LINEARIZED WENGER GRAPHS

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Abstract. Motivated by recent extensive studies on Wenger graphs, we introduce a new infinite class of bipartite graphs of the similar type, called linearized Wenger graphs. The spectrum, diameter and girth of these linearized Wenger graphs are determined.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of order q such that p is prime and $q = p^e$ a prime power. All graph theory notions can be found in Bollobás [2]. Recently, a class of bipartite graphs called *Wenger graphs* which are defined over finite field $\mathbb{F}_q$ has attracted a lot of attention because of their nice graphical properties [5, 11, 12, 16, 18, 19, 20, 21]. For example, the number of edges of these graphs meets the lower bound of Turán number of the cycle with length $4, 6, 10$ [21]. The original definition was introduced by Wenger [21] for p-regular bipartite graphs and then was extended by Lazbnik and Ustimenko [11] for arbitrary prime power q. An equivalent representation of these graphs appeared later in Lazebnik and Viglione [13] and then a more general class of graphs was defined in [19], on which we concentrate in this paper.

Let m be a positive integer and $g_k(x, y) \in \mathbb{F}_q[x, y]$ for $2 \leq k \leq m+1$. Let $\mathfrak{P} = \mathbb{F}_q^{m+1}$ and $\mathfrak{L} = \mathbb{F}_q^{m+1}$ be two copies of the $(m+1)$-dimenional vector space over finite field $\mathbb{F}_q$, which are called the point set and the line set respectively. Let $\mathfrak{G} = G_q(g_2, \cdots, g_{m+1}) = (V, E)$ be the graph with vertex set $V = \mathfrak{P} \cup \mathfrak{L}$ and the edge set E is defined as follow: there is an edge from a point $P = (p_1, p_2, \cdots, p_{m+1}) \in \mathfrak{P}$ to a line $L = [l_1, l_2, \cdots, l_{m+1}] \in \mathfrak{L}$, denoted by $P \sim L$ (we force $\mathfrak{G}$ to be a undirected graph by removing the arrows), if the following m equalities hold:

$$\begin{aligned} l_2 + p_2 &= g_2(p_1, l_1) \\ l_3 + p_3 &= g_3(p_1, l_1) \\ \vdots \; & \vdots \; \vdots \\ l_{m+1} + p_{m+1} &= g_{m+1}(p_1, l_1). \end{aligned} \tag{1.1}$$

If $g_k(x, y), k = 2, \cdots, m+1$, are all monomials, the graph is called a *monomial graph*; see [6]. If $g_k(x, y) = x^{k-1}y, k = 2, \cdots, m+1$, then the graph is just the original Wenger graph in [5], also denoted by $W_m(q)$. It was shown in [11] that the automorphism group of $W_m(q)$ acts transitively on each of $\mathfrak{P}$ and $\mathfrak{L}$, and on the set of edges of $W_m(q)$. In other words, the graphs $W_m(q)$ are point-, line-, and edge-transitive. It is also shown that, see [12], $W_1(q)$ is

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vertex-transitive for all q, and that $W_2(q)$ is vertex-transitive for even q. For all $m \geq 3$ and $q \geq 3$, and for $m = 2$ and all odd q, the graphs $W_m(q)$ are not vertex-transitive. Another result of [12] is that $W_m(q)$ is connected when $1 \leq m \leq q-1$, and disconnected when $m \geq q$, in which case it has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. In [20], Viglione proved that the diameter of $W_m(q)$ is $2m+2$ when $1 \leq m \leq q-1$. In [5], Cioabă, Lazebnik and Li determined the spectrum of $W_m(q)$.

In this paper we focus on the basic properties of some extensions of Wenger graphs defined as in Equation (1.1). In Section 2 we first study the spectrum of a general class of graphs such that polynomials $g_k(x,y) \in \mathbb{F}_q[x,y]$ are defined by $g_k(x,y) = f_k(x)y$, and the mapping $\vartheta : \mathbb{F}_q \to \mathbb{F}_q^{m+1}; u \mapsto (1, f_2(u), \cdots, f_{m+1}(u))$ is injective. The eigenvalues of such a graph are determined, however, their multiplicities are reduced to counting certain polynomials with a given number of roots over finite fields. The latter problem is an interesting number theoretical problem, which is expected to be difficult in general. A complete solution in interesting special cases is already significant. In particular, we introduce a new class of bipartite graphs called linearized Wenger graphs. These graphs are denoted by $L_m(q)$, which are defined by Equation (1.1) together with $g_k(x,y) = x^{p^{k-2}}y, k = 2, \cdots, m+1$. Using results on linearized polynomials over finite fields, we are able to explicitly determine the spectrum of such graphs when $m \geq e$ in Section 3. Finally we obtain the diameter and girth of linearized Wenger graphs in Section 4 and Section 5, respectively. As a consequence, when $m = e$, this provides a new class of infinitely many connected p^e-regular expander graphs of q^{2m+2} vertices with optimal diameter $2(m+1)$ when either the prime p or the exponent e goes to infinity.

