ON KATZ'S BOUND FOR NUMBER OF ELEMENTS WITH GIVEN TRACE AND NORM

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ABSTRACT. In this note an improvement of the Katz's bound on the number of elements in a finite field with given trace and norm is given. The improvement is obtained by reducing the problem to estimating the number of rational points on certain toric Calabi-Yau hypersurface, and then to use detailed cohomological calculations by Rojas-Leon and the second author for such toric hypersurfaces.

1. INTRODUCTION

Let p be a prime and $\mathbb{F}_q$ be the finite field of q elements of characteristic p. Given $a, b \in \mathbb{F}_q$, and positive integer $m \geq 2$, let

$$N_m(a,b) = \#\{\alpha \in \mathbb{F}_{q^m} | \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = a, \mathrm{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = b\}.$$

Motivated by various applications, it is of interest to give a sharp estimate for the number $N_m(a,b)$. The case $b = 0$ is trivial.

Katz [2] proved the following bound:

Theorem 1.1. *Let $a, b \in \mathbb{F}_q^*$ and $n \geq 1$. Then*

$$|N_{n+1}(a,b) - \frac{q^{n+1}-1}{q(q-1)}| \leq (n+1)q^{\frac{n-1}{2}}.$$

This bound was used by Moisio [3] to improve some cases of the explicit bound in Wan [5] on the number of irreducible polynomials in an arithmetic progression of $\mathbb{F}_q[x]$. In the case $n+1 = 3$, the Katz bound also plays a significant role in Cohen and Huczynska [1] for their proof of the existence of a cubic primitive normal polynomial with given norm and trace.

If $a = 0$, Katz's bound can be improved in an elementary way using character sums [3]:

$$|N_{n+1}(0,b) - \frac{q^n-1}{q-1}| \leq (d-1)q^{\frac{n-1}{2}},$$

where $d = \gcd(n+1, q-1)$.

In this note, we give a uniform improvement of Katz's bound in the case $a \neq 0$.

Theorem 1.2. *Let $a, b \in \mathbb{F}_q^*$ and $n \geq 1$. Then*

$$|N_{n+1}(a,b) - \frac{q^n-1}{q-1}| \leq nq^{\frac{n-1}{2}}.$$

2000 *Mathematics Subject Classification.* 11T99, 14G10.

In the case that $n+1$ is a power of p, this improvement was first proved by Moisio [3] using Deligne's estimate for hyper-Kloosterman sums. Moreover, in the case $n+1=3$ also the bounds

$$3\left\lceil \frac{q+1-2\sqrt{q}}{3} \right\rceil \leq N_3(a,b) \leq 3\left\lfloor \frac{q+1+2\sqrt{q}}{3} \right\rfloor .$$

were obtained in [3] by using the Hasse's bound for elliptic curves together with a divisibility result. In corollary 2.4, we extend such divisibility bounds to $N_\ell(a,b)$, where $\ell \geq 3$ is any prime.

In the general case, our proof of Theorem 1.2 consists of two steps. The first step is to reduce it to estimating the number of $\mathbb{F}_q$-rational points on certain toric Calabi-Yau hypersurface over $\mathbb{F}_q$. The second step is to use the detailed cohomological calculations in Rojas-Leon and Wan [4] for such toric hypersurfaces. In the case $n+1=3$, the above improved bounds should significantly reduce the amount of calculations in [1].

2. Proof of Theorem 1.2

Let $u = b/a^{n+1} \in \mathbb{F}_q^*$. Let $N(u)$ denote the number of $\mathbb{F}_q$-rational points on the toric hypersurface

$$Y_u: \ X_1 + \cdots + X_n + \frac{u}{X_1 \cdots X_n} - 1 = 0.$$

Lemma 2.1.

$$N_{n+1}(a,b) = \frac{q^n-1}{q-1} + (-1)^n \left(N(u) - \frac{(q-1)^n - (-1)^n}{q} \right).$$

Proof. Write the equation of Y_u in the form

$$\begin{aligned} X_1 + \cdots + X_{n+1} &= 1 \\ X_1 \cdots X_{n+1} &= u. \end{aligned}$$

Let ψ be the canonical additive character of $\mathbb{F}_q$. Now

$$q(q-1)N(u) = \sum_{x_1,\ldots,x_{n+1}} \sum_v \psi(v(x_1+\cdots+x_{n+1}-1)) \sum_\chi \chi(u^{-1}x_1\cdots x_{n+1}),$$

where $x_1,\ldots,x_{n+1}$ run over $\mathbb{F}_q^*$, v runs over $\mathbb{F}_q$, and χ runs over the multiplicative character group of $\mathbb{F}_q$.

