Class numbers and p-ranks in $\mathbb{Z}_p^d$-towers

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A R T I C L E   I N F O

A B S T R A C T

To extend Iwasawa's classical theorem from $\mathbb{Z}_p$-towers to $\mathbb{Z}_p^d$-towers, Greenberg conjectured that the exponent of p in the n-th class number in a $\mathbb{Z}_p^d$-tower of a global field K ramified at finitely many primes is given by a polynomial in p^n and n of total degree at most d for sufficiently large n. This conjecture remains open for $d \geq 2$. In this paper, we prove that this conjecture is true in the function field case. Further, we propose a series of general conjectures on p-adic stability of zeta functions in a p-adic Lie tower of function fields.

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1. Introduction

Let K be a global field, that is, either a number field or a global function field with constant field $\mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of q elements with characteristic p. In the second case, K is simply the function field of a smooth projective geometrically irreducible curve C defined over $\mathbb{F}_q$. For a positive integer d, consider a $\mathbb{Z}_p^d$-tower of global fields

$$K_\infty = \bigcup_{n=0}^{\infty} K_n \supset \cdots \supset K_n \supset \cdots \supset K_1 \supset K_0 = K,$$

where for all $n \geq 0$,

$$\mathrm{Gal}(K_n/K) = (\mathbb{Z}_p/p^n\mathbb{Z}_p)^d.$$

In this paper, we always assume that the tower is ramified only at finitely many primes of K. This condition is automatically satisfied in the number field case. The condition is natural and necessary in the function field case in order for many definitions and theorems to make sense. Let h_n denote the class number of the field K_n. The basic problem in Iwasawa theory is to understand the stable behavior of the p-part of h_n, namely, the p-adic valuation $v_p(h_n)$ as a function of n. In the literature, the number $v_p(h_n)$ is usually denoted by e_n, which is the exponent of p in the class number h_n. To extend Iwasawa's classical result for $\mathbb{Z}_p$-extensions, Ralph Greenberg made the following conjecture about 40 years ago, see section 7 in [CM].

Conjecture 1.1. *Let K_∞/K be a $\mathbb{Z}_p^d$-extension of a global field K which ramifies at finitely many primes. There is a polynomial $E(x,y) \in \mathbb{Q}[x,y]$ of total degree at most d depending on the tower such that for all sufficiently large n, we have*

$$v_p(h_n) = E(p^n, n).$$

In the case $d = 1$ and K is a number field, this is precisely the classical Iwasawa theorem (1959) which says that $v_p(h_n) = \mu p^n + \lambda n + \nu$ for all sufficiently large n, where μ, λ, ν are constants depending on the tower. Historically, Iwasawa first observed that in the case $d = 1$ and K_∞ is a constant field $\mathbb{Z}_p$-extension of a function field, the statement is true. This motivated him to develop the theory of $\mathbb{Z}_p$ extensions, which led to his proof of the case $d = 1$ for number fields. It was realized in Mazur–Wiles [MW] (1983) and Gold–Kisilevsky [GK] (1988) that Iwasawa's ideas carry over to the function field case as well for $d = 1$, not necessarily constant field extensions. Thus, Greenberg's conjecture was long known to be true in the case $d = 1$, for both number fields and function fields.

To understand how the μ and λ invariants vary as K_∞ varies over all $\mathbb{Z}_p$-extensions of a fixed number field K, Greenberg [Gr] (1973) initiated the study of Iwasawa theory for $\mathbb{Z}_p^d$-extensions of K. The subsequent development of $\mathbb{Z}_p^d$-extensions led Monsky [Mo2] to prove that the μ invariant $\mu(K_\infty/K)$ is bounded by a constant depending only on K and p, as K_∞ varies over all $\mathbb{Z}_p$-extensions of a number field K. It seems still open if the λ invariant $\lambda(K_\infty/K)$ is also bounded by a constant depending only on K and p. Recall that for a number field K, the composite of all $\mathbb{Z}_p$-extensions of K is a $\mathbb{Z}_p^r$-extension of K, where $r_2 + 1 \leq r \leq [K : \mathbb{Q}]$ and r_2 is the number of complex primes of K. Leopoldt's conjecture is equivalent to saying that $r = r_2 + 1$. In contrast, for a function field K, the composite of all $\mathbb{Z}_p$-extensions of K is a huge $\mathbb{Z}_p^\infty$-extension, and the μ-invariant $\mu(K_\infty/K)$ is not known to be bounded as K_∞ varies over all $\mathbb{Z}_p$-extensions of K, see [LZ].

In the number field case with $d \geq 2$, Greenberg's conjecture was initially studied by Cuoco [Cu] (1980) in the case $d = 2$, and later in a series of papers by Cuoco–Monsky [CM] (1981), and Monsky [Mo1][Mo2][Mo3][Mo4] (1981–1989) for general d.

The strongest result so far is from Monsky [Mo4] (1989) who proves the number field case of the following asymptotic theorem.

