = 1,2,3\}.$$

Is there a continuity result as $\{d_1, d_2, d_3\}$ *vary?*

8. Zeta Functions of Toric Affine Hypersurfaces

Let $\triangle \subset \mathbb{R}^n$ be an n-dimensional integral polytope. Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with

$$f = \sum_{u \in \triangle \cap \mathbb{Z}^n} a_u X^u, a_u \in \mathbb{F}_q$$

such that $\triangle(f) = \triangle$. That is, $a_u \neq 0$ for each u which is a vertex of $\triangle$.

Question 8.1 *Consider the toric affine hypersurface*

$$U_f : \{f(x_1, \ldots, x_n) = 0\} \hookrightarrow \mathbb{G}_m^n.$$

1. $\#U_f(\mathbb{F}_q) =$?
2. $Z(U_f, T) =$?

Definition 8.2

1. *If* $\triangle' \subset \triangle$ *is a face of* $\triangle$*, define*

$$f^{\triangle'} = \sum_{u \in \triangle' \cap \mathbb{Z}^n} a_u X^u.$$

2. f *is* $\triangle$*-regular if for every face* $\triangle'$ *(of any dimension) of* $\triangle$*, the system*

$$f^{\triangle'} = x_1 \frac{\partial f^{\triangle'}}{\partial x_1} = \ldots = x_n \frac{\partial f^{\triangle'}}{\partial x_n} = 0$$

has no common zeros in $\mathbb{G}_m^n(\overline{\mathbb{F}_q})$.

Theorem 8.3 (GKZ, [12])

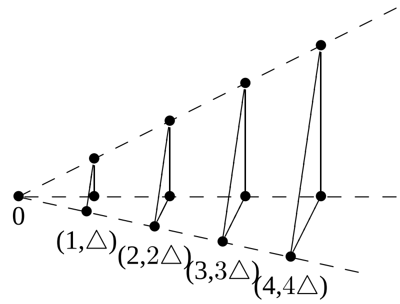

Figure 5. $C(\triangle)$

1. *There is a nonzero polynomial* $\mathrm{disc}_\triangle \in \mathbb{Z}[a_u | u \in \triangle \cap \mathbb{Z}^n]$ *such that* f *is* $\triangle$*-regular if and only if* $\mathrm{disc}_\triangle(f) \neq 0$ *in* $\mathbb{F}_q$*. In other words,* $\mathrm{disc}_\triangle$ *is an integer coefficient polynomial that will determine* $\triangle$*-regularity.*
2. $\triangle(\mathrm{disc}_\triangle)$ *is determined. This is referred to as the secondary polytope.*

Question 8.4 *For which* p*,* $\mathrm{disc}_\triangle \otimes \mathbb{F}_p \neq 0$*?*

Definition 8.5 *Let* $C(\triangle)$ *be the cone in* $\mathbb{R}^{n+1}$ *generated by* 0 *and* $(1, \triangle)$

1. *Define*

$$W_\triangle(k) = \#\{(k, k\triangle) \cap \mathbb{Z}^{n+1}\}, k = 0, 1, \ldots$$

The Hodge numbers of Δ *are defined by*

$$h_\triangle(k) = W_\triangle(k) - \binom{n+1}{1} W_\triangle(k-1) + \binom{n+2}{2} W_\triangle(k-2) - \ldots,$$

$$h_\triangle(k) = 0, \text{if } k \geq n+1.$$

2. $\deg(\triangle) = d(\triangle) = n! Vol(\triangle) = \sum_{k=0}^{n} h_\triangle(k)$.