Let $G(\chi)$ denote the Gauss sum

$$G(\chi) = \sum_{x \in \mathbb{F}_q^*} \psi(x)\chi(x).$$

It follows that

$$\begin{aligned}
q(q-1)N(u) &= (q-1)^{n+1} + \sum_{v\neq 0}\psi(-v)\sum_{\chi}\bar{\chi}(u)\prod_{i=1}^{n+1}\sum_{x_i}\psi(vx_i)\chi(x_i) \\
&\overset{x_i\mapsto x_i/v}{=} (q-1)^{n+1} + \sum_{v\neq 0}\psi(-v)\sum_{\chi}\bar{\chi}(uv^{n+1})G(\chi)^{n+1} \\
&= (q-1)^{n+1} + \sum_{\chi}G(\chi)^{n+1}\bar{\chi}(u)\sum_{v\neq 0}\psi(-v)\bar{\chi}^{n+1}(v) \\
&= (q-1)^{n+1} + \sum_{\chi}G(\chi)^{n+1}G(\bar{\chi}^{n+1})\bar{\chi}((-1)^{n+1}u). \qquad (2.0.1)
\end{aligned}$$

Next we express $N_{n+1}(a,b)$ in terms of Gauss sums. We use the abbreviated notations Tr and Norm in place of $\mathrm{Tr}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$ and $\mathrm{Norm}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$. Let $\psi_{n+1} = \psi\circ\mathrm{Tr}$ be the canonical additive character of $\mathbb{F}_{q^{n+1}}$ and let $\alpha\in\mathbb{F}_{q^{n+1}}$ with $\mathrm{Tr}(\alpha)=1$. Now,

$$\begin{aligned}
q(q-1)N_{n+1}(a,b) &= \sum_{x\in\mathbb{F}^*_{q^{n+1}}}\sum_{v}\psi(v(\mathrm{Tr}(x-\alpha a))\sum_{\chi}\chi(b^{-1}\mathrm{Norm}(x)) \\
&= \sum_{v}\psi(-av)\sum_{\chi}\bar{\chi}(b)\sum_{x}\psi_{n+1}(vx)\chi(\mathrm{Norm}(x)) \\
&= q^{n+1}-1+\sum_{v\neq 0}\psi(-av)\sum_{\chi}\bar{\chi}(b)\sum_{x}\psi_{n+1}(vx)\chi(\mathrm{Norm}(x)) \\
&\overset{x\mapsto x/v}{=} q^{n+1}-1+\sum_{v\neq 0}\psi(-av)\sum_{\chi}\bar{\chi}(bv^{n+1})\sum_{x}\psi_{n+1}(x)\chi(\mathrm{Norm}(x)),
\end{aligned}$$

since $\mathrm{Norm}(v) = v^{n+1}$.

By the Davenport-Hasse identity the inner sum

$$\sum_{x}\psi_{n+1}(x)\chi(\mathrm{Norm}(x)) = (-1)^n G(\chi)^{n+1},$$

and therefore

$$\begin{aligned}
q(q-1)N_{n+1}(a,b) &= q^{n+1}-1+(-1)^n\sum_{\chi}G(\chi)^{n+1}\bar{\chi}(b)\sum_{v\neq 0}\psi(-av)\bar{\chi}^{n+1}(v) \\
&= q^{n+1}-1+(-1)^n\sum_{\chi}G(\chi)^{n+1}G(\bar{\chi}^{n+1})\bar{\chi}((-1)^{n+1}b/a^{n+1}).
\end{aligned}$$

Comparing this expression with (2.0.1), one finds that

$$N_{n+1}(a,b) = \frac{q^{n+1}-1}{q(q-1)} + (-1)^n\left(N(u) - \frac{(q-1)^{n+1}}{q(q-1)}\right).$$