Theorem 1.2. *Let K_∞/K be a $\mathbb{Z}_p^d$-extension of a global field K which ramifies at finitely many primes. There are integers m_0, ℓ_0 and a real number α depending on the tower such that for all sufficiently large n, we have*

$$v_p(h_n) = (m_0 p^n + \ell_0 n + \alpha)p^{(d-1)n} + O(np^{(d-2)n}).$$

As later remarked in Li-Zhao [LZ] (1997), Monsky's proof carries over to function fields too, and thus the above asymptotic theorem is true for both number fields and function fields. Since then, progress on Greenberg's conjecture has stopped in both the number field case and the function field case, perhaps due to the following two reasons. First, Cuoco–Monsky [CM] wrote "Despite the evidence of this paper and of [CM] we believe this (Greenberg conjecture) to be false, and show that a related module-theoretic conjecture fails, even when $d = 2$". In another paper, Monsky [Mo1] wrote "Greenberg's conjecture is probably false; the last section of [Cr] presents module-theoretic evidence against it". Second, Li-Zhao wrote "Classically, there are several approaches to this problem: (A) via class number formula and p-adic L-functions; (B) via class number formula and p-adic measures on $\mathbb{Z}_p$; (C) via the theory of noetherian modules over Iwasawa algebra. Approach (A) works particularly well for the so called cyclotomic $\mathbb{Z}_p$-extensions. But it apparently fails for more general fields which do not possess a good class number formula. (B) was found by Sinnott when he studied the work of Ferrero and Washington. Neither (A) nor (B) can be used to handle the function field case because good analogues of p-adic L-functions and p-adic measures are still unborn for function fields. Approach (C) was the one initiated by Iwasawa and further developed by Serre, Greenberg, Cuoco, Monsky and many others. This method can be combined with class field theory and Kummer theory to get a lot of nice results. The last approach of the above also has the merit of being generalizable to function field case, which is what we shall use in the present paper". This was the status up to 1997, and no further work was done on this conjecture during the last twenty years.

Our aim of this paper is to prove that Greenberg's conjecture is true in the function field case for all $d \geq 1$. Namely, we have

Theorem 1.3. *Let K_∞/K be a $\mathbb{Z}_p^d$-tower of function fields of characteristic p which ramifies at finitely many primes. Then, there is a polynomial $E(x,y) \in \mathbb{Q}[x,y]$ of total degree at most d (and degree at most 1 in y) depending on the tower such that for all sufficiently large n, we have*

$$v_p(h_n) = E(p^n, n).$$

We use the class number formula and p-adic L-function approach. The required universal p-adic L-function in function fields is the T-adic L-function introduced in [LW]

(2009) for $d = 1$ and again in [RWXY] (2017) for $d > 1$. The T-adic L-function is instrumental in the study of slope stability of zeta functions in $\mathbb{Z}_p^d$-tower, which is a new direction, see [DWX,RWXY]. Its analytic property on the closed unit disk (the main conjecture) follows from the work of Crew [Cr] (1987) on a conjecture of Katz [Ka]. The class number is a special value of the zeta function, which can be decomposed in terms of L-functions of the finite characters of the Galois group. To complete the proof, we partition the finite character space of the Galois group into a disjoint union of subspaces in terms of the exact ramification locus of the character. These disjoint subspaces turn out to be semi-algebraic in the sense of [Mo1], and one can then apply Monsky's crucial power series lemma [Mo1]. Contrary to Monsky's belief, philosophically our result suggests that Greenberg's conjecture should be true in the number field case.

A bye-product of our proof also gives a stable formula for the geometric p-rank $r_p(n)$ of K_n. Recall that K_n is the function field of a smooth projective geometrically irreducible curve C_n defined over some finite extension field over $\mathbb{F}_q$. The Jacobian variety J_n of C_n is an abelian variety of dimension g_n, where g_n is the genus of C_n. The p-adic Tate module is

$$T_p(J_n) = \varprojlim_k J_n[p^k],$$

where $J_n[p^k]$ denotes the group of p^k-torsion points of J_n over an algebraic closure of $\mathbb{F}_q$. The Tate module $T_p(J_n)$ is a free $\mathbb{Z}_p$-module whose rank is called the p-rank of K_n, denoted by $r_p(n)$. It is well known that $0 \leq r_p(n) \leq g_n$, where g_n is the genus of K_n.

Theorem 1.4. *Let K_∞/K be a $\mathbb{Z}_p^d$-tower of function fields of characteristic p which ramifies at finitely many primes. Then, there is a polynomial $R(x) \in \mathbb{Z}[x]$ of degree at most d depending on the tower such that for all sufficiently large n, we have*

$$r_p(n) = R(p^n).$$

Finally, we note that there is no stable formula for the genus sequence g_n in the function field case, even for $d = 1$. In fact, the genus g_n can grow as fast as one wants as a function of n. A simple necessary and sufficient condition for the stability of g_n is given in [KW] in the additive framework of Artin–Schreier–Witt construction of $\mathbb{Z}_p$-towers. For natural $\mathbb{Z}_p^d$-towers (those coming from algebraic geometry), we conjecture that the genus g_n is periodically given by several polynomials in p^n for all sufficiently large n. This is a small part of a series of general conjectures on p-adic stability of zeta functions for p-adic Lie towers of function fields that we shall state and discuss in the last section. It is hoped that these conjectures would motivate further work in both the function field case and the number field case.

