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Mirror Symmetry For Zeta Functions

Daqing Wan

Abstract. In this paper, we study the relation between the zeta function of a Calabi-Yau hypersurface and the zeta function of its mirror. Two types of arithmetic relations are discovered. This motivates us to formulate two general arithmetic mirror conjectures for the zeta functions of a mirror pair of Calabi-Yau manifolds.

Contents

1. Introduction

In this section, we describe two mirror relations between the zeta function of a Calabi-Yau hypersurface in a projective space and the zeta function of its mirror manifold. Along the way, we make comments and conjectures about what to expect in the general case.

Let d be a positive integer. Let X and Y be two d-dimensional smooth projective Calabi-Yau varieties over $\mathbb{C}$. A necessary condition (the topological mirror test) for X and Y to be a mirror pair is that their Hodge numbers satisfy the Hodge

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symmetry:

$$h^{i,j}(X) = h^{d-i,j}(Y),\ 0 \le i, j \le d. \tag{1}$$

In particular, their Euler characteristics are related by

$$e(X) = (-1)^d e(Y). \tag{2}$$

In general, there is no known rigorous algebraic geometric definition for a mirror pair, although many examples of mirror pairs are known at least conjecturally. Furthermore, it does not make sense to speak of "the mirror" of X as the mirror variety usually comes in a family. In some cases, the mirror does not exist. This is the case for rigid Calabi-Yau 3-folds X, since the rigid condition $h^{2,1}(X) = 0$ would imply that $h^{1,1}(Y) = 0$ which is impossible.

We shall assume that X and Y are a given mirror pair in some sense and are defined over a number field or a finite field. We are interested in how the zeta function of X is related to the zeta function of Y. Since there is no algebraic geometric definition for X and Y to be a mirror pair, it is difficult to study the possible symmetry between their zeta functions in full generality. On the other hand, there are many explicit examples and constructions which at least conjecturally give a mirror pair, most notably in the toric hypersurface setting as constructed by Batyrev [**1**]. Thus, we shall first examine an explicit example and see what kind of relations can be proved for their zeta functions in this case. This would then suggest what to expect in general.

Let $n \ge 2$ be a positive integer. We consider the universal family of Calabi-Yau complex hypersurfaces of degree $n+1$ in the projective space $\mathbb{P}^n$. Its mirror family is a one parameter family of toric hypersurfaces. To construct the mirror family, we consider the one parameter subfamily X_λ of complex projective hypersurfaces of degree $n+1$ in $\mathbb{P}^n$ defined by

$$f(x_1, \cdots, x_{n+1}) = x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0,$$

where $\lambda \in \mathbb{C}$ is the parameter. The variety X_λ is a Calabi-Yau manifold when X_λ is smooth. Let μ_{n+1} denote the group of $(n+1)$-th roots of unity. Let

$$G = \{(\zeta_1, \cdots, \zeta_{n+1}) | \zeta_i^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}/\mu_{n+1} \cong (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1},$$

where μ_{n+1} is embedded in G via the diagonal embedding. The finite group G acts on X_λ by

$$(\zeta_1, \cdots, \zeta_{n+1})(x_1, \cdots, x_{n+1}) = (\zeta_1 x_1, \cdots, \zeta_{n+1} x_{n+1}).$$

The quotient X_λ/G is a projective toric hypersurface Y_λ in the toric variety $\mathbb{P}_\Delta$, where $\mathbb{P}_\Delta$ is the simplex in $\mathbb{R}^n$ with vertices $\{e_1, \cdots, e_n, -(e_1 + \cdots + e_n)\}$ and the e_i's are the standard coordinate vectors in $\mathbb{R}^n$. Explicitly, the variety Y_λ is the projective closure in $\mathbb{P}_\Delta$ of the affine toric hypersurface in $\mathbb{G}_m^n$ defined by

$$g(x_1, \cdots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$

Assume that X_λ is smooth. Then, Y_λ is a (singular) mirror of X_λ. It is an orbifold. If W_λ is a smooth crepant resolution of Y_λ, then the pair (X_λ, W_λ) is called a mirror pair of Calabi-Yau manifolds. Such a resolution exists for this example but not unique if $n \ge 3$. The number of rational points and the zeta function are independent of the choice of the crepant resolution. We are interested in understanding how the arithmetic of X_λ is related to the arithmetic of W_λ, in particular how the zeta function of X_λ is related to the zeta function of W_λ. Our main concern in this paper is to consider Calabi-Yau manifolds over finite fields, although we shall mention some conjectural implications for Calabi-Yau manifolds defined over number fields.

In this example, we see two types of mirror pairs. The first one is the generic mirror pair $\{X_\Lambda, W_\lambda\}$, where X_Λ is the generic member in the moduli space of smooth projective Calabi-Yau hypersurfaces of degree $(n+1)$ in $\mathbb{P}^n$ and W_λ is the generic member in the above one parameter family of Calabi-Yau manifolds. Note that X_Λ and Y_λ are parameterized by different parameter spaces (of different dimensions). The possible zeta symmetry in this case would then have to be a relation between certain generic property of their zeta functions.

The second type of mirror pairs is the one parameter family of mirror pairs $\{X_\lambda, W_\lambda\}$ parameterized by the same parameter λ. This is a stronger type of mirror pair than the first type. For $\lambda \in \mathbb{C}$, we say that W_λ is a **strong mirror** of X_λ. For such a strong mirror pair $\{X_\lambda, W_\lambda\}$, we can really ask for the relation between the zeta function of X_λ and the zeta function of W_λ. If $\lambda_1 \neq \lambda_2$, W_{λ_1} would not be called a strong mirror for X_{λ_2}, although they would be an usual **weak mirror** pair. Apparently, we do not have a definition for a strong mirror pair in general, as there is not even a definition for a generic or weak mirror pair in general.

Let $\mathbb{F}_q$ be a finite field of q elements, where $q = p^r$ and p is a prime. For a scheme X of finite type of dimension d over $\mathbb{F}_q$, let $\#X(\mathbb{F}_q)$ denote the number of $\mathbb{F}_q$-rational points on X. Let

$$Z(X,T) = \exp(\sum_{k=1}^{\infty} \frac{T^k}{k} \#X(\mathbb{F}_{q^k})) \in 1 + T\mathbb{Z}[[T]]$$

be the zeta function of X. It is well known that $Z(X,T)$ is a rational function in T whose reciprocal zeros and reciprocal poles are Weil q-integers. Factor $Z(X,T)$ over the p-adic numbers $\mathbb{C}_p$ and write

$$Z(X,T) = \prod_i (1 - \alpha_i T)^{\pm 1}$$

in reduced form, where the algebraic integers $\alpha_i \in \mathbb{C}_p$. One knows that the slope $\mathrm{ord}_q(\alpha_i)$ is a rational number in the interval $[0,d]$. For two real numbers $s_1 \leq s_2$, we define the slope $[s_1, s_2]$ part of $Z(X,T)$ to be the partial product

$$Z_{[s_1,s_2]}(X,T) = \prod_{s_1 \leq \mathrm{ord}_q(\alpha_i) \leq s_2} (1 - \alpha_i T)^{\pm 1}. \tag{3}$$

For a half open and half closed interval $[s_1, s_2)$, the slope $[s_1, s_2)$ part $Z_{[s_1,s_2)}(X,T)$ of $Z(X,T)$ is defined in a similar way. These are rational functions with coefficients in $\mathbb{Z}_p$ by the p-adic Weierstrass factorization. It is clear that we have the decomposition

$$Z(X,T) = \prod_{i=0}^{d} Z_{[i,i+1)}(X,T).$$

Our main result of this paper is the following arithmetic mirror theorem.

THEOREM 1.1. *Assume that $\lambda \in \mathbb{F}_q$ such that (X_λ, W_λ) is a strong mirror pair of Calabi-Yau manifolds over $\mathbb{F}_q$. For every positive integer k, we have the congruence formula*

$$\#X_\lambda(\mathbb{F}_{q^k}) \equiv \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \#W_\lambda(\mathbb{F}_{q^k}) \pmod{q^k}.$$

Equivalently, the slope $[0,1)$ *part of the zeta function is the same for the mirror varieties* $\{X_\lambda, Y_\lambda, W_\lambda\}$*:*

$$Z_{[0,1)}(X_\lambda, T) = Z_{[0,1)}(Y_\lambda, T) = Z_{[0,1)}(W_\lambda, T).$$

We now discuss a few applications of this theorem. In terms of cohomology theory, this suggests that the semi-simplification of the DeRham-Witt cohomology (in particular, the p-adic etàle cohomology) for $\{X_\lambda, Y_\lambda, W_\lambda\}$ are all the same. A corollary of the above theorem is that the unit root parts (slope zero parts) of their zeta functions are the same:

$$Z_{[0,0]}(X_\lambda, T) = Z_{[0,0]}(Y_\lambda, T) = Z_{[0,0]}(W_\lambda, T).$$

The p-adic variation of the rational function $Z_{[0,0]}(X_\lambda, T)$ as λ varies is closely related to the mirror map which we do not discuss it here, but see [**4**] for the case $n \leq 3$. From arithmetic point of view, the p-adic variation of the rational function $Z_{[0,0]}(X_\lambda, T)$ as λ varies is explained by Dwork's unit root zeta function [**5**]. We briefly explain the connection here.

Let B be the parameter variety of λ such that (X_λ, W_λ) form a strong mirror pair. Let $\Phi : X_\lambda \to B$ (resp. $\Psi : W_\lambda \to B$) be the projection to the base by sending X_λ (resp. W_λ) to λ. The pair (Φ, Ψ) of morphisms to B is called a **strong mirror pair of morphisms to** B. Each of its fibres gives a strong mirror pair of Calabi-Yau manifolds. Recall that Dwork's unit root zeta function attached to the morphism Φ is defined to be the formal infinite product

$$Z_{\mathrm{unit}}(\Phi, T) = \prod_{\lambda \in |B|} Z_{[0,0]}(X_\lambda, T^{\deg(\lambda)}) \in 1 + T\mathbb{Z}_p[[T]],$$

where $|B|$ denotes the set of closed points of B over $\mathbb{F}_q$. This unit root zeta function is no longer a rational function, but conjectured by Dwork in [**5**] and proved by the author in [**11**][**12**][**13**] to be a p-adic meromorphic function in T. The above theorem immediately implies

COROLLARY 1.2. *Let* (Φ, Ψ) *be the above strong mirror pair of morphisms to the base* B*. Then, their unit root zeta functions are the same:*

$$Z_{\mathrm{unit}}(\Phi, T) = Z_{\mathrm{unit}}(\Psi, T).$$

If λ is in a number field K, then Theorem 1.1 implies that the Hasse-Weil zeta functions of X_λ and Y_λ differ essentially by the L-function of a pure motive $M_n(\lambda)$ of weight $n-3$. That is,

$$\zeta(X_\lambda, s) = \zeta(Y_\lambda, s) L(M_n(\lambda), s-1).$$

In the quintic case $n = 4$, the pure weight 1 motive $M_4(\lambda)$ would come from a curve. This curve has been constructed explicitly by Candelas, de la Ossa and F. Rodriguez-Villegas [**3**]. The relation between the Hasse-Weil zeta functions of X_λ and W_λ are similar, differing by a few more factors consisting of Tate twists of the Dedekind zeta function of K.

