Moment Zeta Functions in Arithmetic Geometry

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1 Introduction

In this expository paper, we discuss several fundamental properties on variation of the Hasse-Weil zeta function attached to an algebraic variety. The zeta function is a suitable generating function which counts the (closed) points of the algebraic variety. Among the most important properties are the conjectural meromorphic continuation and the generalized Riemann hypothesis. In characteristic zero case, both conjectures are wide open in general. In positive characteristic case, both conjectures are well understood by the results of Dwork and Deligne on the Weil conjectures.

Our main concern here is how the zeta function varies when the algebraic variety moves in an algebraic family. This leads to the introduction of a higher moment zeta function. Again, among the most important properties are the conjectural meromorphic continuation and the generalized Riemann hypothesis. In characteristic zero case, everything remains largely conjectural even for an elliptic curve. In positive characteristic case, much can be proved in view of the various tools developed for the Weil conjectures.

A further basic question is to understand how the moment zeta function varies when the moment parameter goes to infinity. This naturally leads to the possible introduction of a limiting moment zeta function, which measures the arithmetic variation of the moment zeta function. In characteristic zero case, we do not yet know how to rigorously define such a limiting moment zeta function, which would presumably be related to the conjectural p-adic zeta function and higher dimensional non-abelian Iwasawa theory. In positive characteristic case, we can indeed define such a limiting moment zeta function. This raises new fundamental open problems (in terms of meromorphic continuation and a suitable Riemann hypothesis) concerning the limiting moment zeta function, generalizing Dwork's p-adic meromorphy conjecture. We shall give an elementary formulation of some of these problems.

2 Riemann zeta function

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots,\}$ be the ring of integers. One of the most important questions in number theory is to understand the distribution of prime numbers. For a positive real number t, let $\pi(t)$ denote the number of primes p such that $p \leq t$. The function $\pi(t)$ is the counting function for prime numbers. To understand its nature as $t \to \infty$, one attaches a suitable generating function to it. In this case, the right generating function is the Riemann zeta function defined by the formal Euler product

$$\zeta(\mathbb{Z}, s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}. \tag{1}$$

By the unique factorization of integers, one deduces the Dirichlet expansion

$$\zeta(\mathbb{Z}, s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

This series converges absolutely for all complex numbers $s \in \mathbb{C}$ whose real part is greater than 1. In particular, it defines a complex analytic function in the half plane $\mathrm{Re}(s) > 1$. The first fundamental property about $\zeta(\mathbb{Z}, s)$ is given by

Theorem 2.1 *(Riemann, 1859). The function $\zeta(\mathbb{Z}, s)$ extends to a meromorphic function in the whole complex plane $s \in \mathbb{C}$ with a simple pole at $s = 1$.*

By the Weierstrass factorization of entire functions, the meaning of the above theorem is that there exists an explicit formula for $\pi(t)$ in terms of zeros and poles of $\zeta(\mathbb{Z}, s)$. More precisely, let $\Lambda(n)$ be the von-Mangoldt function which equals $\ln p$ if n is a positive integral power of p and zero otherwise. The logarithmic derivative of $\zeta(\mathbb{Z}, s)$ gives the Dirichlet series of $\Lambda(n)$:

$$-\frac{\zeta'(\mathbb{Z}, s)}{\zeta(\mathbb{Z}, s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Let

$$\psi(t) = \sum_{1 \leq n \leq t} \Lambda(n).$$

This is a weighted counting of primary numbers, each positive power of p is counted with weight $\ln p$. It is equivalent and more convenient to work with $\psi(t)$ instead of $\pi(t)$. If $t > 0$ is not a prime power, then von Mangoldt (1895) obtained the explicit formula

$$\psi(t) = t - \sum_{\rho} \frac{t^\rho}{\rho} - \frac{\zeta'(\mathbb{Z}, 0)}{\zeta(\mathbb{Z}, 0)} - \frac{1}{2} \ln(1 - t^{-2}),$$

where ρ runs over all non-trivial zeros of $\zeta(\mathbb{Z}, s)$ in the symmetric critical strip $0 < \mathrm{Re}(s) < 1$. Note that the two terms for ρ and $\bar{\rho}$ need to be summed together.