2. The case with no constant $\mathbb{Z}_p$-extension

Let K be a global function field with constant field $\mathbb{F}_q$. In this section and next section, we assume that the $\mathbb{Z}_p^d$-tower K_∞/K contains no constant subextension. The tower gives a continuous group isomorphism

$$\rho : G_\infty := \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^d.$$

For each integer d-tuple $(n_1, ..., n_d) \in \mathbb{Z}_{\geq 0}^d$, reduction modulo $(p^{n_1}, ..., p^{n_d})$ produces a subextension $K_{n_1,...,n_d}$ such that

$$G_{n_1,...,n_d} := \mathrm{Gal}(K_{n_1,...,n_d}/K) \cong \prod_{i=1}^{d} \mathbb{Z}_p/p^{n_i}\mathbb{Z}_p.$$

This makes sense even if some of the n_i are ∞. For $1 \leq i \leq d$, let $K_n^{(i)} = K_{0,..,n,...,0}$, where n lies in the i-th coordinate. For each i, $K_\infty^{(i)}/K$ is a $\mathbb{Z}_p$-tower and thus it is totally ramified at a prime x if the first layer $K_1^{(i)}$ is ramified at x. It is clear that for $0 \leq n \leq \infty$,

$$K_n = K_{n,...,n} = K_{n,0,...,0} \cdots K_{0,...,0,n} = K_n^{(1)} \cdots K_n^{(d)}.$$

Let P be the set of primes of K. Let U be the unramified locus of the tower, which is a subset of P. The tower is ramified on $P - U$. By our assumption, $P - U$ is finite. Furthermore, by class field theory, the set $P - U$ is non-empty since there is no constant subextension. Each p-adic character $\chi : G_\infty \to \mathbb{C}_p^*$ has a unique decomposition

$$\chi = \chi_1 \otimes \cdots \otimes \chi_d,$$

where each χ_i is a p-adic character of G_∞ factoring through the quotient

$$G_\infty^{(i)} = \mathrm{Gal}(K_\infty^{(i)}/K) \cong \mathbb{Z}_p.$$

To recall the class number formula, we need to introduce the zeta function.

Let P_n denote the set of primes of K_n. Recall that the zeta function of K_n is defined by

$$Z(K_n, s) = \prod_{x \in P_n} \frac{1}{1 - s^{\deg(x)}} \in 1 + s\mathbb{Z}[[s]].$$

The Riemann–Roch theorem implies that the zeta function is a rational function in s of the form

$$Z(K_n, s) = \frac{P(K_n, s)}{(1-s)(1-qs)}, \ \ P(K_n, s) \in 1 + s\mathbb{Z}[s],$$

where the numerator $P(K_n, s)$ is a polynomial in s of degree $2g_n$ and $g_n = g(K_n)$ denotes the genus of K_n. By the celebrated theorem of Weil, the polynomial $P(K_n, s)$ is pure of q-weight 1, that is, the reciprocal roots of $P(K_n, s)$ all have complex absolute value equal to $\sqrt{q}$. The class number h_n of K_n is given by the residue formula

$$h_n = P(K_n, 1).$$

The Weil bound implies a good estimate for the complex absolute value of the class number h_n:

$$(\sqrt{q} - 1)^{2g_n} \le h_n \le (\sqrt{q} + 1)^{2g_n}.$$

Thus, the growth of h_n depends very much on the growth of the genus g_n which can be complicated in general, unless the genus sequence g_n becomes stable in some sense, see the last section for the genus stability conjecture for towers coming from algebraic geometry. In this paper, we shall mainly be interested in the p-adic absolute value of the class number, equivalently, $v_p(h_n)$ as a function of n. For this purpose, we use L-functions to decompose it.

Recall that for positive integer $n \ge 1$, the Galois group

$$G_n = \mathrm{Gal}(K_n/K) \cong (\mathbb{Z}/p^n\mathbb{Z})^d.$$

For a finite continuous p-adic character $\chi : G_\infty \to \mathbb{C}_p^*$, χ will factor through G_n for some finite n. That is, $\chi : G_\infty \to G_n \to \mathbb{C}_p^*$. The L-function of χ over K is

$$L(\chi, s) = \prod_{\substack{x \in P, \chi \text{ unramified at } x}} \frac{1}{1 - \chi(\mathrm{Frob}_x)s^{\deg(x)}} \in 1 + s\mathbb{C}_p[[s]],$$

where Frob_x denotes the arithmetic Frobenius element of G_∞ at x. Note that the ramification locus of the character χ could be smaller than $P - U$. If $\chi = 1$, it is unramified everywhere and the L-function $L(\chi, s)$ is just the zeta function of K. If $\chi \ne 1$, the L-function $L(\chi, s)$ is a polynomial in s, pure of weight 1. One has the decomposition

$$Z(K_n, s) = \prod_{\chi : G_\infty \to G_n \to \mathbb{C}_p^*} L(\chi, s), \quad \frac{P(K_n, s)}{P(K_0, s)} = \prod_{\chi \ne 1 : G_\infty \to G_n \to \mathbb{C}_p^*} L(\chi, s).$$

In order to understand each L-function $L(\chi, s)$, we need to know the ramification information of each finite character χ.

