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HODGE TYPE OF THE EXOTIC COHOMOLOGY OF COMPLETE INTERSECTIONS

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ABSTRACT. If $X \subset \mathbb{P}^n$ is a smooth complete intersection, its cohomology modulo the one of $\mathbb{P}^n$ is supported in middle dimension. If the complete intersection is singular, it might also carry exotic cohomology beyond the middle dimension. We show that for this exotic cohomology, one can improve the known bound for the Hodge type of its de Rham cohomology.

Type de Hodge de la cohomologie exotique des intersections complètes.

Résumé: Si $X \subset \mathbb{P}^n$ est une intersection complète lisse, sa cohomologie modulo celle de $\mathbb{P}^n$ est supportée en dimension moitié. Si l'intersection complète est singulière, elle peut aussi avoir de la cohomologie exotique en dimension supérieure. Nous montrons qu' on peut améliorer le type de Hodge de cette cohomologie de de Rham exotique.

Version française abrégée. Si $X \subset \mathbb{P}^n$ est une intersection complète lisse sur un corps k, sa cohomologie modulo celle de $\mathbb{P}^n$ est supportée en dimension moitié. Si l'intersection complète est singulière, elle peut aussi avoir de la cohomologie exotique en dimension supérieure. Si le degré de X vérifie $d_1 \geq d_2 \geq \ldots \geq d_r$ et si $0 < \kappa \leq \frac{n-d_2-\ldots-d_r}{d_1}$, alors nous disposons du théorème de Ax-Katz [7] pour $k = \mathbb{F}_q$ qui affirme que les zéros et pôles réciproques de la fonction zêta de $\mathbb{P}^n \setminus X$ sont divisibles par q^κ en tant que nombres algébriques. Son pendant en théorie de Hodge dit que le type de Hodge est alors $\geq \kappa$ ([1], [3], [5], [6]). Ces valeurs sont optimales. Si X n'est plus lisse, après normalisation convenable de la fonction zêta, on peut améliorer la divisibilité ([8]). Dans cette note, nous utilisons la méthode de [5] pour montrer l'amélioration correspondante du côté Hodge.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be a complete intersection of multi-degree

$$d_1 \geq d_2 \geq \ldots \geq d_r \geq 1 \tag{1.1}$$

over a field k of characteristic zero, where $0 \leq r \leq n$. It is well known that

$$\begin{aligned} H^i(X)/H^i(\mathbb{P}^n) &\simeq H_c^{i+1}(\mathbb{P}^n \setminus X), \text{ for } i \leq 2(n-r), \\ H_c^i(\mathbb{P}^n \setminus X) &\simeq H^i(\mathbb{P}^n), \text{ for } i \geq 2(n-r)+2. \end{aligned} \tag{1.2}$$

Furthermore, one has the vanishing

$$H^i(X)/H^i(\mathbb{P}^n) \simeq H_c^{i+1}(\mathbb{P}^n \setminus X) = 0, \text{ for } i < (n-r). \tag{1.3}$$

If X is also smooth, one has the further vanishing

$$H^i(X)/H^i(\mathbb{P}^n) \simeq H_c^{i+1}(\mathbb{P}^n \setminus X) = 0, \text{ for } 2(n-r) \geq i > (n-r), \tag{1.4}$$

and so the cohomology $H^i(X)/H^i(\mathbb{P}^n)$ is concentrated in the middle dimension $i = \dim(X) = (n-r)$ in the smooth case ([1]). If the complete intersection is singular, it may have exotic cohomology for $i > \dim(X)$. That is, some of the groups $H_c^{i+1}(\mathbb{P}^n \setminus X) = H^i(X)/H^i(\mathbb{P}^n)$ might not vanish for some i with $\dim(X) = (n-r) < i \leq 2(n-r)$. The Hodge type of $H_c^{i+1}(\mathbb{P}^n \setminus X)$, that is the largest non-negative integer μ for which the Hodge filtration fulfills $F^\mu H_c^{i+1}(\mathbb{P}^n \setminus X) = H_c^{i+1}(\mathbb{P}^n \setminus X)$, is closely related to the first slope of the zeta function of $\mathbb{P}^n \setminus X$ over a finite field.