One checks that this is the same as the expression in Lemma 2.1. □

This lemma reduces Theorem 1.2 to the following

Theorem 2.2. *Let $u\in\mathbb{F}_q^*$. Then*

$$|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq nq^{\frac{n-1}{2}}.$$

Proof. Over the algebraic closure $\bar{\mathbb{F}}_q$, we can write $u = \lambda^{-(n+1)}$ for some non-zero element λ. Then Y_u is isomorphic to the toric hypersurface

$$X_\lambda : \ X_1 + \cdots + X_n + \frac{1}{X_1 \cdots X_n} - \lambda = 0$$

whose zeta function over a finite field was studied in detail in [4], see [6] for more elementary description of the results. For a prime $\ell \neq p$, the ℓ-adic cohomology

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \cong H_c^j(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$$

was calculated in Theorem 2.1 in [4]. In particular, we have

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) = 0, \ j < n-1 \text{ or } j > 2n-1,$$

$$H_c^j(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\binom{n}{j-n+2}}(n-1-j), \ n \leq j \leq 2n-2,$$

and there is an exact sequence of Galois modules

$$0 \to \mathbb{Q}_\ell^n \to H_c^{n-1}(Y_u \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \to M_u \to 0,$$

where M_u is of rank at most n and mixed of weight at most $n-1$. It follows that

$$|\mathrm{Tr}(\mathrm{Frob}_u|M_u)| \leq nq^{\frac{n-1}{2}}.$$

By the ℓ-adic trace formula,

$$N(u) = \sum_{j=n}^{2n-2} (-1)^j \binom{n}{j-n+2} q^{(j-(n-1))} + (-1)^{n-1}n + (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_u|M_u).$$

Replacing j by $j+n-2$, one finds

$$N(u) = \sum_{j=2}^{n} (-1)^{j-n} \binom{n}{j} q^{(j-1)} + (-1)^{n-1}n + (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_u|M_u).$$

The theorem follows. □

Remark. If $u \neq (n+1)^{-(n+1)}$, i.e., $\lambda \notin \{(n+1)\zeta | \zeta^{n+1} = 1\}$, then M_u is pure of weight $n-1$ and of rank n. If $u = (n+1)^{-(n+1)}$ (necessarily $p \nmid n+1$), then the rank of M_u drops by 1 and thus

$$|\mathrm{Tr}(\mathrm{Frob}_u|M_u)| \leq (n-1)q^{\frac{n-1}{2}}.$$

If $u = (n+1)^{-(n+1)}$ and n is even, then one of the Frobenius eigenvalues has weight $n-2$ (instead of $n-1$), and thus

$$|\mathrm{Tr}(\mathrm{Frob}_u|M_u)| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.$$

All these follow from Proposition 2.6 in [4].

Corollary 2.3. *Let* $u = (n+1)^{-(n+1)}$. *Then*

$$|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-1)q^{\frac{n-1}{2}}.$$

If n *is also even, then*

$$|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.$$

Corollary 2.4. *Let $\ell \geq 3$ be a prime number. Let $a, b \in \mathbb{F}_q^*$. Then, we have*

$$\ell \left\lceil \frac{\frac{q^{\ell-1}-1}{q-1} - (\ell-1)q^{(\ell-2)/2}}{\ell} \right\rceil \leq N_\ell(a,b) \leq \ell \left\lfloor \frac{\frac{q^{\ell-1}-1}{q-1} + (\ell-1)q^{(\ell-2)/2}}{\ell} \right\rfloor .$$

Proof. Let R be the number of $c \in \mathbb{F}_q$ such that $\ell c = a$ and $c^\ell = b$. It is clear that R is either 0 or 1. Since ℓ is a prime, $N_\ell(a,b) - R$ is divisible by ℓ. If $R = 0$, the corollary is the consequence of Theorem 1.2 and the divisibility of $N_\ell(a,b)$ by ℓ.

Assume now that $R = 1$. Since $a \neq 0$, ℓ cannot be p. In this case, we have $a = \ell c$, $b = c^\ell$ and thus $u = b/a^\ell = \ell^{-\ell} \in \mathbb{F}_q^*$. We can apply the stronger estimate in the previous corollary to deduce the desired inequalities for $N_\ell(a,b)$.

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