2. The spectrum of general Wenger graphs

In this section we study the basic properties of the class of graphs $\mathfrak{G}$ defined by $g_k(x,y) = f_k(x)y$, where $g_k(x,y)$ is a product of a polynomial in terms of x and the linear polynomial y, for $2 \leq k \leq m+1$.

Proposition 2.1. *The graph $\mathfrak{G} = G_q(f_2(x)y, \ldots, f_{m+1}(x)y)$ is q-regular.*

Proof. Given a point P and a line L in V, by definition, $P = (p_1, p_2, \cdots, p_{m+1})$ is adjacent to $L = [l_1, l_2, \cdots, l_{m+1}]$ if and only if the following m equalities hold:

$$\begin{cases} l_2 + p_2 & = & f_2(p_1)l_1 \\ l_3 + p_3 & = & f_3(p_1)l_1 \\ \vdots & \vdots & \vdots \\ l_{m+1} + p_{m+1} & = & f_{m+1}(p_1)l_1. \end{cases} \tag{2.1}$$

When the point P is prescribed, (2.1) implies that one can uniquely solve l_k $(k \geq 2)$ from l_1, and thus (2.1) has q solutions. Similarly, when the point L is prescribed, (2.1) implies that one can uniquely solve p_k $(k \geq 2)$ from p_1, and thus (2.1) has q solutions. □

Since $\mathfrak{G}$ is a bipartite graph, its adjacency matrix is of the form:

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

with a matrix N and

$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^TN \end{pmatrix}. \tag{2.2}$$

In order to consider the properties of $\mathfrak{G}$, we define a graph H as follows: the vertex set is $\mathbb{F}_q^{m+1}$ containing all lines in $\mathfrak{G}$, any two lines $L = [l_1, l_2, \cdots, l_{m+1}]$ and $L' = [l'_1, l'_2, \cdots, l'_{m+1}]$ are adjacent if and only if they share a common neighbor point $P = (p_1, p_2, \cdots, p_{m+1})$ in the graph $\mathfrak{G}$ defined above.

Moreover, one can check that the graph H is a Cayley graph with the generating set

$$S = \{(t, tf_2(u), \cdots, tf_{m+1}(u)) | \, t \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}.$$

Indeed, $L \sim L'$ if and only if $l_k - l'_k = f_k(p_1, l_1) - f_k(p_1, l'_1) = f_k(p_1)(l_1 - l'_1)$ for $2 \le k \le m+1$.

Furthermore, if B is the adjacency matrix of H then

$$NN^T = B + qI, \tag{2.3}$$

where I is the identity matrix. Let us denote all eigenvalues of H by $\lambda_1(B)$, …, $\lambda_{q^{m+1}}(B)$. Since N^TN and NN^T have the same eigenvalues, one can check that the eigenvalues of $\mathfrak{G}$ are $\pm\sqrt{\lambda_i(B) + q}, i = 1, 2, \cdots, q^{m+1}$.

Now let us assume the mapping $\vartheta : \mathbb{F}_q \to \mathbb{F}_q^{m+1}; u \mapsto (1, f_2(u), \cdots, f_{m+1}(u))$ is injective. Then we know that $|S| = q(q-1)$. Our first result is the following

Theorem 2.2. *Let $\mathfrak{G}$ be defined in (1.1) with the assumptions that $g_k(x, y) = f_k(x)y$ for $k = 2, \cdots, m+1$ and the mapping $\vartheta : \mathbb{F}_q \to \mathbb{F}_q^{m+1}$ defined by $u \mapsto (1, f_2(u), \cdots, f_{m+1}(u))$ is injective. For all prime power q and positive integer m, the eigenvalues of $\mathfrak{G}$, counted with multiplicities, are*

$$\pm\sqrt{qN_{F_w}}, w = (w_1, w_2, \cdots, w_{m+1}) \in \mathbb{F}_q^{m+1},$$

where $F_w(u) = w_1 + w_2 f_2(u) + \cdots + w_{m+1} f_{m+1}(u)$ and $N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}|$. For $0 \le i \le q$, the multiplicity of $\pm\sqrt{qi}$ is

$$n_i = |\{w \in \mathbb{F}_q^{m+1} : N_{F_w} = i\}|.$$

Moreover, the number of connected components of $\mathfrak{G}$ is

$$q^{m+1-\mathrm{rank}_{\mathbb{F}_q}(1, f_2, \cdots, f_{m+1})}.$$

Therefore $\mathfrak{G}$ is connected if and only if $1, f_2, \cdots, f_{m+1}$ are $\mathbb{F}_q$-linearly independent.