Let W denote the group of all p-power roots of unity in $\mathbb{C}_p$. Let X denote the group of finite continuous p-adic characters $\chi : G_\infty \to W$. There is a perfect pairing of $\mathbb{Z}_p$-modules

$$\mathbb{Z}_p^d \times W^d \longrightarrow W, \quad ((a_1, \cdots, a_d), (\eta_1, \cdots, \eta_d)) \longrightarrow \prod_{i=1}^{d} \eta_i^{a_i}.$$

It follows that the group X is isomorphic to W^d. Monsky [Mo1] defines a Noetherian topology on $X = W^d$ with the property that the closed subsets of W^d are finite unions of the basic set of the form

$$\{(\eta_1, \cdots, \eta_d) \in W^d \mid \prod_{i=1}^{d} \eta_i^{a_i} = \eta\},$$

where $\eta \in W$ and $(a_1, \cdots, a_d) \in \mathbb{Z}_p^d$. The open sets in W^d are simply the compliments of closed subsets in W^d. Open sets and closed sets in W^d are special cases of a more general class of sets, called semi-algebraic subsets in W^d, see [Mo1].

Next, we need to partition the character group X in terms of their exact ramification locus. For each $x \in P - U$, let I_x denote the inertial subgroup of G_∞, which is a non-trivial p-adic subgroup of G_∞. Let X^* denote the "interior piece" of X consisting of those characters χ in X such that $\chi(I_x) \neq 1$ for all $x \in P - U$. For such an interior character $\chi \in X^*$, it is clear that the ramification locus for χ is exactly $P - U$. Since I_x is a finitely generated $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p^d$, it follows that X^* is an open subset of X.

More generally, for any subset $S \subseteq P - U$, let X_S denote the set of characters $\chi \in X$ such that $\chi(I_x) = 1$ for all $x \notin S$. Let X_S^* denote the interior piece of X_S consisting of those characters $\chi \in X_S$ such that $\chi(I_x) \neq 1$ for all $x \in S$. Then, the ramification locus of $\chi \in X_S^*$ is exactly S, and we obtain the following disjoint interior decomposition

$$X = \bigsqcup_{S \subseteq P - U} X_S^*, \ X^* = X_{P-U}^*.$$

For a positive integer n, let X_n (resp., $X_{n,S}^*$) denote the set of characters $\chi \in X$ (resp., $\chi \in X_S^*$) such that $\chi^{p^n} = 1$. Similarly, we have the following disjoint interior decomposition

$$X_n = \bigsqcup_{S \subseteq P - U} X_{n,S}^*, \ X_n^* = X_{n,P-U}^*.$$

The class number h_n then has the interior decomposition

$$\frac{h_n}{h_0} = \prod_{\chi \in X_n, \chi \neq 1} L(\chi, 1) = \prod_{S \neq \phi, S \subseteq P - U} \prod_{\chi \in X_{n,S}^*} L(\chi, 1).$$

For any subset $S \subseteq P - U$, let I_S denote the closure of subgroups I_x for all $x \notin S$. By the structure theorem for finitely generated $\mathbb{Z}_p$-modules, the quotient G_∞ / I_S is of the form

$$G_\infty / I_S \cong \mathbb{Z}_p^{d(S)} \oplus H_S,$$

where H_S is a finite abelian p-group and $0 \leq d(S) \leq d$. Replacing the base field K by K_m for some large m if necessary, we may assume that $H_S = 0$ for all $S \subseteq P - U$. Thus, we have

$$G_\infty / I_S \cong \mathbb{Z}_p^{d(S)}.$$

The characters in X_S^* are exactly the interior piece in the $\mathbb{Z}_p^{d(S)}$-tower $K_{\infty,S}$ of K arising from the fixed field in K_∞ of I_{P-U-S}. Thus, without loss of generality, we can assume that $S = P - U$ and $X_{n,S}^* = X_n^*$.

The above interior decomposition reduces Greenberg's conjecture for the polynomial stability of $v_p(h_n)$ to the following theorem.

Theorem 2.1. *There is a polynomial $E(x, y) \in \mathbb{Q}[x, y]$ of total degree at most d (and degree at most 1 in y) such that for all sufficiently large n, we have*

$$v_p\left(\prod_{\chi \in X_n^*} L(\chi, 1) \right) = E(p^n, n).$$

Recall that the p-rank $r_p(n)$ of K_n is well known to be the number of p-adic unit roots of $P(K_n, s)$. For $\chi \neq 1$, let $\ell_p(\chi)$ denote the number of p-adic unit roots for $L(\chi, s)$. The interior decomposition of the character space X_n gives the following interior decomposition of the p-rank:

$$r_p(n) - r_p(0) = \sum_{S \neq \phi, S \subseteq P-U} \sum_{\chi \in X_{n,S}^*} \ell_p(\chi).$$

The set X_S^* is an open subset of $X_S = W^{d(S)}$. Theorem 4.1 (or Theorem 4.6) in Monsky [Mo1] implies that the cardinality of the set $X_{n,S}^*$ is a polynomial in p^n of degree bounded by $d(S)$ for all large n. To prove Theorem 1.4, it is enough to prove the following theorem.

Theorem 2.2. *There is a non-negative integer c depending on the tower such that for all sufficiently large n and all $\chi \in X_n^*$, we have*

$$\ell_p(\chi) = c.$$

To prove the above two theorems, we need to package all the L-functions $L(\chi, s)$ for all characters $\chi \in X^*$ into a single universal p-adic L-function and study its variation as χ varies. This is accomplished by the T-adic L-function that we shall introduce next.