The main term t arises from the simple pole at $s = 1$. The term $\frac{1}{2}\ln(1 - t^{-2})$ arises from the trivial zeros at negative even integers of $\zeta(\mathbb{Z}, s)$.

To get further information about $\pi(t)$ and $\psi(t)$, one needs to know the nature of the zeros and poles of $\zeta(\mathbb{Z}, s)$. This is given by the following well known Riemann Hypothesis.

Conjecture 2.2 *(RH, 1859). If $\zeta(\mathbb{Z}, s) = 0$ or ∞ for some $s \in \mathbb{C}$, then $\mathrm{Re}(s) \in \frac{1}{2}\mathbb{Z}$. That is, all non-trivial zeros lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.*

The meaning of this conjecture is that there is a sharp asymptotic formula for $\pi(t)$ and $\psi(t)$. Namely, the Riemann Hypothesis implies that for any $\epsilon > 0$, we have the estimate

$$\psi(t) = t + O_\epsilon(t^{\frac{1}{2}+\epsilon}).$$

Equivalently, for any $\epsilon > 0$, we have

$$\pi(t) = \int_2^t \frac{1}{\ln u} du + O_\epsilon(t^{\frac{1}{2}+\epsilon}). \tag{3}$$

By the prime number theorem as proved by Hadamard and de la Vallé Poussin in 1899, the probability that a randomly chosen positive integer u is a prime is given by $1/\ln(u)$. This explains the above main term which was first suggested by Gauss in 1792.

The RH is undoubtedly very profound. No promising ideas of proving it have emerged yet although there are many conjectural approaches. In terms of our formulation of RH as above, one could try to prove the RH in three stages. The first stage is to prove that $\mathrm{Re}(s)$ for $\zeta(\mathbb{Z}, s) = 0$ is a rational number. The second stage is to prove that the denominator of $\mathrm{Re}(s)$ is uniformly bounded. The third stage is to prove that the common denominator of the $\mathrm{Re}(s)$ is bounded by 2. One reason to cut the RH into three stages comes from certain p-adic analogues [19], where the corresponding p-adic RH analogue fits naturally in the three stages.

At negative odd integer $s = 1 - 2k$, one can compute that

$$\zeta(\mathbb{Z}, 1 - 2k) = -\frac{B_{2k}}{2k},$$

where the B_k are the Bernoulli numbers given by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k}.$$

For a prime number p, Kummer's congruence shows that the modified special value $(1 - p^{2k-1})\zeta(\mathbb{Z}, 1 - 2k)$ is p-adically continuous in k when $2k$ varies in a fixed congruence class modulo $(p - 1)$. This gives rise to a p-adic zeta function $\zeta_p(\mathbb{Z}, s)$ of $\mathbb{Z}$ for each congruence class modulo $(p-1)$. Such p-adic zeta functions motivated much of the modern study in number theory and p-adic modular forms, see [13].

3 Hasse-Weil zeta functions

From algebraic geometric point of view, instead of the ring $\mathbb{Z}$ of integers, we can more generally consider a finitely generated commutative $\mathbb{Z}$-algebra A with 1. By the Hilbert basis theorem, such an algebra can be written explicitly in the form

$$A = \mathbb{Z}[X_1, \cdots, X_n]/(f_1, \cdots, f_m), \tag{4}$$

where the f_i are polynomials in integer coefficients. The algebra A defines an affine arithmetic scheme $X = \mathrm{Spec}(A)$, whose points are simply the set of prime ideals in A. Let $|X|$ denote the subset of closed points of X, namely, the subset of maximal ideals in A. Generalizing the counting of prime numbers (closed points in $\mathrm{Spec}(\mathbb{Z})$), we would like to count the closed points in $|X|$.