Theorem 1.1 motivates the following more general conjecture.

CONJECTURE 1.3 (Congruence mirror conjecture). *Suppose that we are given a strong mirror pair* $\{X, Y\}$ *of Calabi-Yau manifolds defined over* $\mathbb{F}_q$*. Then, for every positive integer* k*, we have*

$$\#X(\mathbb{F}_{q^k}) \equiv \#Y(\mathbb{F}_{q^k}) \pmod{q^k}.$$

Equivalently,

$$Z_{[0,1)}(X, T) = Z_{[0,1)}(Y, T).$$

Equivalently (by functional equation),

$$Z_{(d-1,d]}(X,T) = Z_{(d-1,d]}(Y,T).$$

The condition in the congruence mirror conjecture is vague since one does not know at present an algebraic geometric definition of a strong mirror pair of Calabi-Yau manifolds, although one does know many examples such as the one given above. Thus, a major part of the problem is to make the definition of a strong mirror pair mathematically precise. For an additional evidence of the congruence mirror conjecture, see Theorem 6.2 which can be viewed as a generalization of Theorem 1.1. As indicated before, this conjecture implies that Dwork's unit root zeta functions for the two families forming a strong mirror pair are the same p-adic meromorphic functions. This means that under the strong mirror family involution, Dwork's unit root zeta function stays the same.

Just like the zeta function itself, its slope $[0,1)$ part $Z_{[0,1)}(X_\lambda, T)$ depends heavily on the algebraic parameter λ, not just on the topological properties of X_λ. This means that the congruence mirror conjecture is really a continuous type of arithmetic mirror symmetry. This continuous nature requires the use of a strong mirror pair, not just a generic mirror pair.

Assume that $\{X, Y\}$ forms a mirror pair, not necessarily a strong mirror pair. A different type of arithmetic mirror symmetry reflecting the Hodge symmetry, which is discrete and hence generic in nature, is to look for a suitable quantum version $Z_Q(X,T)$ of the zeta function such that

$$Z_Q(X,T) = Z_Q(Y,T)^{(-1)^d},$$

where $\{X, Y\}$ is a mirror pair of Calabi-Yau manifolds over $\mathbb{F}_q$ of dimension d. This relation cannot hold for the usual zeta function $Z(X,T)$ for obvious reasons, even for a strong mirror pair as it contradicts with the congruence mirror conjecture for odd d. No non-trivial candidate for $Z_Q(X,T)$ has been found. Here we propose a p-adic quantum version which would have the conjectural properties for most (and hence generic) mirror pairs. We will call our new zeta function to be the slope zeta function as it is based on the slopes of the zeros and poles.

Definition 1.4. *For a scheme X of finite type over $\mathbb{F}_q$, write as before*

$$Z(X,T) = \prod_i (1-\alpha_i T)^{\pm 1}$$

in reduced form, where $\alpha_i \in \mathbb{C}_p$. Define the slope zeta function of X to be the two variable function

$$S_p(X,u,T) = \prod_i (1-u^{\mathrm{ord}_q(\alpha_i)}T)^{\pm 1}. \tag{4}$$

Note that

$$\alpha_i = q^{\mathrm{ord}_q(\alpha_i)}\beta_i,$$

where β_i is a p-adic unit. Thus, the slope zeta function $S_p(X,u,T)$ is obtained from the p-adic factorization of $Z(X,T)$ by dropping the p-adic unit parts of the roots and replacing q by the variable u. This is not always a rational function in u and T. It is rational if all slopes are integers. Note that the definition of the slope zeta function is independent of the choice of the ground field $\mathbb{F}_q$ where X is defined. It depends only on $X \otimes \bar{\mathbb{F}}_q$ and thus is also a geometric invariant. It would be interesting to see if there is a diophantine interpretation of the slope zeta function. If we have a smooth proper family of varieties, the Grothendieck specialization theorem implies that the generic Newton polygon on each cohomology exists and hence the generic slope zeta function exists as well.

If X is a scheme of finite type over $\mathbb{Z}$, then for each prime number p, the reduction $X \otimes \mathbb{F}_p$ has the p-adic slope zeta function $S_p(X \otimes \mathbb{F}_p, u, T)$. At the first glance, one might think that this gives infinitely many discrete invariants for X as the set of prime numbers is infinite. However, it can be shown that the set $\{S_p(X \otimes \mathbb{F}_p, u, T) | p \text{ prime}\}$ contains only finitely many distinct elements. In general, it is a very interesting but difficult problem to determine this set $\{S_p(X \otimes \mathbb{F}_p, u, T) | p \text{ prime}\}$.

Suppose that X and Y form a mirror pair of d-dimensional Calabi-Yau manifolds over $\mathbb{F}_q$. For simplicity and for comparison with the Hodge theory, we always assume in this paper that X and Y can be lifted to characteristic zero (to the Witt ring of $\mathbb{F}_q$). In this good reduction case, the modulo p Hodge numbers equal the characteristic zero Hodge numbers. Taking $u = 1$ in the definition of the slope zeta function, we see that the specialization $S_p(X, 1, T)$ already satisfies the desired relation

$$S_p(X, 1, T) = (1 - T)^{-e(X)} = (1 - T)^{-(-1)^d e(Y)} = S_p(Y, 1, T)^{(-1)^d}.$$

This suggests that there is a chance that the slope zeta function might satisfy the desired slope mirror symmetry

$$S_p(X, u, T) = S_p(Y, u, T)^{(-1)^d}. \tag{5}$$

In section 7, we shall show that the slope zeta function satisfies a functional equation. Furthermore, the expected slope mirror symmetry does hold if both X and Y are ordinary. If either X or Y is not ordinary, the expected slope mirror symmetry is unlikely to hold in general.

If $d \leq 2$, the congruence mirror conjecture implies that the slope zeta function does satisfy the expected slope mirror symmetry for a strong mirror pair $\{X, Y\}$, whether X and Y are ordinary or not. For $d \geq 3$, we believe that the slope zeta function is still a little bit too strong for the expected symmetry to hold in general, even if $\{X, Y\}$ forms a strong mirror pair. And it should not be too hard to find a counter-example although we have not done so. However, we believe that the expected slope mirror symmetry holds for a generic mirror pair of 3-dimensional Calabi-Yau manifolds.

Conjecture 1.5 (Slope mirror conjecture). *Suppose that we are given a generic mirror pair $\{X, Y\}$ of 3-dimensional Calabi-Yau manifolds defined over $\mathbb{F}_q$. Then, we have the slope mirror symmetry for their generic slope zeta functions:*

$$S_p(X, u, T) = \frac{1}{S_p(Y, u, T)}. \tag{6}$$

A main point of this conjecture is that it holds for all prime numbers p. For arbitrary $d \geq 4$, the corresponding slope mirror conjecture might be false for some prime numbers p, but it should be true for all primes $p \equiv 1 \pmod{D}$ for some positive integer D depending on the mirror family, if the family comes from the reduction modulo p of a family defined over a number field. In the case $d \leq 3$, one could take $D = 1$ and hence get the above conjecture.

Again the condition in the slope mirror conjecture is vague as it is not presently known an algebraic geometric definition of a mirror family, although many examples are known in the toric setting. In a future paper, using the results in [**10**][**14**], we shall prove that the slope mirror conjecture holds in the toric hypersurface case if

$d \leq 3$. For example, if X is a generic quintic hypersurface, then X is ordinary by the results in [**8**][**10**] for every p and thus one finds

$$S_p(X \otimes \mathbb{F}_p, u, T) = \frac{(1-T)(1-uT)^{101}(1-u^2T)^{101}(1-u^3T)}{(1-T)(1-uT)(1-u^2T)(1-u^3T)}.$$

This is independent of p. Note that we do not know if the one parameter subfamily X_λ is generically ordinary for every p. The ordinary property for every p was established only for the universal family of hypersurfaces, not for a one parameter subfamily of hypersurfaces such as X_λ. If Y denotes the generic mirror of X, then by the results in [**10**] [**14**], Y is ordinary for every p and thus we obtain

$$S_p(Y \otimes \mathbb{F}_p, u, T) = \frac{(1-T)(1-uT)(1-u^2T)(1-u^3T)}{(1-T)(1-uT)^{101}(1-u^2T)^{101}(1-u^3T)}.$$

Again, it is independent of p. The slope mirror conjecture holds in this example.

Remark: The slope zeta function is completely determined by the Newton polygon of the Frobenius acting on cohomologies of the variety in question. The converse is not true, as there may be cancellations coming from different cohomology dimensions in the slope zeta function.

For a mirror pair over a number field, we have the following harder conjecture.

CONJECTURE 1.6 (Slope mirror conjecture over $\mathbb{Z}$). *Let $\{X, Y\}$ be two schemes of finite type over $\mathbb{Z}$ such that their generic fibres $\{X \otimes \mathbb{Q}, Y \otimes \mathbb{Q}\}$ form a usual (weak) mirror pair of d-dimensional Calabi-Yau manifolds defined over $\mathbb{Q}$. Then there are infinitely many prime numbers p (with positive density) such that*

$$S_p(X \otimes \mathbb{F}_p, u, T) = S_p(Y \otimes \mathbb{F}_p, u, T)^{(-1)^d}.$$

Remarks. If one uses the weight $2\log_q |\alpha_i|$ instead of the slope $\mathrm{ord}_q \alpha_i$, where $|\cdot|$ denotes the complex absolute value, one can define a two variable weight zeta function in a similar way. It is easy to see that the resulting weight zeta function does not satisfy the desired symmetry as the weight has nothing to do with the Hodge symmetry, while the slopes are related to the Hodge numbers as the Newton polygon (slope polygon) lies above the Hodge polygon.

In practice, one is often given a mirror pair of singular Calabi-Yau orbifolds, where there may not exist a smooth crepant resolution. In such a case, one could define an orbifold zeta function, which would be equal to the zeta function of the smooth crepant resolution whenever such a resolution exists. Similar results and conjectures should carry over to such orbifold zeta functions.

In the appendix, D. Haessig (my student at UC Irvine) proves some additional congruence results for the strong mirror pair (X_λ, Y_λ), some of which is used in Section 7.