3. T-adic L-functions

We now define the T-adic L-function first introduced in [LW] for $d = 1$ and later in [RWXY] for general d. Instead of just finite characters $\chi_n : G_\infty \to G_n \to \mathbb{C}_p^*$, we will also consider all continuous p-adic characters $\chi : G_\infty \longrightarrow \mathbb{C}_p^*$, not necessarily of finite order. The isomorphism

$$\rho : G_\infty \cong \mathbb{Z}_p^d$$

is crucial for us. Let $\mathbb{Z}_p[[T]] = \mathbb{Z}_p[[T_1, \cdots, T_d]]$ be the power series ring in d variables over $\mathbb{Z}_p$. Consider the universal continuous T-adic character

$$\mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p[[T]]^*, \ (a_1, \cdots, a_d) \longrightarrow \prod_{i=1}^{d}(1+T_i)^{a_i} \in \mathbb{Z}_p[[T]]^*.$$

Composing this universal T-adic character of $\mathbb{Z}_p^d$ with the isomorphism ρ, we get the universal T-adic character of G_∞:

$$\rho_T : G_\infty \longrightarrow \mathbb{Z}_p^d \longrightarrow \mathrm{GL}_1(\mathbb{Z}_p[[T]]) = \mathbb{Z}_p[[T]]^*.$$

Let $D_p(1)$ denote the open unit disk $\{t = (t_1, \cdots, t_d) | \|t_i\|_p < 1\}$ in $\mathbb{C}_p^d$. For any element $t = (t_1, \cdots, t_d) \in D_p(1)$, we have a natural evaluation map $\mathbb{Z}_p[[T]]^* \to \mathbb{C}_p^*$ sending $T = (T_1, \cdots, T_d)$ to $t = (t_1, \cdots, t_d)$. Composing all these maps, we get, for fixed $t \in D_p(1)$, a continuous character

$$\rho_t : G_\infty \longrightarrow \mathbb{C}_p^*. \tag{3.0.1}$$

The open unit disk $D_p(1)$ parametrizes all continuous $\mathbb{C}_p$-valued characters $\chi = \chi_1 \otimes \cdots \otimes \chi_d$ of G_∞ via the relation $t = t_\chi = (\chi_1(1) - 1, \cdots, \chi_d(1) - 1)$, where χ_i factors through the quotient $G_\infty^{(i)} := \mathrm{Gal}(K_\infty^{(i)}/K) \cong \mathbb{Z}_p$. The L-function of ρ_t is defined in the usual way:

$$L(\rho_t, s) = \prod_{x \in P, \rho_t \text{ unramified at } x} \frac{1}{1 - \rho_t(\mathrm{Frob}_x)s^{\deg(x)}} \in 1 + s\mathbb{C}_p[[s]].$$

For a general $t \in D_p(1)$, this L-function $L(\rho_t, s)$ does not have a p-adic meromorphic continuation to the p-adic plane $|s|_p < \infty$, see [Wa0] for results in this direction.

In the case that $\chi = \chi_1 \otimes \cdots \otimes \chi_d$ is a finite p-adic character, we have

$$t = (\chi_1(1) - 1, \cdots, \chi_d(1) - 1) , \ L(\chi, s) = L(\rho_t, s).$$

Elements of the form $t_\chi = (\chi_1(1) - 1, \cdots, \chi_d(1) - 1)$ with each χ_i finite order, are called the classical points in $D_p(1)$. The classical points are exactly the elements in

$$W^d - 1 = \{(\eta_1 - 1, \cdots, \eta_d - 1) | \eta_i \in W\}.$$

For a classical point $t \in W^d - 1$, the ramification locus of the corresponding character $\rho_t \in X$ may be strictly smaller than $P-U$. Let D^* be the set of classical points $t \in W^d-1$ such that the character $\rho_t \in X^*$ (the interior characters). We call D^* as the set of interior classical points. Then, the ramification locus of ρ_t is exactly $P-U$ for all interior classical points $t \in D^*$.

As the order of χ goes to infinity, t_χ approaches to the boundary of the disk $D_p(1)$. Thus, to understand the behavior of $L(\chi, s)$ as $\mathrm{ord}(\chi)$ grows, it is enough to understand the L-function $L(\rho_t, s)$ for all classical points t near the boundary of $D_p(1)$. More precisely, we should understand the following universal L-function.

Definition 3.1. *The T-adic L-function of the tower K_∞/K is the L-function of the T-adic character ρ_T:*

$$L_\rho(T, s) := L(\rho_T, s) = \prod_{x \in U} \frac{1}{1 - \rho_T(\mathrm{Frob}_x)s^{\deg(x)}} \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

If x is a prime of K, then the universal character ρ_T is unramified at x if and only if $\rho_T(I_x) = 1$. This is true if and only if $I_x = 1$. Thus, the ramification locus of ρ_T is exactly $P - U$. So, the above definition agrees with the usual definition of L-functions. The set U is the set of closed points of a smooth affine curve over $\mathbb{F}_q$. Using Crew's work [Cr] on the Katz conjecture in the abelian case, we deduce the following decomposition of the universal L-function, which gives the analytic continuation on the closed unit disk.