Proof. Let ζ_p be a primitive p-th root of unity, and for every $w := (w_1, w_2, \cdots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, we define a character $\psi_w : \mathbb{F}_q^{m+1} \to \mathbb{C}^*$ by

$$\psi_w : u = (u_1, u_2, \cdots, u_{m+1}) \mapsto \zeta_p^{\mathrm{tr}(w_1u_1 + w_2u_2 + \cdots + w_{m+1}u_{m+1})},$$

where tr is the absolute trace map. As described in [1, 14], the eigenvalues of the Cayley graph H are

$$\psi_w(S) := \sum_{t\in\mathbb{F}_q^*,u\in\mathbb{F}_q} \zeta_p^{\mathrm{tr}(t(w_1+w_2f_2(u)+\cdots+w_{m+1}f_{m+1}(u)))}, w\in\mathbb{F}_q^{m+1}. \tag{2.4}$$

Denote by $F_w(u)$ the function $w_1+w_2f_2(u)+\cdots+w_{m+1}f_{m+1}(u)$ and $N_{F_w}=|\{u\in\mathbb{F}_q : F_w(u)=0\}|$. Then it follows that

$$\begin{aligned}\psi_w(S) &= \sum_{t\in\mathbb{F}_q^*,u\in\mathbb{F}_q} \zeta_p^{\mathrm{tr}(tF_w(u))}\\ &= \sum_{t\in\mathbb{F}_q^*,F_w(u)=0} \zeta_p^{\mathrm{tr}(tF_w(u))} + \sum_{t\in\mathbb{F}_q^*,F_w(u)\neq 0} \zeta_p^{\mathrm{tr}(tF_w(u))}\\ &= (q-1)N_{F_w} + (-1)(q-N_{F_w})\\ &= q\,(N_{F_w}-1)\,.\end{aligned}$$

Thus this derives that the eigenvalues of $\mathfrak{G}$ are

$$\pm\sqrt{qN_{F_w}}, w\in\mathbb{F}_q^{m+1}, \tag{2.5}$$

where $N_{F_w}=|\{u\in\mathbb{F}_q : F_w(u)=0\}|$. For example, when $w=(0,\ldots,0)$ we have $N_{F_0}=q$ which implies that $\mathfrak{G}$ has $\pm q$ as its eigenvalues. Moreover, for any $w\neq 0$, it is easy to see that $N_{F_w}\leq \deg(F_w)\leq \max\{\deg(f_2),\ldots,\deg(f_{m+1})\}$.

The number of connected components of $\mathfrak{G}$ is

$$|\{w : F_w(x)\equiv 0 \text{ for all } x\in\mathbb{F}_q\}| = q^{m+1-\mathrm{rank}_{\mathbb{F}_q}(1,f_2,\cdots,f_{m+1})}. \tag{2.6}$$

Therefore $\mathfrak{G}$ is connected if and only if $1, f_2,\cdots,f_{m+1}$ are $\mathbb{F}_q$-linearly independent. □

Remark 1. *The computation of the multiplicities n_i's is obviously an interesting number theoretical problem. One cannot expect a simple closed formula for n_i's in general. Among the most interesting case is when the $f_k(x)$'s are given by monomials in x. When the f_k's are consecutive monomials (the original Wenger graph), there is indeed a simple formula for n_i's. When the f_k's are not consecutive monomials, the problem is more difficult. The linearized Wenger graph considered in next section deals with the first non-trivial example of non-consecutive monomials.*

3. The spectrum of linearized Wenger graphs

Let $q=p^e$ and m be a positive integer as before. We focus on the linearized Wenger graph $L_m(q)$ from now on where $f_k(x)=x^{p^{k-2}}$, $k=2,\cdots,m+1$. The goal of this section is to explicitly compute the spectum of $L_m(q)$ by determining the explicit formula of N_{F_w} and n_i in Theorem 2.2. The computation involved in linearized Wenger graphs is more complicated since the degrees of $f_k(x)=x^{p^{k-2}}, k=2,\ldots,m+1$ are high and not consecutive as in Wenger graphs.

We first give a basic lemma which will be used in the rest of the paper. It is an old result with the first derivation of the formula due to Landsberg [9, p.455]; see also Lemma 2.1 in [10].

Lemma 3.1. *The number of $l\times n$ matrices over $\mathbb{F}_q$ with rank k is* $\frac{\prod_{i=0}^{k-1}(q^l-q^i)(q^n-q^i)}{\prod_{i=0}^{k-1}(q^k-q^i)}$.

Proof. For a fixed k-dimensional subspace $W \in \mathbb{F}_q^l$, the number of $l \times n$ matrices with W as the column space is equal to the number of $k \times n$ matrices of rank k. Such a matrix is given by the k linearly independent row vectors of length n. The number of those is $\prod_{i=0}^{k-1}(q^n - q^i)$. The number of k-dimensional subspaces of $\mathbb{F}_q^l$ is $\frac{\prod_{i=0}^{i}(q^l - q^i)}{\prod_{i=0}^{i}(q^k - q^i)}$ and the product is the number of rank k matrices. □