2. A counting formula via Gauss sums

Let $V_1, \cdots, V_m$ be m distinct lattice points in $\mathbb{Z}^n$. For $V_j = (V_{1j}, \cdots, V_{nj})$, write

$$x^{V_j} = x_1^{V_{1j}} \cdots x_n^{V_{nj}}.$$

Let f be the Laurent polynomial in n variables written in the form:

$$f(x_1, \cdots, x_n) = \sum_{j=1}^{m} a_j x^{V_j}, a_j \in \mathbb{F}_q,$$

where not all a_j are zero. Let M be the $n \times m$ matrix

$$M = (V_1, \cdots, V_m),$$

where each V_j is written as a column vector. Let N_f^* denote the number of $\mathbb{F}_q$-rational points on the affine toric hypersurface $f=0$ in $\mathbb{G}_m^n$. If each $V_j \in \mathbb{Z}_{\geq 0}^n$, we let N_f denote the number of $\mathbb{F}_q$-rational points on the affine hypersurface $f=0$ in $\mathbb{A}^n$. We first derive a well known formula for both N_f^* and N_f in terms of Gauss sums.

For this purpose, we now recall the definition of Gauss sums. Let $\mathbb{F}_q$ be the finite field of q elements, where $q=p^r$ and p is the characteristic of $\mathbb{F}_q$. Let χ be the Teichmüller character of the multiplicative group $\mathbb{F}_q^*$. For $a \in \mathbb{F}_q^*$, the value $\chi(a)$ is just the $(q-1)$-th root of unity in the p-adic field $\mathbb{C}_p$ such that $\chi(a)$ modulo p reduces to a. Define the $(q-2)$ Gauss sums over $\mathbb{F}_q$ by

$$G(k) = \sum_{a \in \mathbb{F}_q^*} \chi(a)^{-k} \zeta_p^{\mathrm{Tr}(a)} \quad (1 \leq k \leq q-2),$$

where ζ_p is a primitive p-th root of unity in $\mathbb{C}_p$ and Tr denotes the trace map from $\mathbb{F}_q$ to the prime field $\mathbb{F}_p$.

Lemma 2.1. *For all $a \in \mathbb{F}_q$, the Gauss sums satisfy the following interpolation relation*

$$\zeta_p^{\mathrm{Tr}(a)} = \sum_{k=0}^{q-1} \frac{G(k)}{q-1} \chi(a)^k,$$

where

$$G(0) = q-1, \; G(q-1) = -q.$$

Proof. By the Vandermonde determinant, there are numbers $C(k)$ $(0 \leq k \leq q-1)$ such that for all $a \in \mathbb{F}_q$, one has

$$\zeta_p^{\mathrm{Tr}(a)} = \sum_{k=0}^{q-1} \frac{C(k)}{q-1} \chi(a)^k.$$

It suffices to prove that $C(k)=G(k)$ for all k. Take $a=0$, one finds that $C(0)/(q-1)=1$. This proves that $C(0)=q-1=G(0)$. For $1 \leq k \leq q-2$, one computes that

$$G(k) = \sum_{a \in \mathbb{F}_q^*} \chi(a)^{-k} \zeta_p^{\mathrm{Tr}(a)} = \frac{C(k)}{q-1}(q-1) = C(k).$$

Finally,

$$0 = \sum_{a \in \mathbb{F}_q} \zeta_p^{\mathrm{Tr}(a)} = \frac{C(0)}{q-1} q + \frac{C(q-1)}{q-1}(q-1).$$

This gives $C(q-1) = -q = G(q-1)$. The lemma is proved.

We also need to use the following classical theorem of Stickelberger.

Lemma 2.2. *Let $0 \leq k \leq q-1$. Write*

$$k = k_0 + k_1 p + \cdots + k_{r-1} p^{r-1}$$

in p-adic expansion, where $0 \leq k_i \leq p-1$. Let $\sigma(k) = k_0 + \cdots + k_{r-1}$ be the sum of the p-digits of k. Then,

$$\mathrm{ord}_p G(k) = \frac{\sigma(k)}{p-1}.$$

Now we turn to deriving a counting formula for N_f in terms of Gauss sums. Write $W_j = (1, V_j) \in \mathbb{Z}^{n+1}$. Then,

$$x_0 f = \sum_{j=1}^{m} a_j x^{W_j} = \sum_{j=1}^{m} a_j x_0 x_1^{V_{1j}} \cdots x_n^{V_{nj}},$$

where x now has $n+1$ variables $\{x_0, \cdots, x_n\}$. Using the formula

$$\sum_{t \in \mathbb{F}_q} t^k = \begin{cases} 0, & \text{if } (q-1) \not| k, \\ q-1, & \text{if } (q-1)|k \text{ and } k > 0, \\ q, & \text{if } k = 0, \end{cases}$$

one then calculates that

$$\begin{aligned} qN_f &= \sum_{x_0, \cdots, x_n \in \mathbb{F}_q} \zeta_p^{\mathrm{Tr}(x_0 f(x))} \\ &= \sum_{x_0, \cdots, x_n \in \mathbb{F}_q} \prod_{j=1}^{m} \zeta_p^{\mathrm{Tr}(a_j x^{W_j})} \\ &= \sum_{x_0, \cdots, x_n \in \mathbb{F}_q} \prod_{j=1}^{m} \sum_{k_j=0}^{q-1} \frac{G(k_j)}{q-1} \chi(a_j)^{k_j} \chi(x^{W_j})^{k_j} \\ &= \sum_{k_1=0}^{q-1} \cdots \sum_{k_m=0}^{q-1} \Big(\prod_{j=1}^{m} \frac{G(k_j)}{q-1} \chi(a_j)^{k_j}\Big) \sum_{x_0, \cdots, x_n \in \mathbb{F}_q} \chi(x^{k_1 W_1 + \cdots + k_m W_m}) \\ &= \sum_{\sum_{j=1}^{m} k_j W_j \equiv 0 (\mathrm{mod}\ q-1)} \frac{(q-1)^{s(k)} q^{n+1-s(k)}}{(q-1)^m} \prod_{j=1}^{m} \chi(a_j)^{k_j} G(k_j), \end{aligned} \tag{7}$$

where $s(k)$ denotes the number of non-zero entries in $k_1 W_1 + \cdots + k_m W_m$.

Similarly, one calculates that

$$\begin{aligned} qN_f^* &= \sum_{x_0 \in \mathbb{F}_q, x_1, \cdots, x_n \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}(x_0 f(x))} \\ &= (q-1)^n + \sum_{x_0, \cdots, x_n \in \mathbb{F}_q^*} \prod_{j=1}^{m} \zeta_p^{\mathrm{Tr}(a_j x^{W_j})} \\ &= (q-1)^n + \sum_{\sum_{j=1}^{m} k_j W_j \equiv 0 (\mathrm{mod}\ q-1)} \frac{(q-1)^{n+1}}{(q-1)^m} \prod_{j=1}^{m} \chi(a_j)^{k_j} G(k_j). \end{aligned} \tag{8}$$

We shall use these two formulas to study the number of $\mathbb{F}_q$-rational points on certain hypersurfaces in next two sections.

3. Rational points on Calabi-Yau hypersurfaces

In this section, we apply formula (7) to compute the number of $\mathbb{F}_q$-rational points on the projective hypersurface X_λ in $\mathbb{P}^n$ defined by

$$f(x_1, \cdots, x_{n+1}) = x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0,$$

where λ is an element of $\mathbb{F}_q^*$. We shall handle the easier case $\lambda = 0$ separately. Let M be the $(n+2) \times (n+2)$ matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ n+1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & n+1 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n+1 & 1 \end{pmatrix} \tag{9}$$

Let $k = (k_1, \cdots, k_{n+2})$ written as a column vector. Let N_f denote the number of $\mathbb{F}_q$-rational points on the affine hypersurface $f = 0$ in $\mathbb{A}^{n+1}$. By formula (7), we deduce that

$$qN_f = \sum_{Mk \equiv 0 (\mathrm{mod}\ q-1)} \frac{(q-1)^{s(k)} q^{n+2-s(k)}}{(q-1)^{n+2}} (\prod_{j=1}^{n+2} G(k_j)) \chi(\lambda)^{k_{n+2}},$$

where $s(k)$ denotes the number of non-zero entries in $Mk \in \mathbb{Z}^{n+2}$. The number of $\mathbb{F}_q$-rational points on the projective hypersurface X_λ is then given by the formula

$$\frac{N_f - 1}{q-1} = \frac{-1}{q-1} + \sum_{Mk \equiv 0 (\mathrm{mod}\ q-1)} \frac{q^{n+1-s(k)}}{(q-1)^{n+3-s(k)}} (\prod_{j=1}^{n+2} G(k_j)) \chi(\lambda)^{k_{n+2}}.$$

If $k = (0, \cdots, 0, q-1)$, then $Mk = (q-1, \cdots, q-1)$ and $s(k) = n+2$. In this case, the corresponding term in the above expression is $-(q-1)^n$ which is $(-1)^{n-1}$ modulo q. If $k = (0, ..., 0)$, then $s(k) = 0$ and the corresponding term is $q^{n+1}/(q-1)$ which is zero modulo q.

Thus, we obtain the congruence formula modulo q:

$$\frac{N_f - 1}{q-1} \equiv 1 + (-1)^{n-1} + \sideset{}{^*}\sum_{Mk \equiv 0 (\mathrm{mod}\ q-1)} \frac{q^{n+1-s(k)}}{(q-1)^{n+3-s(k)}} (\prod_{j=1}^{n+2} G(k_j)) \chi(\lambda)^{k_{n+2}},$$

where $\sum^*$ means summing over all those solutions $k = (k_1, \cdots, k_{n+2})$ with $0 \le k_i \le q-1$, $k \neq (0, \cdots, 0)$, and $k \neq (0, \cdots, 0, q-1)$.

Lemma 3.1. *If $k \neq (0, \cdots, 0)$, then $\prod_{j=1}^{n+2} G(k_j)$ is divisible by q.*

Proof. Let k be a solution of $Mk \equiv 0 (\mathrm{mod}\ q-1)$ such that $k \neq (0, \cdots, 0)$. Then, there are positive integers $\ell_0, \cdots, \ell_{r-1}$ such that

$$k_1 + \cdots + k_{n+2} = (q-1)\ell_0,$$
$$< pk_1 > + \cdots + < pk_{n+2} >= (q-1)\ell_1,$$
$$\cdots$$
$$< p^{r-1}k_1 > + \cdots + < p^{r-1}k_{n+2} >= (q-1)\ell_{r-1},$$

where $< pk_1 >$ denotes the unique integer in $[0, q-1]$ congruent to pk_1 modulo $(q-1)$ and which is 0 (resp. $q-1$) if $pk_1 = 0$ (resp., if pk_1 is a positive multiple of $q-1$). By the Stickelberger theorem, we deduce that

$$\mathrm{ord}_p \prod_{j=1}^{n+2} G(k_j) = \frac{\sum_j \sigma(k_j)}{p-1} = \frac{1}{q-1} \sum_{i=0}^{r-1} (q-1)\ell_i = \sum_{i=0}^{r-1} \ell_i.$$

Since $\ell_i \ge 1$, it follows that

$$\mathrm{ord}_q \prod_{j=1}^{n+2} G(k_j) = \frac{1}{r} \sum_{i=0}^{r-1} \ell_i \ge 1$$

with equality holding if and only if all $\ell_i = 1$. The lemma is proved.