Theorem 3.2. *There is a polynomial $B(T, s) \in 1 + \mathbb{Z}_p[[T]][s]$ in s and a power series $g(T, s) \in 1 + (p, T)\mathbb{Z}_p[[T]][[s]]$ which converges for all $s \in \mathbb{Z}_p[[T]]$ such that*

$$L_\rho(T, s) = B(T, s)g(T, s).$$

As a consequence, we deduce that $g(T, 1) \in 1 + (p, T)\mathbb{Z}_p[[T]]$,

$$B(T, 1) \in \mathbb{Z}_p[[T]], \ L(T) := L_\rho(T, 1) = B(T, 1)g(T, 1) \in \mathbb{Z}_p[[T]]$$

are well defined power series in T over $\mathbb{Z}_p$. Furthermore, for $t \in D_p(1)$ and $s \in \mathbb{C}_p$ with $|s|_p \leq 1$, we have

$$v_p(L_\rho(t, s)) = v_p(B(t, s)).$$

Recall that for each interior classical point $t = (t_1, \cdots, t_d) \in D^*$, the ramification locus of ρ_t is exactly $P - U$. Using the definition of L-functions, one checks that for $t \in D^*$, we have

$$L(\rho_t, s) = L_\rho(T, s)|_{T=t} = L_\rho(t, s).$$

In particular, for every interior finite character $\chi = \chi_1 \otimes \cdots \otimes \chi_d \in X_n^*$, we have $t = (\chi_1(1) - 1, \cdots, \chi_d(1) - 1) \in D^*$ and thus

$$L(\chi, s) = L(\rho_t, s) = L_\rho(t, s), \ L(\chi, 1) = L_\rho(t, 1) = L(t).$$

We conclude that

$$\sum_{\chi \in X_n^*} v_p(L(\chi, 1)) = \sum_{\chi \in X_n^*} v_p(L(\chi_1(1) - 1, \cdots, \chi_d(1) - 1)).$$

Since the set X^* is open (and hence semi-algebraic) in W^d, Theorem 5.6 in [Mo1] implies that the right side is a polynomial in p^n and n of total degree at most d, and of degree at most 1 in n. Theorem 2.1 is proved. Thus, Greenberg's conjecture is true in the function field case when there is no constant subextension.

We now explain how the above proof implies the result on the p-rank of K_n. Since the character ρ_T is trivial modulo T, the L-function $L_\rho(T, s)$ modulo T is the same as the zeta function $Z(U, s)$ of U. This gives the congruence

$$L_\rho(T, s) \equiv Z(K_0, s) \prod_{x \in P - U} (1 - s^{\deg(x)}) = \frac{P(K_0, s)}{(1 - s)(1 - qs)} \prod_{x \in P - U} (1 - s^{\deg(x)}) \mod T.$$

Replacing T by an interior classical point $t = (t_1, \cdots, t_n)$ in D^*, we deduce

$$L(\rho_t, s) = L_\rho(t, s) \equiv \frac{P(K_0, s)}{(1 - s)} \prod_{x \in P - U} (1 - s^{\deg(x)}) \mod (t_1, \cdots, t_d).$$

Let $\ell_p(t)$ denote the number of p-adic unit roots of $L(\rho_t, s)$. Comparing the number of p-adic unit roots on both sides, one finds that for every interior classical point $t \in D^*$,

$$\ell_p(t) = r_p(0) - 1 + \sum_{x \in P - U} \deg(x)$$

is a constant c independent of t, where $r_p(0)$ is the p-rank of K_0 which is the number of p-adic unit roots of $P(K_0, s)$. Theorem 2.2 is proved. Since the p-rank is a geometric invariant, that is, independent of the constant extension, it follows that Theorem 1.4 is also proved.

4. The case with constant $\mathbb{Z}_p$-extension

Assume now that the $\mathbb{Z}_p^d$-tower K_∞ / K has a $\mathbb{Z}_p$-subtower of constant extension. Then, we can write

$$K_\infty = \mathbb{F}_{q^{p^\infty}} L_\infty, \ K_n = \mathbb{F}_{q^{p^n}} L_n,$$

where L_∞ / K is a $\mathbb{Z}_p^{d-1}$-tower with no constant extension. To avoid triviality, we can assume that $d \geq 2$. Replacing K by K_m for some m, we may and will assume that the tower L_∞ / K satisfies the normalization in the previous section and is ramified on the non-empty finite set $P - U$. The zeta function of K_n is then given by p^n-th Adams operation of the zeta function of L_n. That is,

$$Z(K_n, s) = \Phi^{p^n}(Z(L_n, s)) = \frac{\Phi^{p^n}(P(L_n, s))}{(1 - s)(1 - q^{p^n} s)},$$

where Φ^{p^n} denotes the p^n-th Adams operation. Recall that for a polynomial

$$H(s) = \prod_{i=1}^{m}(1 - \alpha_i s) = \det(I - As) \in \mathbb{Z}_p[s],$$

we have

$$\Phi^{p^n}(H(s)) = \prod_{i=1}^{m}(1 - \alpha_i^{p^n} s) = \det(I - A^{p^n} s).$$

That is, it raises each reciprocal root (or eigenvalue) to its p^n-th power. It follows that

$$\Phi^{p^n}(H(s))|_{s=1} = \det(I - A^{p^n}) = \prod_{\eta^{p^n}=1} \det(I - \eta A) = \prod_{\eta^{p^n}=1} H(\eta).$$