When $m = e$, the functions $1, x, \cdots, x^{p^{m-1}}$ are $\mathbb{F}_q$-linearly independent and so $L_m(q)$ is connected. For every $w = (w_1, w_2, \cdots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, define $F_w(x) = w_1 + w_2 x + w_3 x^p + \cdots + w_{m+1} x^{p^{m-1}}$. By Theorem 2.2, the eigenvalues of the linearized Wenger graph $L_m(q)$, counting multiplicities, are

$$\pm\sqrt{qN_{F_w}}, w \in \mathbb{F}_q^{m+1},$$

where $N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}| = |\{u \in \mathbb{F}_q : \bar{F}_w(u) = -w_1\}|$, where $\bar{F}_w(x) = w_2 x + \cdots + w_{m+1} x^{p^{m-1}}$ is an $\mathbb{F}_p$-linearized polynomial. If $-w_1 \notin \mathrm{Im}(\bar{F}_w)$, then $N_{F_w} = 0$. Otherwise, this also implies that

$$N_{F_w} = p^{\dim_{\mathbb{F}_p}(\ker(\bar{F}_w))}.$$

Choosing a fixed basis of $\mathbb{F}_q/\mathbb{F}_p$ as $\alpha_1, \cdots, \alpha_e$, we know that every p-linear polynomial $\bar{F}_w(x)$ can be written as

$$\bar{F}_w(x) = \mathrm{tr}(\beta_1 x)\alpha_1 + \mathrm{tr}(\beta_2 x)\alpha_2 + \cdots + \mathrm{tr}(\beta_e x)\alpha_e, \quad (3.1)$$

where $\beta_1, \cdots, \beta_e$ are elements in $\mathbb{F}_q$ uniquely determined by $w_2, \ldots, w_{m+1}$. By Theorem 2.2 in [10], we have $\dim(\ker(\bar{F}_w)) = i$ if and only if $\mathrm{rank}_{\mathbb{F}_p}(\beta_1, \cdots, \beta_e) = e - i$. For $0 \le i \le e$, there are exactly

$$\frac{\prod_{j=0}^{e-i-1}(p^e - p^j)^2}{\prod_{j=0}^{e-i-1}(p^{e-i} - p^j)}$$

different $w_2, \ldots, w_{m+1}$ such that $\dim(\ker \bar{F}_w) = i$ by Lemma 3.1. There are p^{e-i} choices for $-w_1$ in the image set of $\bar{F}_w$, therefore the multiplicity of the eigenvalue $\pm\sqrt{qp^i}$ is

$$n_{p^i} = p^{e-i} \frac{\prod_{j=0}^{e-i-1}(p^e - p^j)^2}{\prod_{j=0}^{e-i-1}(p^{e-i} - p^j)}. \quad (3.2)$$

Now, counting each $-w_1$ not in the image set of $\bar{F}_w$ such that $\dim(\ker \bar{F}_w) = i$ for $1 \le i \le e$, the multiplicity of the eigenvalue 0 is

$$n_0 = \sum_{i=1}^{e} (p^e - p^{e-i}) \frac{\prod_{j=0}^{e-i-1}(p^e - p^j)^2}{\prod_{j=0}^{e-i-1}(p^{e-i} - p^j)}. \quad (3.3)$$

When $m > e$, one checks that $\mathrm{rank}_{\mathbb{F}_q}(1, x, x^p, \cdots, x^{p^{m-1}}) = e + 1$ and thus we obtain the following result:

Theorem 3.2. *Let $m \ge e$. The linearized Wenger graph $L_m(q)$ has q^{m-e} components. The distinct eigenvalues are*

$$0, \ \pm\sqrt{qp^i}, 0 \le i \le e.$$

For $0 \le i \le e$, the multiplicity of the eigenvalue $\pm\sqrt{qp^i}$ is $q^{m-e} n_{p^i}$ where n_{p^i} is given by (3.2). The multiplicity of the eigenvalue 0 is $q^{m-e} n_0$ where n_0 is given by (3.3).

When $m = e$, these linearized Wenger graphs are connect q-regular (q, ϵ)-expander graphs with edge expansion $\epsilon > \frac{q-\sqrt{qp^{e-1}}}{2} = \frac{q^{1/2}p^{(e-1)/2}(p^{1/2}-1)}{2}$. As to expander graphs, we refer to [7, 8] for more details.

When $m < e$, the linearized Wenger graph $L_m(q)$ is connected, however, we do not know a closed formula for the multiplicities of the eigenvalues $\pm\sqrt{qp^i}$. We leave this as an open problem.

4. The diameter of linearized Wenger graphs

Recall that a sequence of vertices $v_1, \cdots, v_s$ in a simple graph $\mathfrak{G} = (V, E)$ defines a *path* of length $s-1$ if $(v_i, v_{i+1}) \in E$ for every $i, 1 \le i \le s-1$. The *distance* between v_i and v_j is the number of edges in a shortest path joining v_i and v_j. The *diameter* of a graph $\mathfrak{G}$ is the maximum distance between any two vertices of $\mathfrak{G}$. In [20] it is shown that the diameter of the Wenger graph $W_m(q)$ is $2m+2$. In this section, we assume that $m \le e$ so that the linearized Wenger graphs are connected. We now explicitly determine the diameter of the linearized Wenger graph $L_m(q)$.

Theorem 4.1. *If $m \le e$, the diameter of the linearized Wenger graph $L_m(q)$ is $2(m+1)$.*

Before proceeding to the proof of the above theorem, we give the following lemma.