Theorem 8.6 (Adolphson-Sperber [1], Denef-Loesser [4]) *Assume* $f/\mathbb{F}_q$ *is* $\triangle$*-regular. Then*

1. $Z(U_f, T) = \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i}\binom{n}{i+1}} P_f(T)^{(-1)^n}$ *with* $P_f(T) \in 1 + T\mathbb{Z}[T]$ *is of degree* $d(\triangle) - 1$.
2. $P_f(T) = \prod_{i=1}^{d(\triangle - 1)} (1 - \alpha_i(f)T), |\alpha_i(f)| \leq \sqrt{q}^{n-1}$*. In particular,*

$$|\#U_f(\mathbb{F}_q) - \frac{(q-1)^n - (-1)^n}{q}| \leq (d(\triangle) - 1)\sqrt{q}^{n-1}.$$

The precise weights of the $\alpha_i(f)$*'s were also determined by Denef-Loesser.*

Question 8.7 *For* $i = 1, 2, \ldots, d(\triangle) - 1$*, what is* $\mathrm{ord}_q(\alpha_i(f)) =$*?*

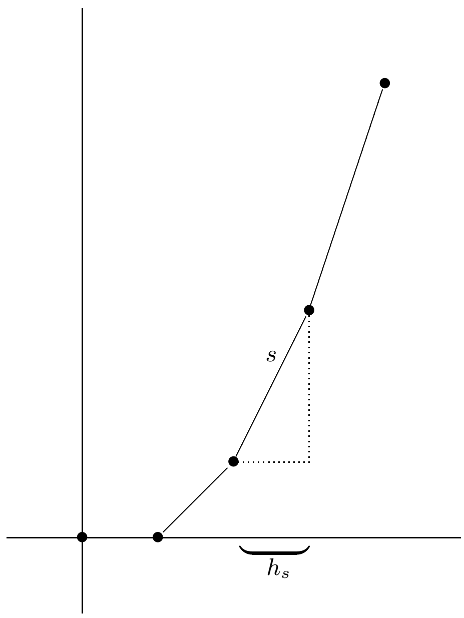

Figure 6. Newton Polygon

9. Newton and Hodge Polygons

Write

$$P_f(T) = 1 + c_1 T + c_2 T^2 + \dots .$$

The q-adic Newton polygon of $P_f(T)$ is the lower convex closure in $\mathbb{R}^2$ of the points $(k, \mathrm{ord}_q(c_k)), (k = 0, 1, \dots, d(\triangle) - 1)$. Denote this Newton polygon by $NP(f)$. Note that $NP(f) = NP(f \otimes \mathbb{F}_{q^k})$ for all k.

Proposition 9.1 *Let h_s denote the horizontal length of the slope s side in $NP(f)$. Then, $P_f(T)$ has exactly h_s reciprocal zeros $\alpha_i(f)$ such that $ord_q(\alpha_i(f)) = s$ for each $s \in \mathbb{Q}_{\geq 0}$.*

Definition 9.2 *The Hodge polygon of $\triangle$, denoted by $HP(\Delta)$, is the polygon in $\mathbb{R}^2$ with a side of slope $k-1$ with horizontal length $h_\triangle(k)$ for $1 \leq k \leq n$ and vertices*

$$(0,0), \left(\sum_{m=1}^{k} h_\triangle(m), \sum_{m=1}^{k} (m-1) h_\triangle(m) \right), k = 1, 2, \dots, n.$$

Theorem 9.3 (Adolphson-Sperber [1]) *The q-adic Newton polygon lies above the Hodge polygon, i,e. $NP(f) \geq HP(\triangle)$. In addition, the endpoints of the two coincide.*

Definition 9.4 *If $NP(f) = HP(\Delta)$, then f is called ordinary.*

Question 9.5 *When is f ordinary? One hopes this is often.*

Let

$$M_p(\triangle) = \{f \in \overline{\mathbb{F}_p}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] | \triangle(f) = \triangle,\ f\ \triangle - \text{regular}\}.$$

Theorem 9.6 (Grothendieck, [18]) *There exists a generic Newton polygon, denoted by $GNP(\Delta, p)$, such that*

$$GNP(\triangle, p) = \inf\{NP(f) | f \in M_p(\triangle)\}$$

Hence for any $f \in M_p(\triangle)$,

$$NP(f) \geq GNP(\triangle, p) \geq HP(\triangle),$$

where the first inequality is an equality for most f (generic f).

Question 9.7 *Given $\triangle$, for which p, is $GNP(\triangle, p) = HP(\triangle)$? In other words, when is f generically ordinary?*

This suggests the following conjecture.