Using this lemma and the previous congruence formula, we deduce

LEMMA 3.2. *Let $\lambda \in \mathbb{F}_q^*$. We have the congruence formula modulo q:*

$$\#X_\lambda(\mathbb{F}_q) \equiv 1 + (-1)^{n-1} + \sum^*_{\substack{Mk\equiv 0(\mathrm{mod}\ q-1)\\ s(k)=n+2}} \frac{1}{q(q-1)}(\prod_{j=1}^{n+2} G(k_j))\chi(\lambda)^{k_{n+2}}.$$

4. Rational points on the mirror hypersurfaces

In this section, we apply formula (8) to compute the number of $\mathbb{F}_q$-rational points on the affine toric hypersurface in $\mathbb{G}_m^n$ defined by the Laurent polynomial equation

$$g(x_1, \cdots, x_n) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0,$$

where λ is an element of $\mathbb{F}_q^*$. Let N be the $(n+1) \times (n+2)$ matrix

$$N = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots\vdots & \\ 0 & 0 & \cdots & 1 & -1 & 0 \end{pmatrix} \tag{10}$$

Let $k = (k_1, \cdots, k_{n+2})$ written as a column vector. By formula (8), we deduce that

$$qN_g^* = (q-1)^n + \sum_{Nk\equiv 0(\mathrm{mod}\ q-1)} \frac{1}{(q-1)}(\prod_{j=1}^{n+2} G(k_j))\chi(\lambda)^{k_{n+2}},$$

where $k = (k_1, \cdots, k_{n+2})$ with $0 \le k_i \le q-1$.

The contribution of those trivial terms k (where each k_i is either 0 or $q-1$) is given by

$$\frac{1}{q-1}\sum_{s=0}^{n+2}(-q)^s(q-1)^{n+2-s}\binom{n+2}{s} = \frac{(-1)^n}{q-1}.$$

Since

$$(q-1)^n + \frac{(-1)^n}{q-1} = \frac{(q-1)^{n+1}+(-1)^n}{q-1} \equiv q(n+1)(-1)^{n-1}(\mathrm{mod} q^2),$$

we deduce

LEMMA 4.1. *For $\lambda \in \mathbb{F}_q^*$, we have the following congruence formula modulo q:*

$$N_g^* \equiv (n+1)(-1)^{n-1} + \sum'_{Nk\equiv 0(\mathrm{mod}\ q-1)} \frac{1}{q(q-1)}(\prod_{j=1}^{n+2} G(k_j))\chi(\lambda)^{k_{n+2}},$$

where $\sum'$ means summing over all those non-trivial solutions k.

5. The mirror congruence formula

THEOREM 5.1. *For $\lambda \in \mathbb{F}_q^*$, we have the congruence formula*

$$\#X_\lambda(\mathbb{F}_q) \equiv N_g^* + 1 - n(-1)^{n-1} \pmod q.$$

Proof. If k is a non-trivial solution of $Nk \equiv 0(\bmod q-1)$, then we have

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \equiv k_{n+1} (\bmod q-1)$$

and

$$k_1 + \cdots + k_{n+1} + k_{n+2} \equiv 0(\bmod q-1).$$

Since k is non-trivial, we must have

$$0 < k_1 = k_2 = \cdots = k_{n+1} < q-1,$$

$$k_1 + \cdots + k_{n+2} = (n+1)k_1 + k_{n+2} = (n+1)k_2 + k_{n+2} = \cdots \equiv 0(\bmod q-1).$$

This gives all solutions of the equation $Mk \equiv 0(\bmod q-1)$ with $k_1 = \cdots = k_{n+1}$, $0 < k_1 < q-1$ and $s(k) = n+2$. The corresponding terms for these k's in $(N_f - 1)/(q-1)$ and N_g^* are exactly the same.

A solution of $Mk \equiv 0(\bmod\ q-1)$ is called **admissible** if $s(k) = n+2$ and its first $k+1$ coordinates $\{k_1, \cdots, k_{n+1}\}$ contain at least two distinct elements. The above results show that we have

$$\frac{N_f - 1}{q-1} - 1 - (-1)^{n-1} - (N_g^* - (n+1)(-1)^{n-1})$$

$$\equiv \sum_{\text{admissible } k} \frac{1}{q(q-1)} (\prod_{j=1}^{n+2} G(k_j)) \chi(\lambda)^{k_{n+2}} \ (\bmod\ q).$$

This congruence together with the following lemma completes the proof of the theorem.

Lemma 5.2. *If k is an admissible solution of $Mk \equiv 0(\bmod\ q-1)$, then*

$$\mathrm{ord}_q(\prod_{j=1}^{n+2} G(k_j)) \geq 2.$$

Proof. If k is an admissible solution, then $< pk >, \cdots, < p^{r-1}k >$ are also admissible solutions. For each $1 \leq i \leq n+1$, write

$$(n+1)k_i + k_{n+2} = (q-1)\ell_i,$$

where ℓ_i is a positive integer. Adding these equations together, we get

$$(n+1)(k_1 + \cdots + k_{n+1}) + (n+1)k_{n+2} = (q-1)(\ell_1 + \cdots + \ell_{n+1}).$$

Thus, the integer

$$\frac{k_1 + \cdots + k_{n+2}}{q-1} = \frac{\ell_1 + \cdots + \ell_{n+1}}{n+1} = \ell \in \mathbf{Z}_{>0}.$$

It is clear that $\ell = 1$ if and only if each $\ell_i = 1$ which would imply that $k_1 = \cdots = k_{n+1}$ contradicting with the admissibility of k. Thus, we must have that $\ell \geq 2$.

Similarly, for each $0 \leq i \leq r-1$, we have

$$< p^i k_1 > + \cdots + < p^i k_{n+2} > = (q-1)j_i,$$

where $j_i \geq 2$ is a positive integer. We conclude that

$$\mathrm{ord}_q(\prod_{j=1}^{n+2} G(k_j)) = \frac{j_0 + \cdots + j_{r-1}}{r} \geq 2.$$

The lemma is proved.

6. Rational points on the projective mirror

Let Δ be the convex integral polytope associated with the Laurent polynomial g. It is the n-dimensional simplex in $\mathbb{R}^n$ with the following vertices:

$$\{e_1, \cdots, e_n, -(e_1 + \cdots + e_n)\},$$

where the e_i's are the standard unit vectors in $\mathbb{R}^n$.

Let $\mathbb{P}_\Delta$ be the projective toric variety associated with the polytope Δ, which contains $\mathbb{G}_m^n$ as an open dense subset. Let Y_λ be the projective closure in $\mathbb{P}_\Delta$ of the affine toric hypersurface $g = 0$ in $\mathbb{G}_m^n$. The variety Y_λ is then a projective toric hypersurface in $\mathbb{P}_\Delta$. We are interested in the number of $\mathbb{F}_q$-rational points on Y_λ.

The toric variety $\mathbb{P}_\Delta$ has the following disjoint decomposition:

$$\mathbb{P}_\Delta = \bigcup_{\tau \in \Delta} \mathbb{P}_{\Delta,\tau},$$

where τ runs over all non-empty faces of Δ and each $\mathbb{P}_{\Delta,\tau}$ is isomorphic to the torus $\mathbb{G}_m^{\dim\tau}$. Accordingly, the projective toric hypersurface Y_λ has the corresponding disjoint decomposition

$$Y_\lambda = \bigcup_{\tau \in \Delta} Y_{\lambda,\tau}, \ Y_{\lambda,\tau} = Y_\lambda \cap \mathbb{P}_{\Delta,\tau}.$$

For $\tau = \Delta$, the subvariety $Y_{\lambda,\Delta}$ is simply the affine toric hypersurface defined by $g = 0$ in $\mathbb{G}_m^n$. For zero-dimensional τ, $Y_{\lambda,\tau}$ is empty. For a face τ with $1 \leq \dim\tau \leq n-1$, one checks that $Y_{\lambda,\tau}$ is isomorphic to the affine toric hypersurface in $\mathbb{G}_m^{\dim\tau}$ defined by

$$1 + x_1 + \cdots + x_{\dim\tau} = 0.$$

For such a τ, the inclusion-exclusion principle shows that

$$\#Y_{\lambda,\tau}(\mathbb{F}_q) = q^{\dim\tau - 1} - \binom{\dim\tau}{1} q^{\dim\tau-2} + \cdots + (-1)^{\dim\tau-1}\binom{\dim\tau}{\dim\tau - 1}.$$

Thus,

$$\#Y_{\lambda,\tau}(\mathbb{F}_q) = \frac{1}{q}((q-1)^{\dim\tau} + (-1)^{\dim\tau+1}).$$

This formula holds even for zero-dimensional τ as both sides would then be zero.

Putting these calculations together, we deduce that

$$\#Y_\lambda(\mathbb{F}_q) = N_g^* - \frac{(q-1)^n + (-1)^{n+1}}{q} + \sum_{\tau\in\Delta} \frac{1}{q}((q-1)^{\dim\tau} + (-1)^{\dim\tau+1}),$$

where τ runs over all non-empty faces of Δ including Δ itself. Since Δ is a simplex, one computes that

$$\sum_{\tau\in\Delta}((q-1)^{\dim\tau} + (-1)^{\dim\tau+1}) = \frac{q^{n+1}-1}{q-1} + (-1) = \frac{q(q^n-1)}{q-1}.$$

This implies that

$$\#Y_\lambda(\mathbb{F}_q) = N_g^* - \frac{(q-1)^n + (-1)^{n+1}}{q} + \frac{q^n - 1}{q-1}. \tag{11}$$

This equality holds for all $\lambda \in \mathbb{F}_q$, including the case $\lambda = 0$. Reducing modulo q, we get

$$\#Y_\lambda(\mathbb{F}_q) \equiv N_g^* + 1 - n(-1)^{n-1} \pmod{q}. \tag{12}$$

This and Theorem 5.1 prove the case $\lambda \neq 0$ of the following theorem.