Apply the previous interior decomposition proof in the no constant extension case to the $\mathbb{Z}_p^{d-1}$-tower L_∞/K, we deduce that for each subset $S \subset P - U$, there is a polynomial $P_S(T_S, s) \in 1 + s\mathbb{Z}_p[[T_S]][s]$ such that

$$v_p\left(\frac{h_n}{h_0}\right) = \sum_{S \neq \phi, S \subset P-U} \left(\sum_{\chi \in X_{n,S}^*} \sum_{\eta^{p^n}=1} v_p(P_S(\chi_1 - 1, \cdots, \chi_{d-1}(1) - 1, (\eta - 1) + 1)) \right).$$

The set $X_S^* \times W$ is an open subset of $W^{d(S)} \times W = W^{d(S)+1}$. By Theorem 5.6 in [Mo1] again, the right side is a polynomial in p^n and n of total degree at most $d(S) + 1 \le d$, and of degree at most 1 in n. The theorem is proved.

5. Stability of zeta functions in p-adic Lie towers

In this final section, K is a global function field of characteristic p with constant field $\mathbb{F}_q$. We propose several conjectures on possible stability of zeta functions in various p-adic Lie towers over K, vastly extending the function field Greenberg conjecture in several different directions.

Let $G_K = \text{Gal}(K^{sep}/K)$ denote the absolute Galois group of K. Let

$$\rho : G_K \longrightarrow \text{GL}_d(\mathbb{Z}_p)$$

be a continuous p-adic representation of G_K with infinite image, ramified at finite number of primes of K. The image G_∞ of this representation is a compact p-adic Lie group of dimension $\dim(G) > 0$. The fixed field K_∞ of the kernel $Ker(\rho)$ is a Galois extension of K with Galois group isomorphic to G_∞. That is, $\text{Gal}(K_\infty/K) \cong G_\infty$. For integer $n \geq 0$, the kernel of the reduction

$$\rho_n : G_K \longrightarrow \text{GL}_d(\mathbb{Z}_p) \longrightarrow \text{GL}_d(\mathbb{Z}_p/p^n\mathbb{Z}_p)$$

produces a Galois extension K_n of K whose Galois group is isomorphic to the image G_n of ρ_n. This produces a p-adic Lie tower

$$K_\infty = \bigcup_{n=0}^\infty K_n \supset \cdots \supset K_n \supset \cdots \supset K_1 \supset K_0 = K, \ \mathrm{Gal}(K_n/K) \cong G_n,$$

which ramifies at finite number of primes of K. In the spirit of Iwasawa theory, our basic problem is to understand possible stable behavior of various arithmetic properties of this tower of global fields K_n as n grows. This naturally raises many interesting open problems, already in the simpler abelian case when $G_\infty = \mathbb{Z}_p$, see the survey paper [Wa3]. We extend some of these conjectures to general non-abelian case below. The first one concerns the p-ranks and class numbers of this tower of global fields.

Conjecture 5.1. *Let K_∞/K be a p-adic Lie extension as above. Let $r_p(n)$ denote the p-rank of K_n and let h_n denote the class number of K_n.*
(1) (p-Rank stability). There is a polynomial $R(x) \in \mathbb{Q}[x]$ of degree at most $\dim(G)$ depending on the tower such that for all sufficiently large n, we have

$$r_p(n) = R(p^n).$$

(2) (p-Class number stability). There is a polynomial $E(x,y) \in \mathbb{Q}[x,y]$ of total degree at most $\dim(G)$ depending on the tower such that for all sufficiently large n, we have

$$v_p(h_n) = E(p^n, n).$$

Our previous results for $\mathbb{Z}_p^d$-towers imply that this conjecture is true when the p-adic Lie group G_∞ is abelian. The conjecture can be viewed as an attempt to extend geometric Iwasawa theory to non-abelian extensions. Part (1) of the conjecture can be proved if each ramified prime is totally ramified. This provides a positive evidence to part (2) of the conjecture which seems significantly more difficult.

For a given K, there are too many p-adic Lie towers, most of them are not natural. To state further stability conjectures, we need to assume that the tower K_∞/K is natural in some sense, that is, arising from algebraic geometry. The tower K_∞/K is called algebraic geometric if the corresponding Galois representation ρ arises from a p-adic étale cohomology of a smooth proper variety X defined over the global field K. The algebraic geometric tower K_∞/K is further called ordinary if the variety X is generically ordinary over K. An algebraic geometric representation automatically ramifies at finite number of primes of K. In the case that $X = A$ is an ordinary abelian variety of dimension g over K, the first p-adic étale cohomology can be explicitly constructed from the p-adic Tate module:

$$T_p(A) = \lim_{\leftarrow k} A[p^k],$$

which gives a g-dimensional p-adic representation of G_K. This example is extremely important, already in the case $g = 1$, because of its close connection to p-adic automorphic forms.

It was conjectured by Dwork [Dw] and proved by the author [Wa1][Wa2] that the L-function $L(\rho, s)$ of any algebraic geometric p-adic representation ρ is p-adic meromorphic everywhere in $s \in \mathbb{C}_p$. This nice analytic property suggests that the p-adic Lie tower arising from an algebraic geometric p-adic representation should have good stable geometric and arithmetic properties. The simplest is the following genus stability conjecture.