Let $\kappa = \max\{0, [\frac{n-d_2-\ldots-d_r}{d_1}]\}$, where $[x]$ denotes the integer part of x. If $k = \mathbb{F}_q$ is the finite field with q elements of characteristic p, the theorem of Ax and Katz ([7]) says

$$|(\mathbb{P}^n \setminus X)(\mathbb{F}_q)| \equiv 0 \bmod q^\kappa. \tag{1.5}$$

The zeta function

$$Z(\mathbb{P}^n \setminus X, T) = \exp(\sum_{m=1}^{\infty} \frac{|(\mathbb{P}^n \setminus X)(\mathbb{F}_{q^m})|}{m} T^m) \tag{1.6}$$

is a rational function over $\mathbb{Q}$, by Dwork's rationality theorem ([4]). We factor the zeta function over the algebraic closure of the p-adic rational number field $\mathbb{Q}_p$:

$$Z(\mathbb{P}^n \setminus X, T)^{(-1)^{n-r}} = \frac{\prod_{i=1}^{a}(1-\alpha_i T)}{\prod_{i=1}^{b}(1-\beta_i T)}, \tag{1.7}$$

where the α_i (resp. the β_j) are the reciprocal zeros (resp. the reciprocal poles) of $Z(\mathbb{P}^n \setminus X, T)^{(-1)^{n-r}}$. The Ax-Katz theorem is equivalent to the following lower bound for the first slope of the zeta function:

$$\mathrm{ord}_q(\alpha_i) \geq \kappa, \ \mathrm{ord}_q(\beta_j) \geq \kappa. \tag{1.8}$$

This result has a Hodge theoretical analogue. If k has characteristic 0, then X has Hodge type HT

$$HT(H^i_c(\mathbb{P}^n \setminus X)) \geq \kappa \ \text{ for all } i. \tag{1.9}$$

(See [1] for the smooth case, [3] for the hypersurface case, [5] for the complete intersection case and [6] for the general case.)

The above bounds are sharp ([7], [5]) among all the X defined by r equations of degrees $d_1 \geq d_2 \geq \ldots \geq d_r$. However, it is proven in [8], Theorem 1.3 that if X is a complete intersection, then the above slope bound can be improved for the β_j's. More precisely, if $\kappa_1 = \max\{0, [\frac{n-1-d_2-\ldots-d_r}{d_1}]\}$, then for complete intersections X, one has

$$\mathrm{ord}_q(\beta_j) \geq \kappa_1 + 1. \tag{1.10}$$

If $n - d_2 - \ldots - d_r > 0$ and d_1 divides $(n - d_2 - \ldots - d_r)$, then this bound $\kappa_1 + 1 = \kappa$ does not improve the one by Ax and Katz. But if either $n - d_2 - \ldots - d_r \leq 0$ or d_1 does not divide $(n - d_2 - \ldots - d_r)$, then one obtains the strict improvement:

$$\mathrm{ord}_q(\beta_j) \geq \kappa_1 + 1 = \kappa + 1. \tag{1.11}$$

The purpose of this note is to show the corresponding Hodge theoretical improvement, which then concerns only the exotic cohomology. One has

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a complete intersection of multi-degree $d_1 \geq \ldots \geq d_r \geq 1$. Let*

$$\kappa_1 = \max\{0, [\frac{n-1-d_2-\ldots-d_r}{d_1}]\}.$$

Then for $i \geq 1$, one has

$$HT(H_c^{n+1-r+i}(\mathbb{P}^n \setminus X)) \geq \kappa_1 + 1. \tag{1.12}$$

Note that if $i < 0$, the cohomology $H_c^{n+1-r+i}(\mathbb{P}^n \setminus X)$ vanishes. If $i = 0$, Ax-Katz' bound (1.9) can not be improved in general as noted above. If X is not assumed to be a complete intersection, for example think of X being defined by r times the same equation, then the theorem, obviously, can't be true. Remark 2.2 shows that our bound on the Hodge type can't be improved to $\kappa_1 + 2$.

Finally let us observe that, in the singular case, we do not know whether (1.5) implies that the eigenvalues of Frobenius acting on ℓ-adic cohomology are divisible by q^κ, due to possible cancellations in the representation of the zeta function as a rational function. While we know that the Hodge type is at least κ for all cohomology groups, that is for all i in (1.9). Theorem 1.3 of [8] concerns only the slope improvement for the reciprocal poles, while Theorem 1.1 concerns all cohomology groups beyond the middle dimension, whether of odd or of even dimension.

We shall use the method of [5], Proposition 1.2 to reduce the theorem to a vanishing theorem which is a more precise version than the one shown in loc.cit, Proposition 2.1.