THEOREM 6.1. *For every finite field $\mathbb{F}_q$ with $\lambda \in \mathbb{F}_q$, we have the congruence formula*

$$\#X_\lambda(\mathbb{F}_q) \equiv \#Y_\lambda(\mathbb{F}_q) \pmod q.$$

If furthermore, $\lambda \in \mathbb{F}_q$ such that g is Δ-regular and W_λ is a mirror manifold of X_λ, then

$$\#Y_\lambda(\mathbb{F}_q) \equiv \#W_\lambda(\mathbb{F}_q) \pmod q.$$

Proof. For the first part, it remains to check the case $\lambda = 0$. The proof is similar and in fact somewhat simpler than the case $\lambda \neq 0$. We give an outline. Since $\lambda = 0$, we can take $k_{n+2} = 0$ in the calculations of N_f and N_g^*. One finds then

$$\#X_0(\mathbb{F}_q) \equiv 1 + \sideset{}{^*}\sum_{\substack{Mk\equiv 0 (\mathrm{mod}\ q-1)\\ s(k)=n+2}} \frac{1}{q(q-1)}(\prod_{j=1}^{n+2} G(k_j)),$$

where $\sum^*$ means summing over all those solutions $k = (k_1, \cdots, k_{n+1}, 0)$ with $0 \leq k_i \leq q-1$ and $k \neq (0, \cdots, 0)$.

Similarly, one computes that

$$N_g^* \equiv n(-1)^{n-1} + \sideset{}{'}\sum_{Nk\equiv 0 (\mathrm{mod}\ q-1)} \frac{1}{q(q-1)}(\prod_{j=1}^{n+2} G(k_j)),$$

where $\sum'$ means summing over all those non-trivial solutions k with $k_{n+2} = 0$. By (12), we deduce

$$\#Y_0(\mathbb{F}_q) \equiv 1 + \sideset{}{'}\sum_{Nk\equiv 0 (\mathrm{mod}\ q-1)} \frac{1}{q(q-1)}(\prod_{j=1}^{n+2} G(k_j)).$$

As before, one checks that

$$\sideset{}{^*}\sum_{\substack{Mk\equiv 0 (\mathrm{mod}\ q-1)\\ s(k)=n+2}} (\prod_{j=1}^{n+2} G(k_j)) \equiv \sideset{}{'}\sum_{Nk\equiv 0 (\mathrm{mod}\ q-1)} (\prod_{j=1}^{n+2} G(k_j)) (\mathrm{mod}\ q^2).$$

The first part of the theorem follows.

To prove the second part of the theorem, let Δ^* be the dual polytope of Δ. One checks that Δ^* is the simplex in $\mathbb{R}^n$ with the vertices

$$(n+1)e_i - \sum_{j=1}^{n} e_j \ (i = 1, ..., n), \ -\sum_{j=1}^{n} e_j.$$

This is the $(n+1)$-multiple of a basic (regular) simplex in $\mathbb{R}^n$. In particular, the codimension 1 faces of Δ^* are $(n+1)$-multiples of a basic simplex in $\mathbb{R}^{n-1}$. By the parrallel hyperplane decomposition in [**7**], one deduces that the codimension 1 faces of Δ^* have a triangulation into basic simplices. Fix such a triangulation which produces a smooth crepant resolution $\phi : W_\lambda \to Y_\lambda$. One checks [**2**] that for each point $y \in Y_\lambda(\mathbb{F}_q)$, the fibre $\phi^{-1}(\lambda)$ is stratified by affine spaces over $\mathbb{F}_q$. Since the fibres are connected, it follows that the number of $\mathbb{F}_q$-rational points on $\phi^{-1}(\lambda)$ is congruent to 1 modulo q. Thus, modulo q, we have the congruence

$$\#W_\lambda(\mathbb{F}_q) \equiv \sum_{y\in Y_\lambda(\mathbb{F}_q)} \phi^{-1}(\lambda)(\mathbb{F}_q) \equiv \sum_{y\in Y_\lambda(\mathbb{F}_q)} 1 = \#Y_\lambda(\mathbb{F}_q).$$

The proof is complete.

In terms of zeta functions, the above theorem says that the slope $[0, 1)$ part of the zeta function for X_λ equals the slope $[0, 1)$ part of the zeta function for Y_λ.

The above elementary calculations can be used to treat some other examples of toric hypersurfaces and complete intersections. In a joint work [**6**] with Lei Fu, we have proved the following partial generalization.

THEOREM 6.2. *Let X be a smooth connected Calabi-Yau scheme defined over the ring $W(\mathbb{F}_q)$ of Witt vectors of $\mathbb{F}_q$. Let G be a finite group of W-morphisms acting on X. Assume that G fixes the non-zero global section of the canonical bundle of X. Then, for each positive integer k, we have the congruence formula*

$$\#(X \otimes \mathbb{F}_q)(\mathbb{F}_{q^k}) \equiv \#(X/G \otimes \mathbb{F}_q)(\mathbb{F}_{q^k})(\text{mod } q^k).$$

7. Applications to zeta functions

In this section, we compare the two zeta functions $Z(X_\lambda, T)$ and $Z(Y_\lambda, T)$, where $\{X_\lambda, Y_\lambda\}$ is our strong mirror pair.

First, we recall what is known about $Z(X_\lambda, T)$. Let $\lambda \in \mathbb{F}_q$ such that X_λ is smooth projective. By the Weil conjectures, the zeta function of X_λ over $\mathbb{F}_q$ has the following form

$$Z(X_\lambda, T) = \frac{P(\lambda, T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)}, \tag{13}$$

where $P(\lambda, T) \in 1 + T\mathbb{Z}[T]$ is a polynomial of degree $n(n^n - (-1)^n)/(n+1)$, pure of weight $n-1$. By the results in [**8**][**10**], the universal family of hypersurfaces of degree $n+1$ is generically ordinary for every p (Mazur's conjecture). However, we do not know if the one parameter family X_λ of hypersurfaces is generically ordinary for every p. Thus, we raise

QUESTION 7.1. *Is the one parameter family X_λ of degree $n+1$ hypersurfaces in $\mathbb{P}^n$ generically ordinary for every prime number p not dividing $(n+1)$?*

The answer is yes if $p \equiv 1 \pmod{n+1}$ since the fibre for $\lambda = 0$ is already ordinary if $p \equiv 1 \pmod{n+1}$. It is also true if $n \leq 3$. The first unknown case is when $n = 4$, the quintic case.

Next, we recall what is known about $Z(Y_\lambda, T)$. Let $\lambda \in \mathbb{F}_q$ such that g is Δ-regular. This is equivalent to assuming that $(-\lambda)^{n+1} \neq (n+1)^{n+1}$. Then, the zeta function of the affine toric hypersurface $g = 0$ over $\mathbb{F}_q$ in $\mathbb{G}_m^n$ has the following form (see [**15**])

$$Z(g, T) = Q(\lambda, T)^{(-1)^n} \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i}\binom{n}{i+1}},$$

where $Q(\lambda, T) \in 1 + T\mathbb{Z}[T]$ is a polynomial of degree n, pure of weight $n-1$. The product of the trivial factors in $Z(g, T)$ is simply the zeta function of this sequence

$$\frac{(q^k - 1)^n + (-1)^{n+1}}{q^k}, \ k = 1, 2, \cdots.$$

¿From this and (11), one deduces that the zeta function of the projective toric hypersurface Y_λ has the form

$$Z(Y_\lambda, T) = \frac{Q(\lambda, T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)}. \tag{14}$$

By the results in [**10**][**14**], this one parameter family Y_λ of toric hypersurfaces is generically ordinary for every n and every prime number p.

Now, we are ready to compare the two zeta functions $Z(X_\lambda, T)$ and $Z(Y_\lambda, T)$. Let now $\lambda \in \mathbb{F}_q$ such that X_λ is smooth and g is Δ-regular. The above description shows that

$$\frac{Z(X_\lambda, T)}{Z(Y_\lambda, T)} = (\frac{P(\lambda, T)}{Q(\lambda, T)})^{(-1)^n}.$$

To understand this quotient of zeta functions, it suffices to understand the quotient $P(\lambda, T)/Q(\lambda, T)$.

Lemma 7.2. *Assume that $(n+1)|(q-1)$. Then the polynomial $Q(\lambda, T)$ divides $P(\lambda, T)$.*

Proof. Since $(n+1)|(q-1)$, the finite map $X_\lambda \to Y_\lambda$ is a Galois covering with Galois group G, where $G = (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ is an abelian group. For an ℓ-adic representation $\rho : G \to \mathrm{GL}(V_\rho)$, let $L(X_\lambda, \rho, T)$ denote the corresponding L-function of ρ associated to this Galois covering. Then, we have the standard factorization

$$Z(X_\lambda, T) = \prod_\rho L(X_\lambda, \rho, T),$$

where ρ runs over all irreducible (necessarily one-dimensional) ℓ-adic representations of G. If $\rho = 1$ is the trivial representation, then

$$L(X_\lambda, 1, T) = Z(Y_\lambda, T).$$

For a prime number $\ell \neq p$, the ℓ-adic trace formula for $Z(X_\lambda, T)$ is

$$Z(X_\lambda, T) = \prod_{i=0}^{2(n-1)} \det(I - T\mathrm{Frob}_q | H^i(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell))^{(-1)^{i-1}},$$

where Frob_q denotes the geometric Frobenius element over $\mathbb{F}_q$. Since X_λ is a smooth projective hypersurface of dimension $n-1$, one has the more precise form of the zeta function:

$$Z(X_\lambda, T) = \frac{\det(I - T\mathrm{Frob}_q | H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell))^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)}. \tag{15}$$

Similarly, the ℓ-adic trace formula for the L-function is

$$L(X_\lambda, \rho, T) = \prod_{i=0}^{2(n-1)} \det(I - T(\mathrm{Frob}_q \otimes 1) | (H^i(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G)^{(-1)^{i-1}}.$$

For odd $i \neq n-1$,

$$H^i(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) = 0, \ (H^i(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G = 0.$$

For even $i = 2k \neq n-1$ with $0 \le k \le n-1$,

$$H^{2k}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-k), \ (H^{2k}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G = 0$$

for non-trivial irreducible ρ. This proves that for irreducible $\rho \neq 1$, we have

$$L(X_\lambda, \rho, T) = \det(I - T(\mathrm{Frob}_q \otimes 1) | (H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell) \otimes V_\rho)^G)^{(-1)^n}.$$

Similarly, taking $\rho = 1$, one finds that

$$Z(Y_\lambda, T) = \frac{\det(I - T\mathrm{Frob}_q | (H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell))^G)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)}. \tag{16}$$

Comparing (13)-(16), we conclude that

$$P(\lambda, T) = \det(I - T\mathrm{Frob}_q | H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)),$$

$$Q(\lambda, T) = \det(I - T\mathrm{Frob}_q | (H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell))^G).$$

Furthermore, the quotient

$$\frac{P(\lambda,T)}{Q(\lambda,T)} = \prod_{\rho\neq 1} \det(I - T(\mathrm{Frob}_q \otimes 1)|(H^{n-1}(X_\lambda \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)\otimes V_\rho)^G)$$

is a polynomial with integer coefficients of degree $\frac{n(n^n-(-1)^n)}{n+1} - n$, pure of weight $n-1$. The lemma is proved.