Lemma 4.2. *If $x_1, \ldots, x_m$ in $\mathbb{F}_q$ are $\mathbb{F}_p$-linearly independent, then*

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_m \\ x_1^p & x_2^p & \ldots & x_m^p \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{p^{m-2}} & x_2^{p^{m-2}} & \ldots & x_m^{p^{m-2}} \end{vmatrix} \neq 0.$$

Proof. First it is easy to see that

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_m \\ x_1^p & x_2^p & \ldots & x_m^p \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{p^{m-2}} & x_2^{p^{m-2}} & \ldots & x_m^{p^{m-2}} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 0 & x_2 - x_1 & \ldots & x_m - x_1 \\ 0 & (x_2-x_1)^p & \ldots & (x_m-x_1)^p \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (x_2-x_1)^{p^{m-2}} & \ldots & (x_m-x_1)^{p^{m-2}} \end{vmatrix}.$$

Since $x_1, \ldots, x_m$ are $\mathbb{F}_p$-linearly independent, $x_2 - x_1, \ldots, x_m - x_1$ are $\mathbb{F}_p$-linearly independent.

By induction, $\begin{vmatrix} x_2 - x_1 & \ldots & x_m - x_1 \\ (x_2-x_1)^p & \ldots & (x_m-x_1)^p \\ \vdots & \vdots & \vdots \\ (x_2-x_1)^{p^{m-2}} & \ldots & (x_m-x_1)^{p^{m-2}} \end{vmatrix} \neq 0$, the proof is complete. ☐

Proof of Theorem 4.1. First we consider the distaince between any two vertices L and L' in $\mathfrak{L}$ of the linearized Wenger graph $L_m(q)$. If $L_1P_1 \ldots P_sL_{s+1}$ is a path in $L_m(q)$ between $L = L_1$ and $L' = L_{s+1}$, where $L_i = [l_1^{(i)}, \cdots, l_{m+1}^{(i)}]$ and $P_i = (p_1^{(i)}, \cdots, p_{m+1}^{(i)})$, we have

$$l_k^{(i+1)} - l_k^{(i)} = (l_1^{(i+1)} - l_1^{(i)})(p_1^{(i)})^{p^{k-2}}, k = 2, \cdots, m+1, i = 1, \cdots, s.$$

Therefore there are elements $t_i = l_1^{(i+1)} - l_1^{(i)}$, $x_i = p_1^{(i)} \in \mathbb{F}_q$, $1 \leq i \leq s$, such that

$$(L_{s+1} - L_1)^T = t_1 \begin{pmatrix} 1 \\ x_1 \\ x_1^p \\ \vdots \\ x_1^{p^{m-1}} \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ x_2 \\ x_2^p \\ \vdots \\ x_2^{p^{m-1}} \end{pmatrix} + \cdots + t_s \begin{pmatrix} 1 \\ x_s \\ x_s^p \\ \vdots \\ x_s^{p^{m-1}} \end{pmatrix}. \quad (4.1)$$

Take $s = m+1$ and choose $x_1, \ldots, x_{m+1} \in \mathbb{F}_q$ such that $x_2 - x_1, \ldots, x_{m+1} - x_1$ are $\mathbb{F}_p$-linearly independent. Then by Lemma 4.2, the coefficient matrix of Eq. (4.1) is nonsingular, and thus Eq. (4.1) has a unique solution for $t_1, t_2, \ldots, t_s$. Thus the distance of any two vertices in $\mathfrak{L}$ is at most $2(m+1)$.

Similarly, let us consider any two vertices P and P' in $\mathfrak{P}$ of $L_m(q)$. Let $P_1 L_1 \ldots L_s P_{s+1}$ is a path in $L_m(q)$ between $P = P_1$ and $P' = P_{s+1}$, where $L_i = [l_1^{(i)}, \cdots, l_{m+1}^{(i)}]$ and $P_i = (p_1^{(i)}, \cdots, p_{m+1}^{(i)})$. Then we have

$$p_k^{(i+1)} - p_k^{(i)} = l_1^{(i)} (p_1^{(i+1)} - p_1^{(i)})^{p^{k-2}}, k = 2, \cdots, m+1, i = 1, \cdots, s.$$

Similarly, if we take $s = m+1$ and choose $p_i \in \mathbb{F}_q$ such that $p_1^{(i+1)} - p_1^{(i)}$, $1 \leq i \leq m$ are $\mathbb{F}_p$-linearly independent, then we can find unique solution for $l_1^{(1)}, \ldots, l_1^{(m)}$. Hence the distance of any two vertices in $\mathfrak{P}$ is at most $2(m+1)$.

Finally, we consider the distance between a vertex $P = (p_1, \ldots, p_{m+1}) \in \mathfrak{P}$ and a vertex $L \in \mathfrak{L}$. First we choose any line $L_1 \in \mathfrak{L}$ such that it is adjacent to P. From the earlier discussion, there exists a path from L_1 to L with distince at most $2(m+1)$. We modify the earlier construction so that the path goes through the vertex P. Namely, In Eq. (4.1), we let $x_1 = p_1$ and choose the rest of x_i's so that $x_2 - x_1, \ldots, x_{m+1} - x_1 \in \mathbb{F}_q$ are $\mathbb{F}_p$-linearly independent. Then there is a unique solution $\{t_1, \ldots, t_s\}$ and so there is a path between L_1 and L with length at most $2(m+1)$ passing through P. Therefore the distance of P and L is less than or equal to $2(m+1)$. Hence the diameter of $L_m(q)$ is always at most $2(m+1)$.