Conjecture 9.8 (Adolphson-Sperber [1]) *For each $p \gg 0$, $GNP(\triangle, p) = HP(\Delta)$.*

This is false in general. Some counterexamples can be found in [22].

Definition 9.9

1. $S(\triangle) =$*the semigroup* $C(\triangle) \cap \mathbb{Z}^{n+1}$.
   $S_1(\triangle) =$ *the semigroup generated by* $(1, \triangle) \cap \mathbb{Z}^{n+1}$.
2. *Define the exponents of $\triangle$ as*

$$\begin{aligned} I(\triangle) &= \inf\{D > 0 | Du \in S_1(\triangle), \forall u \in S(\triangle)\} \\ I_\infty(\triangle) &= \inf\{D > 0 | Du \in S_1(\triangle), \forall u \in S(\triangle), u \gg 0\} \end{aligned}$$

Conjecture 9.10 *If $p \equiv 1$ mod $I(\triangle)$ or if $p \equiv 1$ mod $I_\infty(\triangle)$ for $p \gg 0$, then*

1. $\text{disc}_\triangle \otimes \mathbb{F}_p \neq 0$,
2. $GNP(\triangle, p) = HP(\triangle)$.

Part (2) is a weaker version of the conjecture in [22].

10. Generic Ordinarity

Toric Hypersurface

Let $\triangle \subset \mathbb{R}^n$ be a n-dimensional integral polytope and p a prime. Let $d(\triangle) = n! Vol(\triangle)$. Define

$$M_p(\triangle) = \{f \in \overline{\mathbb{F}_p}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] | \triangle(f) = \triangle,\ f\ \triangle - \text{regular}\}.$$

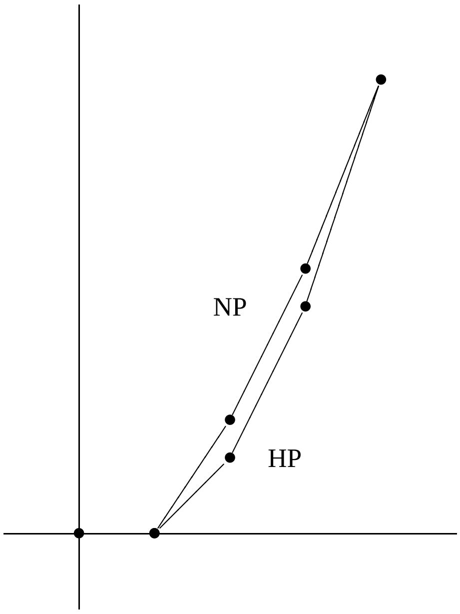

Figure 7. $NP \geq HP$

For each $f \in M_p(\triangle)$, let $NP(f)$ be the Newton polygon of the interesting factor $P_f(T)$ of the zeta function $Z(U_f, T)$. Note that changing the ground field will not change the Newton polygon. Recall that

$$NP(f) \geq GNP(\triangle, p) \geq HP(\triangle).$$

Note that $NP(f)$ is defined in a completely arithmetic fashion and is dependent on the coefficients of the polynomial f. On the other hand, $GNP(\triangle, p)$ is independent of coefficients while $HP(\triangle)$ is obtained combinatorially. If $GNP(\triangle, p) = HP(\triangle)$, we refer to p as ordinary for $\triangle$.

Conjecture 10.1 (Adolphson-Sperber) *For any $\triangle$, p is ordinary for all $p \gg 0$.*

This conjecture is too strong as Example 10.2 illustrates.

Example 10.2 *Let $f = a_0 + a_1x_1 + \ldots + a_nx_n + a_{n+1}x_1x_2\ldots x_n$ and*

$$\triangle = Conv((0, \ldots, 0), (1, \ldots, 0), \ldots, (0, \ldots, 1), (1, 1, \ldots, 1)).$$

Therefore $d(\triangle) = n$ for $n \geq 2$. Furthermore, Δ is an empty simplex, i.e., a simplex with no lattice points other than vertices. It follows that

1. *p is ordinary for $\triangle$ if and only if $p \equiv 1 \bmod (n-1)$. This implies*
2. *If $n \geq 4$, then the Adolphson-Sperber conjecture is false.*