This lemma together with Theorem 6.1 gives the following result.

THEOREM 7.3. *Assume that* $(n+1)|(q-1)$. *There is a polynomial* $R_n(\lambda,T) \in 1+T\mathbb{Z}[T]$ *which is pure of weight* $n-3$ *and of degree* $\frac{n(n^n-(-1)^n)}{n+1} - n$, *such that*

$$\frac{P(\lambda,T)}{Q(\lambda,T)} = R_n(\lambda, qT).$$

We conjecture that the condition $(n+1)|(q-1)$ is not necessary in the above lemma and theorem. In the appendix, D. Haessig proved this divisibility in the case $(n+1, q-1) = 1$ and $n+1$ is a prime. In particular, the divisibility conjecture is always true if $(n+1)$ is a prime.

The polynomial $R_n(\lambda,T)$ measures how far the zeta function of Y_λ differs from the zeta function of X_λ. Being of integral pure weight $n-3$, the polynomial $R_n(\lambda,T)$ should come from the zeta function of a variety (or motive $M_n(\lambda)$) of dimension $n-3$. It would be interesting to find this variety or motive $M_n(\lambda)$ parameterized by λ. In this direction, the following is known.

If $n=2$, then $n-3<0$, $M_2(\lambda)$ is empty and we have $R_2(\lambda,T)=1$. If $n=3$, then $n-3=0$ and

$$R_3(\lambda,T) = \prod_{i=1}^{18}(1-\alpha_i(\lambda)T)$$

is a polynomial of degree 18 with $\alpha_i(\lambda)$ being roots of unity. In fact, Dwork [**4**] proved that all $\alpha_i(\lambda) = \pm 1$ in this case. Thus, $R_3(\lambda,T)$ comes from the the zeta function of a zero-dimensional variety $M_3(\lambda)$ parameterized by λ. What is this zero-dimensional variety $M_3(\lambda)$? For every p and generic λ, the slope zeta function has the form $S_p(Y_\lambda,u,T)=1$ and

$$S_p(X_\lambda,u,T) = \frac{1}{(1-T)^2(1-uT)^{20}(1-u^2T)^2}.$$

Note that Y_λ is singular and not a smooth mirror of X_λ yet. Thus, it is not surprising that the two slope zeta functions $S_p(X_\lambda,u,T)$ and $S_p(Y_\lambda,u,T)$ do not satisfy the expected slope mirror symmetry.

If $n=4$, then $n-3=1$ and

$$R_4(\lambda,T) = \prod_{i=1}^{200}(1-\alpha_i(\lambda)T)$$

is a polynomial of degree 200 with $\alpha_i(\lambda) = \sqrt{q}$. Thus, $M_4(\lambda)$ should come from some curve parameterized by λ. This curves has been constructed explicitly in a recent paper by Candelas, de la Ossa and F. Rodriguez-Villegas [**3**]. For every p and generic λ, we know that $S_p(Y_\lambda,u,T)=1$, but as indicated at the beginning of this section, we do not know if the slope zeta function of X_λ for a generic λ has the form

$$S_p(X_\lambda,u,T) = \frac{(1-T)(1-uT)^{101}(1-u^2T)^{101}(1-u^3T)}{(1-T)(1-uT)(1-u^2T)(1-u^3T)}.$$

For general n and $\lambda \in K$ for some field K, in terms of ℓ-adic Galois representations, the pure motive $M_n(\lambda)$ is simply given by

$$M_n(\lambda) = (\bigoplus_{\rho \neq 1}(H^{n-1}(X_\lambda \otimes \bar{K}, \mathbb{Q}_\ell) \otimes V_\rho)^G) \otimes \mathbb{Q}_\ell(-1),$$

where $\mathbb{Q}_\ell(-1)$ denotes the Tate twist. If λ is in a number field K, this implies that the Hasse-Weil zeta functions of X_λ and Y_λ are related by

$$\zeta(X_\lambda, s) = \zeta(Y_\lambda, s) L(M_n(\lambda), s-1).$$

8. Slope zeta functions

The slope zeta function satisfies a functional equation. This follows from the usual functional equation which in turn is a consequence of the Poincare duality for ℓ-adic cohomology.

PROPOSITION 8.1. *Let X be a connected smooth projective variety of dimension d over $\mathbb{F}_q$. Then the slope zeta function $S_p(X, u, T)$ satisfies the following functional equation*

$$S_p(X, u, \frac{1}{u^d T}) = S_p(X, u, T)(-u^{d/2}T)^{e(X)}, \tag{17}$$

where $e(X)$ denotes the the ℓ-adic Euler characteristic of X.

Proof. Let $P_i(T)$ denote the characteristic polynomial of the geometric Frobenius acting on the i-th ℓ-adic cohomology of $X \otimes \bar{\mathbb{F}}_q$. Then,

$$Z(X, T) = \prod_{i=0}^{2d} P_i(T)^{(-1)^{i+1}}.$$

Let s_{ij} $(j = 1, \cdots, b_i)$ denote the slopes of the polynomial $P_i(T)$, where b_i is the degree of $P_i(T)$ which is the i-th Betti number. Write

$$Q_i(T) = \prod_{j=1}^{b_i}(1 - u^{s_{ij}}T).$$

Then, by the definition of the slope zeta function, we have

$$S_p(X, u, T) = \prod_{i=0}^{2d} Q_i(T)^{(-1)^{i+1}}.$$

For each $0 \leq i \leq 2d$, the slopes of $P_i(T)$ satisfies the determinant relation

$$\sum_{j=1}^{b_i} s_{ij} = \frac{i}{2}b_i.$$

Using this, one computes that

$$Q_i(\frac{1}{T}) = (-1/T)^{b_i} u^{ib_i/2} \prod_{j=1}^{b_i}(1 - u^{-s_{ij}}T).$$

Replacing T by $u^d T$, we get

$$Q_i(\frac{1}{u^d T}) = (\frac{-1}{u^d T})^{b_i} u^{ib_i/2} \prod_{j=1}^{b_i}(1 - u^{d-s_{ij}}T).$$

The functional equation for the usual zeta function $Z(X,T)$ implies that $d - s_{ij}$ ($j = 1, \cdots, b_i$) are exactly the slopes for $P_{2d-i}(T)$. Thus,

$$Q_i(\frac{1}{u^d T}) = (\frac{-1}{u^d T})^{b_i} u^{ib_i/2} Q_{2d-i}(T).$$

We deduce that

$$S_p(X, u, \frac{1}{u^d T}) = \prod_{i=0}^{2d} \left(Q_{2d-i}(T)(\frac{-1}{u^d T})^{b_i} u^{ib_i/2}\right)^{(-1)^{i+1}}.$$

Since $b_i = b_{2d-i}$, it is clear that

$$\sum_{i=0}^{2d} (-1)^i \frac{i}{2} b_i = \frac{d}{2} e(X).$$

We conclude that

$$S_p(X, u, \frac{1}{u^d T}) = S_p(X, u, T)(-T)^{e(X)} u^{\frac{d}{2}e(X)}.$$

The proposition is proved.

In the rest of this section, we assume that X is a smooth projective scheme over $W(\mathbb{F}_q)$. Assume that the reduction $X \otimes \mathbb{F}_q$ is ordinary, i.e., the p-adic Newton polygon coincides with the Hodge polygon [**9**]. This means that the slopes of $P_i(T)$ are exactly j ($0 \leq j \leq i$) with multiplicity $h^{j,i-j}(X)$. In this case, one gets the explicit formula

$$S_p(X \otimes \mathbb{F}_q, u, T) = \prod_{j=0}^{d} (1 - u^j T)^{e_j(X)}, \tag{18}$$

where

$$e_j(X) = (-1)^j \sum_{i=0}^{d} (-1)^{i-1} h^{j,i}(X). \tag{19}$$

If X and Y form a mirror pair over the Witt ring $W(\mathbb{F}_q)$, the Hodge symmetry $h^{j,i}(X) = h^{j,d-i}(Y)$ implies for each j,

$$e_j(X) = (-1)^j \sum_{i=0}^{d} (-1)^{i-1} h^{j,d-i}(Y) = (-1)^d e_j(Y).$$

We obtain the following result.

Proposition 8.2. *Let X and Y be a mirror pair of d-dimensional smooth projective Calabi-Yau schemes over $W(\mathbb{F}_q)$. Assume that both $X \otimes \mathbb{F}_q$ and $Y \otimes \mathbb{F}_q$ are ordinary. Then, we have the following symmetry for the slope zeta function:*

$$S_p(X \otimes \mathbb{F}_q, u, T) = S_p(Y \otimes \mathbb{F}_q, u, T)^{(-1)^d}.$$

The converse of this proposition may not be always true. The slope mirror conjecture follows from the following slightly stronger

Conjecture 8.3 (Generically ordinary conjecture). *Let $d \leq 3$. Suppose that $\{X, Y\}$ form a generic mirror pair of d-dimensional smooth projective Calabi-Yau schemes over $W(\mathbb{F}_q)$. Then, both $X \otimes \mathbb{F}_q$ and $Y \otimes \mathbb{F}_q$ are generically ordinary.*

For $d \leq 3$, it should be possible to prove this conjecture in the toric hypersurface case using the results in [**10**][**14**]. For $d \geq 4$, we expect that the same conjecture holds if $p \equiv 1 \pmod{D}$ for some positive integer D. This should again be provable in the toric hypersurface case using the results in [**10**]. But we do not know if we can always take $D = 1$, even in the toric hypersurface case if $d \geq 4$.

9. Appendix (by C. Douglas Haessig)

The purpose of this appendix is twofold. First, we demonstrate that under certain conditions we may extend the Arithmetic Mirror Theorem (Theorems 1.1 and 6.1). Second, we apply this extension to the study of the quotient of the zeta functions of X_λ and Y_λ.

As in the introduction, with $\lambda \in \mathbb{C}$ we may define a family of complex projective hypersurfaces X_λ in $\mathbb{P}^n_{\mathbb{C}}$ by

$$x_1^{n+1} + \cdots + x_{n+1}^{n+1} + \lambda x_1 \cdots x_{n+1} = 0.$$

With the group

$$G := \{(\zeta_1, \ldots, \zeta_{n+1}) | \zeta_i \in \mathbb{C}, \zeta_i^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\}$$

we may define the (singular) mirror variety Y_λ as the quotient X_λ/G where G acts by coordinate multiplication. It turns out that Y_λ is a toric hypersurface and may be explicitly described as the projective closure in $\mathbb{P}_\Delta$ of the affine toric hypersurface

$$g(x_1, \ldots, x_n) := x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$

Note, $\mathbb{P}_\Delta$ is the toric variety obtained from the polytope in $\mathbb{R}^n$ with vertices $\{e_1, \ldots, e_n, -(e_1 + \cdots + e_n)\}$, where the e_i are the standard basis vectors of $\mathbb{R}^n$. From this description of Y_λ, if we let $\mathbb{F}_q$ denote the finite field with q elements of characteristic p, it makes sense to discuss $\mathbb{F}_{q^k}$-rational points of X_λ and its mirror Y_λ when the parameter λ lies in $\mathbb{F}_q$.