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2. The proof of the Theorem

Let $\mathcal{I}$ denote the ideal sheaf $(f_1, \cdots, f_r)$ in $\mathbb{P}^n$. We know by [5], Proposition 1.2 that it is enough to show

$$\mathbb{H}^{n+1-r+i}(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1} \to \mathcal{I}^{\kappa_1} \otimes \Omega^1 \to \ldots \to \mathcal{I} \otimes \Omega^{\kappa_1}) = 0, i \geq 1. \tag{2.1}$$

This is of course implied by the vanishing

Proposition 2.1. *Let (X, d_i, κ_1, n, i) be as in Theorem 1.1. Then one has*

$$H^{n+1-r+i-s}(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1-s} \otimes \Omega^s) = 0, 0 \leq s \leq \kappa_1. \tag{2.2}$$

Proof. We give here a simplification of our original proof due to Pierre Deligne.

By the standard resolution of $\Omega^s_{\mathbb{P}^n}$

(2.3)
$$0 \to \Omega^s_{\mathbb{P}^n} \to \oplus_{n_s} \mathcal{O}_{\mathbb{P}^n}(-s) \to \oplus_{n_{s-1}} \mathcal{O}_{\mathbb{P}^n}(-(s-1)) \to \ldots \to \oplus_{n_0} \mathcal{O}_{\mathbb{P}^n} \to 0$$

for some positive integers n_i, we are reduced to showing

$$H^{n+1-r+i-s-a}(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1-s}(-(s-a))) = 0, 0 \leq a \leq s \leq \kappa_1. \tag{2.4}$$

For $s = 0$, this reduces to

$$H^{n+1-r+i}(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1}) = 0, i \geq 1. \tag{2.5}$$

The exact sequence

$$0 \to \mathcal{I}^{\kappa_1+1} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}^{\kappa_1+1} \to 0 \tag{2.6}$$

induces the exact sequence

(2.7)
$$H^{n-r+i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}^{\kappa_1+1}) \to H^{n+1-r+i}(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1}) \to H^{n+1-r+i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}).$$

Since $(n-r+i) > \dim(X)$, the term on the left vanishes by cohomological reasons, while the term on the right vanishes since $(n+1-r+i) \geq 1$. This shows Proposition 2.1 for $s = 0$.

Assume now that $s \geq 1$. In particular $\kappa_1 \geq 1$ and thus $\kappa_1 d_1 \leq n - 1 - d_2 - \cdots - d_r$. For $0 \leq a \leq s \leq \kappa_1$, one has

$$\begin{aligned}(\kappa_1 + 1 - s)d_1 + (s - a) &\leq (\kappa_1 + 1 - s)d_1 + s \\ &= \kappa_1 d_1 - (s-1)(d_1 - 1) + 1 \\ &\leq \kappa_1 d_1 + 1 \leq (n - d_2 - \ldots - d_r).\end{aligned} \tag{2.8}$$

Thus Proposition 2.1 of [5] allows to conclude

$$H^m(\mathbb{P}^n, \mathcal{I}^{\kappa_1+1-s}(-(s-a))) = 0, 0 \leq a \leq s \leq \kappa_1, m \geq 0. \tag{2.9}$$

This finishes the proof. $\square$

Remark 2.2. If X is the union of two planes $\mathbb{P}^2 \subset \mathbb{P}^3$, then $\kappa_1 = 1 < \frac{n}{d} = \frac{3}{2}$, and X is a normal crossing divisor to which one can apply directly [2]. One has $H^i_c(\mathbb{P}^3 \setminus X) = \mathbb{H}^i(\mathbb{P}^3, \Omega^\bullet(\log X)(-X))$. However $\Omega^2(\log X)(-X) = \mathcal{O}(-3) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4)$. Since the Hodge to de Rham spectral sequence degenerates, one obtains $H^5_c(\mathbb{P}^3 \setminus X) = H^3(\mathcal{O}(-4)) \neq 0$, $\kappa_1 + 1 = 2$, and the Hodge type of $H^5_c(\mathbb{P}^3 \setminus X)$ is 2 but not 3, which can also be checked by writing that $\mathbb{P}^3 \setminus X$ is isomorphic to $\mathbb{G}_m \times \mathbb{A}^2$. In contrast to this, if $\kappa_1 = 0 < \frac{n}{d} = \frac{3}{3}$, and X is the union of three planes $\mathbb{P}^2 \subset \mathbb{P}^3$ in general position, one has $\Omega^1(\log X)(-X) = \mathcal{O}(-3) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4)$, thus by the degeneration of the Hodge to de Rham spectral sequence again, one sees $H^4_c(\mathbb{P}^3 \setminus X) = \mathbb{H}^4(\mathbb{P}^3, \Omega^\bullet(\log X)(-X)) = H^3(\mathcal{O}(-4)) \neq 0$. Thus the Hodge type of $H^4_c(\mathbb{P}^3 \setminus X)$ is 1 but not 2, which can also be checked by writing that $\mathbb{P}^3 \setminus X$ is isomorphic to $(\mathbb{G}_m)^2 \times \mathbb{A}^1$.