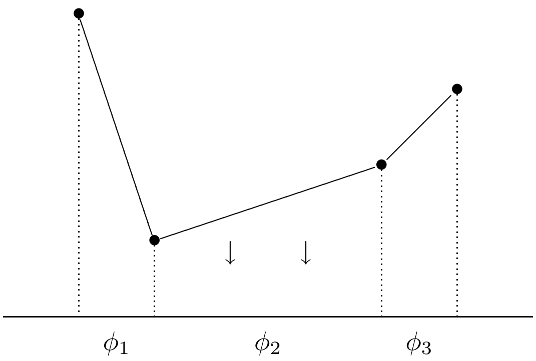

Figure 8. Piecewise projection down

Convex Triangulation

Definition 10.3

1. *A triangulation of* $\triangle$ *is a decomposition*

$$\triangle = \bigcup_{i=1}^{m} \triangle_i,$$

such that each $\triangle_i$ *is a simplex,* $\triangle_i \cap \triangle_j$ *is a common face for both* $\triangle_i$ *and* $\triangle_j$*.*
2. *The triangulation is called* **convex** if there is a piecewise linear function $\phi : \triangle \mapsto \mathbb{R}$ such that

   (a) ϕ is convex i.e. $\phi(\frac{1}{2}x + \frac{1}{2}x') \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x')$, for all $x, x' \in \triangle$.
   (b) The domains of linearity of ϕ are precisely the n-dimensional simplices $\triangle_i$ for $1 \leq i \leq m$.

Examples of convex triangulations include the star decomposition, the hyperplane decomposition and the collasping decomposition [27].

Basic Decomposition Theorem

The decomposition methods in [22][27] generalize to prove the following decomposition theorem.

Theorem 10.4

1. *Let* $\triangle = \cup_{i=1}^{m} \triangle_i$ *be a convex integral triangulation of* $\triangle$*. If* p *is ordinary for each* $\triangle_i$*,* $1 \leq i \leq m$*, then* p *is ordinary for* $\triangle$*.*
2. *If* $\triangle$ *is a simplex and* $p \equiv 1 \ mod \ d(\triangle)$*, then* p *is ordinary.*

Corollary 10.5 *If* $p \equiv 1 \ mod \ (lcm(d(\triangle_1), \ldots, d(\triangle_m)))$*, then* p *is ordinary.*

Example 10.6 *Let* A *be the convex closure of* $(-1, -1)$*,* $(1, 0)$ *and* $(0, 1)$ *in* $\mathbb{R}^2$*. The star decomposition in Figure 9 is convex and integral.*

More generally,

Example 10.7 *Consider* $f : \{x_1 + x_2 + \ldots + x_n + 1/x_1x_2\ldots x_n - y = 0\}$ *over* $\mathbb{F}_p$. *This is generically ordinary for all* p. *The proof uses the same star decomposition.*

Example 10.8 *Let* $\triangle = \{(d, 0, \ldots, 0), (0, d, 0, \ldots, 0), \ldots, (0, \ldots, d), (0, \ldots, 0)\}$. *We may make a parallel hyperplane cut as in Figure 10. This will make* $d(\triangle_i) = 1$ *for each piece* $\triangle_i$ *of the decomposition, see [22]. This proves that the universal family of affine (or projective) hypersurfaces of degree* d *and* n *variables over* $\mathbb{F}_p$ *is also generically ordinary for every* p. *The projective hypersurface (complete intersection) case was first proved by Illusie [15].*

Corollary 10.9 *If* $n = \dim(\triangle) = 2$, *then* p *is ordinary for* $\triangle$ *for all* p.

Corollary 10.10 *If* $n = \dim(\triangle) = 3$, *then* p *is ordinary for* $p > 6Vol(\triangle)$.

This corollary is proven by showing stability of the p-action on the weight. This is a different argument than by proving $d(\triangle_i) = 1$ argument.