When the $gcd(n+1, q^k - 1) = 1$, there are no $(n+1)$-roots of unity in the field $\mathbb{F}_{q^k}$. Viewing G as a group scheme over $\mathbb{Z}$, this means there are no $\mathbb{F}_{q^k}$-rational points of G. This leads one to suspect a direct relation between the $\mathbb{F}_{q^k}$-rational points of X_λ and Y_λ:

Theorem 9.1. *For every positive integer k such that $gcd(n+1, q^k - 1) = 1$, we have the equality $\#X_\lambda(\mathbb{F}_{q^k}) = \#Y_\lambda(\mathbb{F}_{q^k})$.*

If W_λ is a smooth crepant resolution of Y_λ, then there is a rational map from W_λ to Y_λ which is injective on rational smooth points. Thus, if none of the $\mathbb{F}_{q^k}$-rational points on Y_λ are singular points, we see that $\#Y_\lambda(\mathbb{F}_{q^k}) = \#W_\lambda(\mathbb{F}_{q^k})$. Consequently, we have:

Corollary 9.2. *Suppose the singular locus of Y_λ contains no $\mathbb{F}_{q^k}$-rational points. If $gcd(n+1, q^k-1) = 1$, then we have $\#X_\lambda(\mathbb{F}_{q^k}) = \#Y_\lambda(\mathbb{F}_{q^k}) = \#W_\lambda(\mathbb{F}_{q^k})$.*

Next, when $gcd(n+1, q^k - 1) > 1$ we may prove:

Theorem 9.3. *Let $d := gcd(n+1, q^k - 1) > 1$. Then*

(1) *$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0$ mod d,*
(2) *if $n+1$ is a power of a prime ℓ, then, writing $\lambda = -(n+1)\psi$ in the new parameter ψ, we have*

$$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \ mod(\ell d) \quad and \quad \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \begin{cases} 1 & \psi^{n+1} = 1 \\ 0 & otherwise \end{cases} \ mod(\ell).$$

Thus, combining Theorems 1.1 and 9.3 with the Chinese Remainder Theorem yields:

Corollary 9.4. *Suppose $n+1$ is a power of a prime ℓ and $gcd(n+1, q) = 1$. Set $\lambda = -(n+1)\psi$. If $\psi^{n+1} \neq 1$, then for every positive integer k, we have $\#X_\lambda(\mathbb{F}_{q^k}) \equiv \#Y_\lambda(\mathbb{F}_{q^k})$ modulo(ℓq^k).*

Before discussing the proofs of Theorems 9.1 and 9.3, let us apply Theorem 9.1 to the quotient of the zeta functions of X_λ and Y_λ.

9.1. Application to zeta functions. From Theorem 7.3, when $(n+1)|q-1$ then the quotient of the zeta functions of X_λ and Y_λ, when raised to the $(-1)^n$ power, is a polynomial of specified degree. As mentioned in Section 7, we conjecture that the divisibility $(n+1)|q-1$ is unnecessary and may be removed without disturbing the conclusion. Evidence for this is the following:

THEOREM 9.5. *Let $n+1$ be a prime such that $gcd(n+1,q)=1$. Let k be the smallest positive integer such that $q^k \equiv 1$ modulo $n+1$. Assume X_λ is non-singular and $\lambda^{n+1} \neq (-(n+1))^{n+1}$. Then there are positive integers $\rho_1,\dots,\rho_s$, each divisible by k, and polynomials $Q_1,\dots,Q_s \in 1+T\mathbb{Z}[T]$ which are pure of weight $n-3$ and irreducible over $\mathbb{Z}$, such that*

$$\left(\frac{Z(X_\lambda/\mathbb{F}_q,T)}{Z(Y_\lambda/\mathbb{F}_q,T)}\right)^{(-1)^n} = Q_1(q^kT^k)^{\rho_1/k}\cdots Q_s(q^kT^k)^{\rho_s/k}$$

Furthermore, $\rho_1+\cdots+\rho_s = \frac{n(n^n-(-1)^n)}{n+1} - n$. (Note, the polynomials Q_i depend on n and λ.)

PROOF. For every nonnegative integer s and $j=1,\dots,k-1$, we have $gcd(n+1, q^{sk+j}-1)=1$. So, by Theorem 9.1, we have $\#X_\lambda(\mathbb{F}_{q^{sk+j}}) = \#Y_\lambda(\mathbb{F}_{q^{sk+j}})$ for every $s\geq 0$ and $j=1,\dots,k-1$. This implies

$$\frac{Z(X_\lambda/\mathbb{F}_q,T)}{Z(Y_\lambda/\mathbb{F}_q)} = \frac{\exp\sum_{s\geq 1}\frac{\#X_\lambda(\mathbb{F}_{q^{ks}})}{ks}T^{ks}}{\exp\sum_{s\geq 1}\frac{\#Y_\lambda(\mathbb{F}_{q^{ks}})}{ks}T^{ks}} = \left(\frac{Z(X_\lambda/\mathbb{F}_{q^k},T^k)}{Z(Y_\lambda/\mathbb{F}_{q^k},T^k)}\right)^{1/k} \tag{20}$$

where the first equality uses the previous sentence and the second equality is simply definition. By Theorem 7.3, there exists a polynomial $R_n(\lambda,T)\in 1+T\mathbb{Z}[T]$ of degree $\frac{n(n^n-(-1)^n)}{n+1}-n$, pure of weight $n-3$, such that

$$\left(\frac{Z(X_\lambda/\mathbb{F}_{q^k},T)}{Z(Y_\lambda/\mathbb{F}_{q^k},T)}\right)^{(-1)^n} = R_n(\lambda,q^kT).$$

Combining this with (20) shows us that

$$\left(\frac{Z(X_\lambda/\mathbb{F}_q,T)}{Z(Y_\lambda/\mathbb{F}_q,T)}\right)^{(-1)^n} = R_n(\lambda,q^kT^k)^{1/k}.$$

Therefore, factorizing $R_n(\lambda,T) = Q_1(T)^{\rho_1}\cdots Q_s(T)^{\rho_s}$ into irreducibles over $\mathbb{Z}$ proves the theorem. ☐

As a side remark, for $n+1=5$, Theorem 9.5 explains the form of the zeta function of $Z(X_\lambda/\mathbb{F}_q,T)$ found in [**3**].

We note that, for the quintic ($n+1=5$), it follows from [**3**, Equation 10.3] and [**3**, Equation 10.7], in which they empirically compute the zeta functions of X_λ and W_λ, that for X_λ smooth,

$$\frac{Z(X_\lambda/\mathbb{F}_q,T)}{Z(W_\lambda/\mathbb{F}_q,T)} = R_{\mathcal{A}}(q^kT^k,\lambda)^{20/k}R_{\mathcal{B}}(q^kT^k,\lambda)^{30/k}$$

where k is the smallest positive integer such that $q^k\equiv 1$ modulo 5 and the R's are quartic polynomials over $\mathbb{Z}$ which are not necessarily irreducible. Note, $k=1$, 2 or 4. Furthermore, they have constructed auxiliary curves $\mathcal{A}$ and $\mathcal{B}$, both of genus 4, whose zeta functions experimentally correspond to $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$, respectively. It would be interesting to find these "auxiliary varieties" for general $n+1$ and see how they fit into the framework of mirror symmetry (if at all).

9.2. The proof of Theorem 9.1. Without loss, we will write q instead of q^k in the following proof.

9.2.1. *Formulas for $X_\lambda(\mathbb{F}_q)$ and $Y_\lambda(\mathbb{F}_q)$ in terms of Gauss sums.* Define the Gauss sums $G(k)$ as in Section 2. Also, let M be the $(n+2)\times(n+2)$-matrix defined in Section 3.

Define the set

$$E := \{k \in \mathbb{Z}^{n+2} | 0 \le k_i \le q-1 \text{ and } Mk \equiv 0 \bmod (q-1)\}.$$

For each $k \in \mathbb{Z}^{n+2}$, define $s(k)$ as the number of non-zero entries in $Mk \in \mathbb{Z}^{n+2}$. Next, define

$$\begin{aligned}
E_1 &:= \{k \in E | \text{not all } k_1, \dots, k_{n+1} \text{ are the same, but } 0 \le k_{n+2} \le q-1\} \\
E_2 &:= \{k \in E | k_1 = k_2 = \cdots = k_{n+1}, 0 \le k_{n+2} \le q-1\} \\
E_2^* &:= \{k \in E_2 | 0 < k_1 < q-1, s(k) = n+2\} \\
S_k &:= \frac{q^{n+1-s(k)}}{(q-1)^{n+3-s(k)}} \left(\prod_{j=1}^{n+2} G(k_j) \right) \chi(\lambda)^{k_{n+2}}.
\end{aligned}$$

Now, Section 3 demonstrated that

$$\#X_\lambda(\mathbb{F}_q) = \frac{-1}{q-1} + \sum_{k \in E_1} S_k + \sum_{k \in E_2} S_k.$$

Consider $k \in E$. Suppose $k_1 = \cdots = k_{n+1} = 0$. If $k_{n+2} = 0$, then $S_k = q^{n+1}/(q-1)$, else, if $k_{n+2} = q-1$, then $S_k = -(q-1)^n$. Similarly, suppose $k_1 = \cdots = k_{n+1} = q-1$. If $k_{n+2} = 0$, then $S_k = (-1)^{n+1} q^n$, else, if $k_{n+2} = q-1$, then $S_k = (-1)^n q^{n+1}/(q-1)$.