On the other hand, we now show that the distance $2(m+1)$ can be reached. Indeed, choose two vertices L_1 and L_{s+1} such that $L_{s+1} - L_1 = [0, \ldots, 0, 1]$. We can show that the distance between them is at least $2(m+1)$. Otherwise, suppose there is a path from L_1 to L_{s+1} with distance $2s \leq 2m$. Then Eq. (4.1) has a solution with $1 \leq s \leq m$. We show that this is impossible.

If either $x_1, \ldots, x_s$ are $\mathbb{F}_p$-linearly independent and $s < m$, or $x_1, \ldots, x_s$ are $\mathbb{F}_p$-linearly dependent, then the last m rows of (4.1) always can be reduced to

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = t_1' \begin{pmatrix} x_1' \\ (x_1')^p \\ \vdots \\ (x_1')^{p^{m-1}} \end{pmatrix} + t_2' \begin{pmatrix} x_2' \\ (x_2')^p \\ \vdots \\ (x_2')^{p^{m-1}} \end{pmatrix} + \cdots + t_k' \begin{pmatrix} x_k' \\ (x_k')^p \\ \vdots \\ (x_k')^{p^{m-1}} \end{pmatrix}, \quad (4.2)$$

where $x'_1, \dots, x'_k$ are $\mathbb{F}_p$-linearly independent and $k < m$. Because the determinant of the coefficient matrix of the system from the first k rows is not zero by Lemma 4.2, we must have $t'_i = 0$ for all i's, which contradicts with $t'_1 x'^{p^{m-1}}_1 + \dots + t'_k x'^{p^{m-1}}_k = 1$.

If $x_1, \dots, x_s$ are $\mathbb{F}_p$-linearly independent and $s = m$, then the determinant of the coefficient matrix of the system from the first m rows in Eq. (4.1) are not zero by Lemma 4.2. Again we must have $t_i = 0$ for all i's, which also contradicts with $t_1 x_1^{p^{m-1}} + \dots + t_s x_s^{p^{m-1}} = 1$. The proof is now complete. □

5. The girth of linearized Wenger graphs

In graph theory, the *girth* of a graph is the length of a shortest cycle contained in the graph. In [18], Shao et al proved the Wenger graphs have girth 8, and moreover, if $m \geq 3$, then for any integer l with $l \neq 5, 4 \leq l \leq 2p$ (where p is the character of the finite field $\mathbb{F}_q$) and any vertex v in the Wenger graph $W_m(q)$, there is a cycle of length $2l$ in $W_m(q)$ passing through the vertex v. The existence of the cycles of certain even length plays an important role in the study of the accurate order of the Turan number $ex(n; C_{2m})$ in extremal graph theory. See [3, 4, 15, 17]. In this section, we consider the girth of linearized Wenger graphs $L_m(q) = (V, E)$.

Let $P = (p_1, \cdots, p_{m+1}), P' = (p'_1, \cdots, p'_{m+1})$ be two distinct points in V. Suppose that P, P' share a common neighbor $L = [l_1, \cdots, l_{m+1}]$, then

$$P - P' = (p_1 - p'_1, l_1(p_1 - p'_1), l_1(p_1 - p'_1)^p, \cdots, l_1(p_1 - p'_1)^{p^{m-1}}). \tag{5.1}$$

In other words, $P - P'$ has the form $(u, lu, lu^p, \cdots, lu^{p^{m-1}})$. Conversely, if $P - P'$ has the form $(u, lu, lu^p, \cdots, lu^{p^{m-1}})$ with $u \neq 0$, we show that there exists a unique $L \in V$ such that L is a common neighbor of P, P'. Indeed, let $l_1 = l$. Since $l_1 p_1^{p^{k-2}} - p_k = l_1 p'^{p^{k-2}}_1 - p'_k, k = 2, \cdots, m+1$, we can define $l_k = l_1 p_1^{p^{k-2}} - p_k, k = 2, \cdots, m+1$, then the point $L = [l_1, \cdots, l_{m+1}]$ is a common neighbor of P, P'. Moreover, if both $L = [l_1, \cdots, l_{m+1}]$ and $L' = [l'_1, \cdots, l'_{m+1}]$ are common neighbors of P, P', then by definition, $l_1 = l'_1 = l$ and $l_k = l'_k = l_1 p_1^{p^{k-2}} - p_k = l_1 p'^{p^{k-2}}_1 - p'_k, k = 2, \cdots, m+1$. Thus $L = L'$.