We conclude the note by a remark which shows the pattern of Theorem 1.1.

Theorem 2.3. *Let $X \subset \mathbb{P}^n$ be a complete intersection of multi-degree $d_1 \geq \ldots \geq d_r \geq 1$. Let ℓ be an integer such that $0 \leq \ell \leq n+1-r$. Let*

$$\kappa_\ell = \max\{0, [\frac{n - \ell - d_2 - \ldots - d_r}{d_1}]\}.$$

Then for $i \geq \ell$, one has

$$HT(H_c^{n+1-r+i}(\mathbb{P}^n \setminus X)) \geq \kappa_\ell + \ell. \tag{2.10}$$

Proof. For $i \geq n-r+1$, we have $n+1-r+i \geq 2(n-r)+2$, thus

$$H_c^{n+1-r+i}(\mathbb{P}^n \setminus X)) = H^{n+1-r+i}(\mathbb{P}^n). \tag{2.11}$$

This is 0 if $(n+1-r+i)$ is odd. Otherwise, it has Hodge type $\frac{n+1-r+i}{2}$. The last number is at least ℓ since $i \geq \ell$ and $n+1-r \geq \ell$. Thus, we may assume that $\kappa_\ell > 0$, in which case $\kappa_\ell + \ell \leq n - d_2 - \cdots - d_r$ and

$$\frac{n+1-r+i}{2} \geq n+1-r \geq n-d_2-\cdots-d_r \geq \kappa_\ell + \ell. \tag{2.12}$$

It remains to consider the case $i \leq (n-r)$. The proof goes as before. By loc. cit. it suffices to show

$$H^{n-r+1+i-s}(\mathbb{P}^n, \mathcal{I}^{\kappa_\ell+\ell-s} \otimes \Omega^s) = 0, s < (\kappa_\ell + \ell). \tag{2.13}$$

Considering the exact sequence

$$0 \to \mathcal{I}^{\kappa_\ell+\ell-s} \otimes \Omega^s \to \Omega^s \to \Omega^s/\mathcal{I}^{\kappa_\ell+\ell-s} \otimes \Omega^s \to 0 \tag{2.14}$$

one obtains

$$H^{n-r+1+i-s}(\mathbb{P}^n, \mathcal{I}^{\kappa_\ell+\ell-s} \otimes \Omega^s) = 0, \text{ for } i > s, n-r+1+i-s \neq s. \tag{2.15}$$

On the other hand, if $(n-r+1-i) = 2s$ and $s \leq (i-1)$, one has $(n-r+1-i) \leq 2(i-1)$ that is $(n-r+3) \leq i$ which contradicts $i \leq (n-r)$.

Let now $s \geq i \geq \ell$. Since $s < \kappa_\ell + \ell$, we must have $\kappa_\ell > 0$. By the standard resolution (2.3) of Ω^s, we are reduced to showing

$$H^{n+1-r+i-s-a}(\mathbb{P}^n, \mathcal{I}^{\kappa_\ell+\ell-s}(-(s-a)) = 0, 0 \leq a \leq s \leq \kappa_\ell. \tag{2.16}$$

One checks that for $s \geq \ell$ and $0 \leq a \leq s \leq \kappa_\ell$,

$$\begin{aligned}(\kappa_\ell + \ell - s)d_1 + d_2 + \ldots + d_r + s - a = \\ \kappa_\ell d_1 + d_2 + \ldots + d_r - (s-\ell)(d_1-1) + (\ell - a) \leq \\ \kappa_\ell d_1 + d_2 + \ldots + d_r + \ell \leq n.\end{aligned} \tag{2.17}$$

The last inequality holds because $\kappa_\ell > 0$. The desired vanishing follows from Proposition 2.1 of [5]. This finishes the proof. ∎

Remark 2.4. The case $\ell = 0$ of Theorem 2.3 is the main result of [5]. The case $\ell = 1$ of Theorem 2.3 reduces to Theorem 1.1 in the above.

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