Next, notice

$$Mk = \begin{pmatrix} k_1 + \cdots + k_{n+2} \\ (n+1)k_1 + k_{n+2} \\ (n+1)k_2 + k_{n+2} \\ \vdots \\ (n+1)k_{n+1} + k_{n+2} \end{pmatrix} \in \mathbb{Z}^{n+2}. \tag{21}$$

If one of the rows equals zero, then we must have $k_i = 0$ for some $1 \le i \le n+1$. Thus, if $k \in E_2$ such that $0 < k_1 < q-1$, then all the rows of Mk must be non-zero; that is, $s(k) = n+2$. Putting this together with the last paragraph, we find that for $\lambda \ne 0$, then

$$\#X_\lambda(\mathbb{F}_q) = \frac{q^{n+1} + (-1)^n q^n - 1 - (q-1)^{n+1}}{q-1} + \sum_{k \in E_1} S_k + \sum_{k \in E_2^*} S_k. \tag{22}$$

If $\lambda = 0$, then Section 2 tells us that k_{n+2} is forced to equal zero. Thus, in the above calculations, we need to neglect all terms in which $k_{n+2} \ne 0$. Doing this, we obtain

$$\#X_0(\mathbb{F}_q) = \sum_{k \in E_1} S_k + N_0^* + \frac{q^{n+1}-1}{q-1} + (-1)^{n+1} q^n + \frac{(-1)^n - (q-1)^n}{q}.$$

Let N_λ^* denote the number of $\mathbb{F}_q$-rational points on the affine (toric) variety defined by

$$g(x_1, \ldots, x_n) := x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} + \lambda = 0.$$

In the proof of Theorem 5.1, we saw that for $\lambda \neq 0$

$$N_\lambda^* = \frac{(q-1)^n}{q} + \frac{(-1)^n}{q(q-1)} + \sum_{k \in E_2^*} S_k. \tag{23}$$

Also, if $\lambda = 0$, we may calculate that

$$N_0^* = \frac{(q-1)^n}{q} + \frac{(-1)^{n+1}}{q} + \sum_{k \in E_2^*} S_k.$$

For ease of reference, recall from Section 2 that, for all $\lambda \in \mathbb{F}_q$), we have

$$\#Y_\lambda(\mathbb{F}_q) = N_\lambda^* - \frac{(q-1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n - 1}{q-1}. \tag{24}$$

9.2.2. *Finishing the proof of Theorem 9.1.* For $\lambda \neq 0$, combining equations (22), (23), and (24) yields

$$\#X_\lambda(\mathbb{F}_q) - \#Y_\lambda(\mathbb{F}_q) = \sum_{k \in E_1} S_k - (q-1)^n + \frac{q^{n+1} + (-1)^n q^n + (-1)^{n+1} - q^n}{q-1}.$$

Similarly, if $\lambda = 0$, then

$$\#X_0(\mathbb{F}_q) - \#Y_0(\mathbb{F}_q) = q^n[(-1)^{n+1} + 1] + \sum_{k \in E_1} S_k.$$

We may now prove Theorem 9.1 by demonstrating that the right-hand sides of the above two formulas equal zero when $gcd(n+1, q-1) = 1$.

LEMMA 9.6. *If $gcd(n+1, q-1) = 1$, then the right-hand sides are zero.*

PROOF. Let $k \in E$. Suppose $k_i \neq 0$ for $1 \leq i \leq q-1$. Then, by (21), we see that $(n+1)k_i \equiv (n+1)k_j$ modulo $q-1$ for every $1 \leq i, j \leq q-1$. By hypothesis, $n+1$ is invertible in $\mathbb{Z}/(q-1)$, and so $k_i = k_j$. This means that, if $k \in E_1$ then at least one of the first $n+1$ coordinates must be zero.

Let $1 \leq i \leq n$. Suppose $k \in E_1$ and its first i coordinates are zero. Then (21) tells us that k_{n+2} is either zero or $q-1$. In the *first case*, we have $k_{i+1} = \cdots = k_{n+1} = q-1$ and $s(k) = n+2$. In the *second case*, again we have $k_{i+1} = \cdots = k_{n+2} = q-1$, but $s(k) = (n+2) - i$. This leads to the following formulas:

When $k_{n+2} = 0$ (first case): $S_k = (-1)^{(n+1)-i} q^n$

When $k_{n+2} = q-1$ (second case): $S_k = (-1)^{n-i}(q-1)^{i-1} q^{(n+1)-i}$.

(Note, if $\lambda = 0$, then k_{n+2} must be zero, and so, the second case never occurs.) Permuting these zeros around in $\binom{n+1}{i}$ ways among the first $n+1$ coordinates gives us all possible points in E_1. That is, if we set

$$A := \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{(n+1)-i} q^n \qquad \text{(counts first case)}$$

and

$$B := \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{n-i} (q-1)^{i-1} q^{(n+1)-i} \qquad \text{(counts second case)},$$

then we have $\sum_{k\in E_1} S_k = A+B$ for $\lambda \neq 0$, else $\sum_{k\in E_1} S_k = A$ for $\lambda = 0$. Now, by the binomial theorem, we see that

$$A = q^n[(1-1)^{n+1} - (-1)^{n+1} - 1] = q^n[(-1)^n - 1]$$

and

$$\begin{aligned} B &= (q-1)^{-1}(-1)^n[(-(q-1)+q)^{n+1} - q^{n+1} - (-1)^{n+1}(q-1)^{n+1}] \\ &= (q-1)^{-1}[(-1)^n + (-1)^{n+1}q^{n+1} + (q-1)^{n+1}]. \end{aligned}$$

Thus, for $\lambda \neq 0$, we have

$$\sum_{k\in E_1} S_k = q^n[(-1)^n - 1] + \frac{(-1)^n + (-1)^{n+1}q^{n+1} + (q-1)^{n+1}}{q-1}$$

which proves the lemma. □

9.3. The proof of Theorem 9.3. Let us recall what we will prove.

Theorem 9.3. *Let $d := gcd(n+1, q^k - 1) > 1$. Then*

1. *$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0$ modulo d.*
2. *Writing $\lambda = -(n+1)\psi$ in the new parameter ψ, if $n+1$ is a power of a prime ℓ, then*

$$\#X_\lambda(\mathbb{F}_{q^k}) \equiv 0 \ mod(\ell d) \quad and \quad \#Y_\lambda(\mathbb{F}_{q^k}) \equiv \begin{cases} 1 & \psi^{n+1} = 1 \\ 0 & otherwise \end{cases} \ mod(\ell).$$

Proof. Without loss, we will write q instead of q^k in the following proof. First, let us prove the congruences on X_λ. We do this by gathering all the points of X_λ that have the same number of coordinates zero. For each $1 \leq i \leq n-1$, define M_i^* as the number of $\mathbb{F}_q$-rational points in $\mathbb{P}^{n-i}_{\mathbb{F}_q^*}$ which lie on the diagonal hypersurface

$$x_{i+1}^{n+1} + \cdots + x_{n+1}^{n+1} = 0.$$

Notice that the group

$$G_i(\mathbb{F}_q) := \{(\zeta_{i+1}, \ldots, \zeta_{n+1}) | \zeta_j \in \mathbb{F}_q, \zeta_j^{n+1} = 1\} / \{(\zeta, \ldots, \zeta) | \zeta^{n+1} = 1\}$$

acts freely on the set of points defining M_i^*. Since there are $d := gcd(n+1, q-1)$ many $(n+1)$-roots of unity in $\mathbb{F}_q$, we have $\#G_i(\mathbb{F}_q) = d^{n+1-i}/d = d^{n-i}$. Consequently, d^{n-i} divides M_i^*. Next, let M_0^* be the number of $\mathbb{F}_q$-rational points in $\mathbb{P}^n_{\mathbb{F}_q^*}$ which lie on X_λ. The group

$$G(\mathbb{F}_q) := \{(\zeta_1, \ldots, \zeta_{n+1}) | \zeta_i \in \mathbb{F}_q, \zeta_i^{n+1} = 1, \zeta_1 \cdots \zeta_{n+1} = 1\} / \{(\zeta, \ldots, \zeta) | \zeta^{n+1} = 1\}$$

acts freely on the points defining M_0^*, and so $\#G(\mathbb{F}_q) = d^n$ divides M_0^*. Putting this together, we have

$$\#X_\lambda(\mathbb{F}_q) = M_0^* + \sum_{i=1}^{n-1} \binom{n+1}{i} M_i^*.$$

This proves the first part of the theorem since each M_i^* is divisible by d. If $n+1$ is a power of a prime ℓ, then not only are the M_i^* divisible by d, but each of the binomial factors are divisible by ℓ; this proves the congruence on X_λ in the second part of the theorem.

We now assume $n+1$ is a power of a prime ℓ. Let us prove the congruence on Y_λ. With $\lambda = -(n+1)\psi$, recall from (24) that

$$\#Y_\lambda(\mathbb{F}_q) = N_\lambda^* - \frac{(q-1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n - 1}{q-1}$$

where N_λ^* is the number of $\mathbb{F}_q$-rational points in $\mathbb{A}^{n+1}_{\mathbb{F}_q}$ that satisfy

$$\begin{cases} x_1 + \cdots + x_{n+1} - (n+1)\psi = 0 \\ x_1 \cdots x_{n+1} = 1 \end{cases}. \tag{25}$$

We claim that $\#Y_\lambda(\mathbb{F}_q) \equiv N_\lambda^*$ modulo ℓ. Since $gcd(n+1, q-1) > 1$, $q \equiv 1$ modulo ℓ. Using the fact that $\frac{q^n-1}{q-1} = q^{n-1} + \cdots + q + 1$, we have: if ℓ is an odd prime (the even case is similar), then

$$-\frac{(q-1)^n}{q} + \frac{(-1)^n}{q} + \frac{q^n-1}{q-1} \equiv -0 + 1 + n \equiv 0 \quad \text{modulo}(\ell).$$

This proves the claim.

Since we now have $\#Y_\lambda(\mathbb{F}_q) \equiv N_\lambda^*$ modulo ℓ, we will concentrate on computing N_λ^*. Consider counting the points on (25) as follows: suppose a point $x := (x_1, \ldots, x_{n+1}) \in \mathbb{A}^{n+1}_{\mathbb{F}_q}$ has two coordinates equal. Then we may permute these two around in $\binom{n+1}{2}$ ways without changing the *order* of the other coordinates. Thus, the orbit of the point x under this type of permutation contains $\binom{n+1}{2}$ points contained in the affine toric variety defined by (25). Note that we are not overcounting the points which have multiple pairs of coordinates being the same, like $(1, 1, 2, 2, 2)$. If all the coordinates of x are different then we may permute these around in $(n+1)!$ ways.

Putting this together, we find, modulo ℓ:

$$N_\lambda^*(\mathbb{F}_q) \equiv \#\{x \in \mathbb{A}^{n+1}_{\mathbb{F}_q} | \text{ all coordinates are equal and } x \text{ satisfies (25)}\}.$$

If all the coordinates are equal then we have the system $(n+1)x - (n+1)\psi = 0$ and $x^{n+1} = 1$. By hypothesis, $n+1$ is invertible in $\mathbb{F}_q$, thus, we have $x = \psi$ for the first equation, and so $\psi^{n+1} = 1$ for the second. Therefore,

$$N_\lambda^* \equiv \begin{cases} 1 & \psi^{n+1} = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{modulo}(\ell).$$

□

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Institute of Mathematics, Chinese Academy of Sciences, Beijing, P.R. China

Department of Mathematics, University of California, Irvine, CA 92697-3875
E-mail address: `dwan@math.uci.edu`