We summarize the above discussion as follows:

Lemma 5.1. *In the linearized Wenger graph $L_m(q)$, two distinct points $P = (p_1, \cdots, p_{m+1})$ and $P' = (p'_1, \cdots, p'_{m+1})$ have a common neighbor if and only if $P - P'$ has the form $(u, lu, lu^p, \cdots, lu^{p^{m-1}})$ with $u \in \mathbb{F}_q^*, l \in \mathbb{F}_q$. Moreover, if $P - P'$ has the form $(u, lu, lu^p, \cdots, lu^{p^{m-1}})$ with $u \in \mathbb{F}_q^*, l \in \mathbb{F}_q$, then P, P' have a unique common neighbor.*

As a consequence, we have

Corollary 5.2. *There is no cycle of length 4 in the linearized Wenger graph $L_m(q)$.*

Proof. If $P_1 L_1 P_2 L_2 P_1$ or $L_1 P_1 L_2 P_2 L_1$ is a cycle of length 4 in the linearized Wenger graph, then L_1, L_2 are common neighbors of P_1, P_2, which is contrary to Lemma 5.1. □

Since the girth of the linearized Wenger graphs is even, the girth of the linearized Wenger graphs is at least 6 by Corollary 5.2. Furthermore, if $P_1 L_1 P_2 L_2 P_3 \dots L_t P_1$ is a cycle of length

$2t$ in the linearized Wenger graph $L_m(q)$, then there are elements $u_1, u_2, \ldots, u_t \in \mathbb{F}_q^*$, and $c_1, c_2, \ldots, c_t \in \mathbb{F}_q$ such that

$$\begin{cases} P_1 - P_2 = (u_1, c_1 u_1, c_1 u_1^p, \cdots, c_1 u_1^{p^{m-1}}) \\ P_2 - P_3 = (u_2, c_2 u_2, c_2 u_2^p, \cdots, c_2 u_2^{p^{m-1}}) \\ \vdots \\ P_t - P_1 = (u_t, c_t u_t, c_t u_t^p, \cdots, c_t u_t^{p^{m-1}}) \end{cases} \tag{5.2}$$

and thus

$$\begin{cases} u_1 + u_2 + \ldots + u_t = 0 \\ c_1 u_1 + c_2 u_2 + \ldots + c_t u_t = 0 \\ \vdots \\ c_1 u_1^{p^{m-1}} + c_2 u_2^{p^{m-1}} + \ldots + c_t u_t^{p^{m-1}} = 0. \end{cases} \tag{5.3}$$

The converse of this result does not hold since $P_1 L_1 P_2 L_2 P_3 \cdots L_t P_1$ may not be a cycle. For example, in linearized Wenger graph $L_1(11)$, choose $P_1 = (0,0)$, $P_2 = (-1,-1)$, $P_3 = (-2,0)$, $P_4 = P_1 = (0,0)$, $P_5 = (-1,-2)$, $P_6 = (-2,-8)$, $L_1 = (1,0)$, $L_2 = (-1,2)$, $L_3 = (0,0)$, $L_4 = (2,0)$, $L_5 = (6,-4)$, and $L_6 = (4,0)$. Then there are $u_1 = u_2 = u_4 = u_5 = 1$, $u_3 = u_6 = -2$, $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $c_4 = 2$, $c_5 = 6$, $c_6 = 4$ such that Eq. (5.2) and (5.3) hold. However, $P_1 L_1 \ldots P_6 P_1$ is not a cycle in $W_1(11)$.

Therefore, in order to study cycles of length $2t$ in linearized Wenger graphs, we first try to solve Eq. (5.2) and (5.3). If there are no u_i's and c_i's satisfying Eq. (5.2) and (5.3), then there is no cycle with length $2t$ in $L_m(q)$. Otherwise, construct $P_1, \ldots, P_t$ and $L_1, \ldots, L_t$ as follows:

Let $P_i = (p_1^{(i)}, \cdots, p_{m+1}^{(i)})$, $L_i = [l_1^{(i)}, \cdots, l_{m+1}^{(i)}]$, $i = 1, \cdots, t$, where

$$p_1^{(i)} - p_1^{(i+1)} = u_i, i = 1, 2, \ldots, t-1, p_1^{(t)} - p_1^{(1)} = u_t$$
$$l_1^{(i)} = c_i, l_k^{(i)} = l_1^{(i)} (p_1^{(i)})^{p^{k-2}} - p_k^{(i)}, k = 2, \cdots, m+1.$$

If both $P_1, \ldots, P_t$ are distinct and $L_1, \ldots, L_t$ are also distinct, then $P_1 L_1 P_2 L_2 P_3 \ldots L_t P_1$ is a cycle of length $2t$ in $W_m(q)$. Otherwise, we choose new solutions u_i's and c_i's, and test these new vertices. If there are always two P_i's (or two L_i's) which are the same in the above construction for all u_i's and c_i's satisfying Eq. (5.2) and (5.3), then there is no cycle with length $2t$ in $L_m(q)$.

Using the above technique, in the following we give the girth of linearized Wenger graphs.