Definition 10.11 *Let* $\triangle$ *be an* n*-dimensional integral convex polytope in* $\mathbb{R}^n$. *Assume that* 0 *(origin) is in the interior of* $\triangle$. *Given such a situation, define* $\triangle^* \subset \mathbb{R}^n$ *by:*

$$\triangle^* = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i y_i \geq -1, \ \forall (y_1, \ldots, y_n) \in \triangle\}$$

Observe $\triangle^*$ is also a convex polytope in $\mathbb{R}^n$, though it may not have integral vertices. Also observe $(\triangle^*)^* = \triangle$.

Definition 10.12 $\triangle$ *is called reflexive if* $\triangle^*$ *is also integral.*

Corollary 10.13 *If* $n = \dim(\triangle) = 4$ *and if* $\triangle$ *is reflexive then* p *is ordinary for* $\triangle$ *for all* $p > 12Vol(\Delta)$.

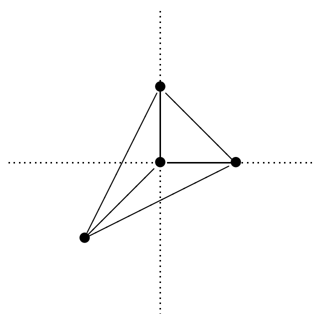

Figure 9. Star decomposition of A

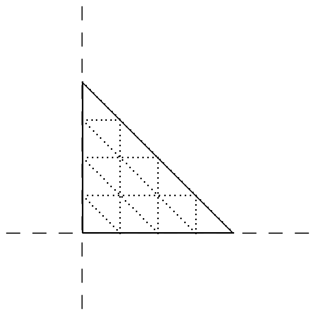

Figure 10. Parallel Hyperplane Decomposition into simplices

Slope Zeta Function

The concept of slope zeta functions was developed for arithmetic mirror symmetry as we will describe here. More information can be found in [29][30].

Let (X, Y) be a mirror pair over $\mathbb{F}_q$. Candelas, de la Ossa and Rodriques-Villegas in [3] desired a possible mirror relation of the type

$$Z(X, T) = \frac{1}{Z(Y, T)}$$

for 3 dimensional Calabi-Yau varieties. This is not true. If this were the case then

$$\sum \frac{T^k}{k} \#X(\mathbb{F}_q) = \sum \frac{T^k}{k}(-\#Y(\mathbb{F}_q)).$$

Therefore

$$\#X(\mathbb{F}_q) = -\#Y(\mathbb{F}_q),$$

which is impossible for large q on nonempty varieties.

The question is then to modify the zeta function suitably so that the desired mirror relation holds. The slope zeta function was introduced for this purpose.

Definition 10.14 *Write* $Z(X, T) = \prod_i (1 - \alpha_i T)^{\pm 1} \in \mathbb{C}_p(T)$.

1. *The slope zeta function of* X *is defined to be the following two variable function:*

$$S(X, U, T) = \prod_i (1 - U^{\mathrm{ord}_q(\alpha_i)} T)^{\pm 1}.$$

2. *If* $f : X \mapsto Y$ *defined over* $\mathbb{F}_q$ *(a nice family) then the slope zeta function of* f *is the generic one among* $S(f^{-1}(y), U, T)$ *from all* $y \in Y$*, denoted by* $S(f, U, T)$.

Conjecture 10.15 *Let* X *be a* 3*-dimensional Calabi-Yau variety over* $\mathbb{Q}$*. Assume that* X *has a mirror over* $\mathbb{Q}$*. Then the generic family containing* X *as a member is generically ordinary for all* $p \gg 0$.

This conjecture implies the following

Conjecture 10.16 (Arithmetic Mirror Conjecture) *Let $\{f, g\}$ be two generic mirror families of a 3-dimensional Calabi-Yau variety over $\mathbb{Q}$. Then for all $p \gg 0$,*

$$S(f \otimes \mathbb{F}_p, U, T) = \frac{1}{S(g \otimes \mathbb{F}_p, U, T)}.$$

References

[1] A. Adolphson and S. Sperber. *Exponential sums and Newton polyhedra: Cohomology and estimates*. Ann. Math., 130 (1989), 367-406.

[2] P. Berthelot and A. Ogus, *Notes on Crystalline Cohomology,* Princeton University Press, 1978.

[3] P. Candelas, X. de la Ossa, F. Rodriques-Villegas, *Calabi-Yau manifolds over finitef fields II,* Fields Instit. of Commun., 38(2003).