Conjecture 5.2 *(Genus stability). Let g_n denote the genus of K_n. Assume that the tower K_∞/K is algebraic geometric and ordinary. Then there is a polynomial $G(x) \in \mathbb{Q}[x]$ of degree at most $\dim(G) + 1$ depending on the tower such that for all sufficiently large n, we have*

$$g_n = G(p^n).$$

A major progress toward this conjecture has been made by Joe Kramer-Miller in [Kr] and in his forthcoming work. Motivated by this conjecture, the number field analogue on discriminant stability has been proved recently by James Upton [Up]. If the algebraic geometric tower is non-ordinary, then a slightly weaker genus stability is still expected to hold as suggested by Kramer-Miller. In this more general case, one would need several polynomials $G_j(x) \in \mathbb{Q}[x]$ $(1 \le j \le m)$ to express the genus g_n such that $g_{mn+j} = G_j(n)$ for all large n.

Next, we move to the deeper slope stability for algebraic geometric p-adic Lie towers. Without loss of essential generality, we can and will assume that the tower has no proper constant subextension below.

The zeta function $Z(K_n, s)$ of K_n is a rational function in s of the form

$$Z(K_n, s) = \frac{P(K_n, s)}{(1 - s)(1 - qs)}, \quad P(K_n, s) \in 1 + s\mathbb{Z}[s],$$

where the zeta polynomial $P(K_n, s)$ is a polynomial of degree $2g_n$ and $g_n = g(K_n)$ denotes the genus of K_n. Recall that the class number is given by the special zeta value $h_n = P(K_n, 1)$ and the p-rank $r_p(n)$ is the number of p-adic unit roots of $P(K_n, s)$. Thus, the above two conjectures are really about the p-adic stability of some aspects of the zeta function $Z(K_n, s)$ as n grows.

A deeper property is the possible stability of higher slopes for the zeta function, not just the p-rank which is the slope zero part. To describe this, we write

$$P(K_n, s) = \prod_{i=1}^{2g_n} (1 - \alpha_i(n)s) \in \mathbb{C}_p[s], \quad 0 \le v_q(\alpha_1(n)) \le \cdots \le v_q(\alpha_{2g_n}(n)) \le 1,$$

where the q-adic valuation is normalized such that $v_q(q) = 1$. These rational numbers $\{v_q(\alpha_1(n)), \cdots, v_q(\alpha_{2g_n}(n))\}$ are called the q-slopes of K_n. They are symmetric in the interval $[0, 1]$ by the functional equation. For fixed rational number $\alpha \in [0, 1]$, let $r_p(n, \alpha)$ denote the multiplicity of the slope α in the slope sequence $\{v_q(\alpha_1(n)), \cdots, v_q(\alpha_{2g_n}(n))\}$. For $\alpha = 0$, $r_p(n, 0) = r_p(n)$ is simply the p-rank of K_n. The integer $r_p(n, \alpha)$ is called the slope α-rank of K_n. We would like to understand how the slope α-rank $r_p(n, \alpha)$ and more generally how the full slope sequence vary when n grows. The following conjecture has three parts with increasing level of difficulty.

Conjecture 5.3. *Assume that the p-adic Lie tower K_∞/K is algebraic geometric and ordinary with no proper constant subextension. To avoid triviality, assume that the genus sequence g_n is unbounded as n grows.*
(1) (Slope uniformity). The q-slopes

$$\{v_q(\alpha_1(n)), \cdots, v_q(\alpha_{2g_n}(n))\} \subset [0, 1] \cap \mathbb{Q} \subset [0, 1]$$

are equi-distributed in the interval $[0, 1]$ as n goes to infinity.
(2) (Slope α-rank stability). For each fixed rational number $\alpha \in [0, 1]$, there is a polynomial $R_\alpha(x) \in \mathbb{Q}[x]$ of degree at most $\dim(G)$ depending on the tower such that for all sufficiently large n, we have

$$r_p(n, \alpha) = R_\alpha(p^n).$$

(3) (Slope stability). There is a positive integer n_0 depending on the tower such that the re-scaled q-slopes $\{p^n v_q(\alpha_1(n)), \cdots, p^n v_q(\alpha_{2g_n}(n))\}$ for all $n > n_0$ are determined explicitly by their values for $0 \le n \le n_0$, using a finite number of arithmetic progressions.

If the tower is non-ordinary, one expects similar and possibly slightly weaker stability results. If the tower does not come from algebraic geometry, the conjecture is false in general. A remarkable recent work of Kosters–Zhu [KZ] suggests that a slightly stronger condition than the genus stability would imply the slope stability conjecture. In fact, they have proved this implication when $K = \mathbb{F}_q(x)$ is the rational function field and $G_\infty = \mathbb{Z}_p$ in [KZ] and in their forthcoming work. We do not have a precise formulation about the harder slope stability conjecture in such a generality yet. The essence is a finiteness property that all the slopes for large n are determined by a finite number of arithmetic progressions. Needless to say that this conjecture (and even the weaker α-rank stability conjecture) is wide open, but interesting progress has been made in various special cases when the p-adic tower K_∞/K is abelian, see [DWX][RWXY][Li][KZ]. In the case that the tower is the Igusa $\mathbb{Z}_p^*$-tower over a modular curve, the above slope stability conjecture is partially related to the ghost conjecture [BP] and the spectral halo conjecture on p-adic modular forms, see [LWX][WXZ] for recent progress and further references.

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