Theorem 5.3. *Let $q = p^e$ and $m \geq 1$, $e \geq 1$ and p be an odd prime, or $m = 1$, $e \geq 2$ and $p = 2$. Then the girth of the linearized Wenger graph $L_m(q)$ is 6.*

Proof. Case 1. $m \geq 1$, $e \geq 1$ and p is an odd prime. By Corollary 5.2, it is enough to construct a cycle with length 6 in this case. Indeed, let $u_1 = u_2 = 1, u_3 = -2$, $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $P_1 = (0, 0, \ldots, 0)$, $P_2 = (-1, -1, \ldots, -1)$, $P_3 = (-2, 0, \ldots, 0)$, $L_1 = [1, 0, \ldots, 0]$, $L_2 = [-1, 2, 2, \ldots, 2]$, $L_3 = [0, 0, \ldots, 0]$. Then $P_1 L_1 P_2\ L_2\ P_3 L_3 P_1$ is a cycle with length 6.

Case 2. $e \geq 2$, $m = 1$ and $p = 2$. For an element $\beta \in \mathbb{F}_q^*$ and $\mathrm{tr}(\beta) = 0$, there exists some $\alpha \in \mathbb{F}_q^*$ such that $\alpha^2 + \alpha = \beta$. Put $u_1 = \alpha^2$, $u_2 = \alpha$, $u_3 = \beta$, $c_1 = 0$, $c_2 = \alpha^{-1}\beta$ and $c_3 = 1$. One can construct a cycle $P_1 L_1 P_2 L_2 P_3 L_3 P_1$ of length 6, where $P_1 = (0,0)$, $P_2 = (\alpha^2, 0)$, $P_3 = (\beta, \beta)$, $L_1 = [0,0]$, $L_2 = [\alpha^{-1}\beta, \alpha\beta]$ and $L_3 = [1,0]$. □

Theorem 5.4. *Let $q = p^e$, $p = 2$ and either $e = m = 1$ or $e \geq 1$, $m \geq 2$. Then the girth of the linearized Wenger graph $L_m(q)$ is 8.*

Proof. We need to show that there is no cycle of length 6 in $L_m(q)$ in these two cases. For the case of $e = 1$ and $p = 2$, there is no $u_i \in \mathbb{F}_q^*$, $1 \leq i \leq 3$, such that Eq (5.3) holds. Hence there is no cycle with length 6 in this case. Assume that there is a cycle $P_1L_1P_2L_2P_3L_3P_1$ of length 6 in $L_m(q)$ for the case of $e \geq 2$, $m \geq 2$ and $p = 2$. Then there are elements $u_1, u_2, u_3 \in \mathbb{F}_q^*$, $c_1, c_2, c_3 \in \mathbb{F}_q$ such that Eq (5.2) and (5.3) hold.

Eliminating c_1 among two successive equations of the last $m - 1$ equations in Eq. (5.3), we get

$$\begin{cases} u_1 + u_2 + u_3 = 0 \\ c_1u_1 + c_2u_2 + c_3u_3 = 0 \\ c_2(u_2^2 - u_2u_1) + c_3(u_3^2 - u_3u_1) = 0 \\ \vdots \\ c_2(u_2^{2^{m-1}} - u_2^{2^{m-2}}u_1^{2^{m-2}}) + c_3(u_3^{2^{m-1}} - u_3^{2^{m-2}}u_1^{2^{m-2}}) = 0. \end{cases} \tag{5.4}$$

Further simplifying Eq. (5.4) by using $u_1 + u_2 + u_3 = 0$ and $u_1, u_2, u_3 \in \mathbb{F}_q^*$, we get

$$\begin{cases} u_1 + u_2 + u_3 = 0 \\ c_1u_1 + c_2u_2 + c_3u_3 = 0 \\ c_2 + c_3 = 0 \\ \vdots \\ c_2 + c_3 = 0. \end{cases} \tag{5.5}$$

Therefore, by symmetry, Eq. (5.3) has only the solution $c_1 = c_2 = c_3$. Then we have $L_1 = L_3$ since they share the common vertex P_1, which contradicts to the earlier assumption.

In the following we can construct a cycle $P_1L_1P_2L_2 \ldots L_4P_1$ in both cases: Put $u_1 = u_2 = u_3 = u_4 = 1$ and $c_1 = c_3 = 0$, $c_2 = c_4 = 1$. Let $P_1 = (0, 0, 0, \ldots, 0)$, $P_2 = (1, 0, 0, \ldots, 0)$, $P_3 = (0, 1, 1, \ldots, 1)$, $P_4 = (1, 1, 1, \ldots, 1)$, $L_1 = [0, 0, 0, \ldots, 0]$, $L_2 = [1, 1, 1, \ldots, 1]$, $L_3 = [0, 1, 1, \ldots, 1]$, $L_4 = [1, 0, 0, \ldots, 0]$. Then it is straightforward to check $P_1L_1P_2L_2 \ldots L_4P_1$ is indeed a cycle of length 8. Hence we complete the proof. □

6. Open Problems

There are several open problems about linearized Wenger graphs. First finding an explicit formula for the eigenvalue multiplicities n_{p^i}'s of the linearized Wenger graphs when $m < e$ is an open problem. Constructing even cycles with specific length in linearized Wenger graphs is also interesting. In addition, it would be desirable to find new classes of $f_k(x)$ such that the explicit spectrum of these new types of Wenger graphs can be determined by Theorem 2.2.

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