[4] J. Denef and F. Loesser, Weights of exponential sums, intersection cohomology, and Newton polyhedra, Invent. Math., 106(1991), no.2, 275-294.

[5] P. Deligne, *Applications de la Formule des Traces aux Sommes Trigonométriques,* in Cohomologie Étale (SGA $4\frac{1}{2}$), 168-232, Lecture Notes in Math. 569, Springer-Verlag 1977.

[6] P. Deligne, *La Conjecture de Weil II,* Publ. Math. IHES 52 (1980), 137-252.

[7] B. Dwork, *On the rationality of the zeta function of an algebraic variety,* Amer. J. Math., 82(1960), 631-648.

[8] B. Dwork, *Normalized period matrices II,* Ann. Math., 98(1973), 1-57.

[9] L. Fu and D. Wan, *Moment L-functions, partial L-functions and partial exponential sums,* Math. Ann., 328(2004), 193-228.

[10] L. Fu and D. Wan, *L-functions for symmetric products of Kloosterman sums,* J. Reine Angew. Math., 589(2005), 79-103.

[11] L. Fu and D. Wan, *Trivial factors for L-functions of symmetric products of Kloosterman sheaves,* Finite Fields & Appl., to appear.

[12] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminatns, Resultants and Multidimensional Determinants,* Birkhüser Boston, Inc., Boston, MA, 1994.

[13] E. Grosse-Klönne, *On families of pure slope L-functions,* Documenta Math., 8(2003), 1-42.

[14] A. Grothendick, *Formule de Lefschetz et rationalité des fonctions L,* Séminaire Bourbaki, exposé 279, 1964/65.

[15] L. Illusie. *Ordinarité des intersections complétes générales,* Grothendieck Festschrift, Vol. II (1990). 375-405.

[16] N. Katz, *On a theorem of Ax,* Amer. J. Math., 93(1971), 485-499.

[17] N. Katz, *Slope filtration of F-crystals,* Astérisque 63(1979), 113-164.

[18] N. Katz, *Frobenius-Schur indicator and the ubiquity of Brock-Granville quadratic excess,* Finite Fields & Appl., 7(2001), 45-69.

[19] K. Kedlaya, *Fourier transforms and p-adic "Weil II",* Compositio Mathematica, 142(2006), 1426-1450.

[20] B. Mazur, *Frobenius and the Hodge filtration,* Bull. Amer. Math. Soc., 78(1972), 653-667.

[21] A. Rojas-Leon and D. Wan, *Moment zeta functions for toric Calabi-Yau hypersurfaces,* Communications in Number Theory and Physics, Vol 1, No. 3, 2007.

[22] D. Wan, *Newton polygons of zeta functions and L functions,* Ann. of Math, Vol. 2, No. 2(1993), 249-293.

[23] D. Wan, *Dwork's conjecture on unit root zeta functions,* Ann. Math., 150(1999), 867-927.

[24] D. Wan, *Higher rank case of Dwork's conjecture,* J. Amer. Math. Soc., 13(2000), 807-852.

[25] D. Wan, *Rank one case of Dwork's conjecture,* J. Amer. Math. Soc., 13(2000), 853-908.

[26] D. Wan, *Rationality of partial zeta functions,* Indagationes Math., New Ser., 14(2003), 285-292.

[27] D. Wan, *Variations of p-adic Newton polygons for L-functions of exponential sums,* Asian J. Math., Vol.8, 3(2004), 427-474.

[28] D. Wan, *Geometric moment zeta functions,* Geometric Aspects of Dwork Theory, Walter de Gruyter, 2004 , Vol II, 1113-1129.

[29] D. Wan, *Arithmetic mirror symmetry,* Pure Appl. Math. Q., 1(2005), 369-378.

[30] D. Wan, *Mirror symmetry for zeta functions,* Mirror Symmetry V, AMS/IP Studies in Advanced Mathematics, Vol.38, (2007), 159-184.
[31] J-D. Yu, *Variation of the unit roots along the Dwork family of Calabi-Yau varieties,* preprint, 2007.