If $x \in |X|$ is a closed point, then the quotient ring A/x is a finite field, called the residue field of X at x. Denote the number of elements in the finite field A/x by $N(x)$. For a positive real number t, we define $\pi_X(t)$ to be the number of closed points $x \in |X|$ such that $N(x) \leq t$. This is a finite integer for each finite t. The function $\pi_X(t)$ is our counting function for closed points of X. To understand its behavior as $t \to \infty$, we similarly attach a generating function. The right generating function is the Hasse-Weil zeta function defined by the following formal Euler product

$$\zeta(X, s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}. \tag{5}$$

It can be written as a Dirichlet series

$$\zeta(X, s) = \sum_{n=1}^{\infty} \frac{a_n(X)}{n^s}, \tag{6}$$

where $a_n(X)$ are some non-negative integers depending on X. This series $\zeta(X, s)$ converges absolutely for all complex numbers $s \in \mathbb{C}$ whose real part is sufficiently large (say, larger than the Krull dimension of X). In particular, it defines a complex analytic function in the half plane $\mathrm{Re}(s) > \dim(X)$. The first fundamental property one would like to have is

Conjecture 3.1 *(Hasse-Weil). The function $\zeta(X, s)$ extends to a meromorphic function in the whole complex plane $s \in \mathbb{C}$.*

The meaning of this conjecture is that there exists an explicit formula for $\pi_X(t)$ in terms of zeros and poles of $\zeta(X, s)$. Without loss of generality, one can assume that X is irreducible. The above meromorphy conjecture is still wide open in general. It is true if $\dim(X) = 1$ by the work of Hecke (1917) on the Dedekind zeta functions. If $\dim(X) = 2$, the conjecture is still open but true in the case that the generic fibre $X \otimes \mathbb{Q}$ is an elliptic curve:

$$y^2 = x^3 + ax + b, \ a, b \in \mathbb{Z}.$$

This follows from the work of Wiles, Taylor and others (2001 [2]) on the modularity of elliptic curves. It is then easy to prove that if the generic fibre $X\otimes\mathbb{Q}$ is an elliptic curve, then for every positive integer k, $\zeta(\mathrm{Sym}^k X, s)$ is meromorphic everywhere, where $\mathrm{Sym}^k X$ is the k-th symmetric power of X, whose generic fibre is a k-dimensional variety over $\mathbb{Q}$. In fact, in this case, $\mathrm{Sym}^k(X\otimes\mathbb{Q})$ is a projective bundle of rank $k-1$ over $X\otimes\mathbb{Q}$ (the Jacobian of the elliptic curve $X\otimes\mathbb{Q}$). Thus, up to possibly finitely many Euler factors, we have the relation

$$\zeta(\mathrm{Sym}^k X, s) = \prod_{i=0}^{k-1} \zeta(X, s-i).$$

More generally, we have

Proposition 3.2 *Let the generic fibre $X\otimes\mathbb{Q}$ be a smooth projective curve of genus g. Assume that $\zeta(\mathrm{Sym}^k X, s)$ is meromorphic for the finitely many integers $1\leq k\leq g$. Then, $\zeta(\mathrm{Sym}^k X, s)$ is meromorphic for every positive integer k.*

This finiteness result can be generalized to certain higher dimensional varieties. Let the generic fibre $X\otimes\mathbb{Q}$ be a smooth projective variety. Let $B^{\mathrm{even}}(X)$ (resp. $B^{\mathrm{odd}}(X)$) denote the even (resp. odd) Betti number of $X\otimes\mathbb{Q}$. Let $R(X)$ denote the rank of the group of algebraic cycles on $X\otimes\mathbb{Q}$ modulo homological equivalence. The cycle class map shows that we have the inequality

$$1+\dim(X\otimes\mathbb{Q}) \leq R(X) \leq B^{\mathrm{even}}(X).$$

These inequalities become equality if $X\otimes\mathbb{Q}$ is a curve or an odd dimensional complete intersection. Using either the Weil conjectures or Deligne's symmetric Künneth formula, one can quickly prove the following finiteness result. We leave the proofs to the reader as an exercise.

Theorem 3.3 *Let $X\otimes\mathbb{Q}$ be a smooth projective variety over $\mathbb{Q}$. Assume $R(X) = B^{\mathrm{even}}(X)$. Assume that $\zeta(\mathrm{Sym}^k X, s)$ is meromorphic for the finitely many integers $1\leq k\leq B^{\mathrm{odd}}/2$. Then, $\zeta(\mathrm{Sym}^k X, s)$ is meromorphic for every positive integer k.*

If $R(X) < B^{\mathrm{even}}(X)$ such as the case of a general surface $X\otimes\mathbb{Q}$, we do not know if there is a similar finiteness property for the sequence of zeta functions $\zeta(\mathrm{Sym}^k X, s)$ parametrized by positive integers k. The above theorem suggests that under the assumption $R(X) = B^{\mathrm{even}}(X)$, the geometry of $\mathrm{Sym}^k X$ for large k can probably be determined from the first few terms $\mathrm{Sym}^k X$ for small k, as in the case of curves. If so, it would be interesting to make this precise geometrically.

Assume the meromorphic continuation of $\zeta(X, s)$. To get further information, one needs to know the nature of the zeros and poles. This is given by the following Generalized Riemann Hypothesis.

Conjecture 3.4 *(GRH). If $\zeta(X, s) = 0$ or ∞ for some complex number $s\in\mathbb{C}$, then* $\mathrm{Re}(s)\in\frac{1}{2}\mathbb{Z}$.

The meaning of this conjecture is that there is a sharp asymptotic formula for $\pi_X(t)$. Although the above two conjectures are apparently extremely difficult in general, they have been proved completely if A has positive characteristic. We shall review this geometric case in next section.

There are many general conjectures on the orders of zeros (resp. poles) of $\zeta(X,s)$ and the special values $\zeta(X,k)$ at various integers or half integers k. These include the classical BSD conjecture and Tate conjectures [17]. For each prime number p, conjecturally there is a p-adic zeta function $\zeta_p(X,s)$ attached to X, which interpolates certain special values of $\zeta(X,s)$ at negative integers. If $\dim(X)=1$, the p-adic zeta function has been constructed. If $\dim(X)=2$, the p-adic zeta function has not been constructed in general, but in the case that $X\otimes\mathbb{Q}$ is an ellliptic curve, this has been done. It is a major open problem to construct the p-adic zeta function $\zeta_p(X,s)$ of a scheme of finite type over $\mathbb{Z}$, see [16].

4 Zeta functions over finite fields

First, we assume that the characteristic of A is a prime number p. Then, A contains the finite prime field $\mathbb{F}_p$ of p elements and it becomes a finitely generated commutative $\mathbb{F}_p$-algebra with 1. The affine scheme X then becomes an affine algebraic variety defined by finitely many polynomial equations

$$f_1(X_1,\cdots,X_n)=\cdots=f_m(X_1,\cdots,X_n)=0,$$

where the f_i are polynomials over $\mathbb{F}_p$.

If $x\in|X|$ is a closed point, the residue field A/x is a finite field containing $\mathbb{F}_p$ as a subfield. Thus, A/x is a finite extension of $\mathbb{F}_p$ of degree denoted by $\deg(x)$. We have

$$N(x)=p^{\deg(x)}.$$

The Hasse-Weil zeta function becomes

$$\zeta(X,s)=\prod_{x\in|X|}(1-(p^{-s})^{\deg(x)})^{-1}. \tag{7}$$

Let $T=p^{-s}$ and define the zeta function $Z(X,T)$ of X over $\mathbb{F}_p$ by

$$Z(X,T)=\prod_{x\in|X|}(1-T^{\deg(x)})^{-1}\in 1+T\mathbb{Z}[[T]]. \tag{8}$$

Then, we have the relation

$$\zeta(X,s)=Z(X,p^{-s}).$$

Since the exponential function p^{-s} is an entire function with no zeros, it suffices to study $Z(X,T)$.

We shall consider a slightly more general situation. Instead of the finite prime field $\mathbb{F}_p$, we shall consider a general finite field $\mathbb{F}_q$. Let $\mathbb{F}_q$ be a finite field of q elements of characteristic p. Let X be a scheme of finite type over $\mathbb{F}_q$. One can similarly define

$$Z(X,T) = \prod_{x\in|X|}(1-T^{\deg(x)})^{-1} \in 1+T\mathbb{Z}[[T]], \tag{9}$$

where $\deg(x)$ denotes the extension degree of the finite residue field at x over the ground field $\mathbb{F}_q$ of X (instead of $\mathbb{F}_p$). An alternative definition for $Z(X,T)$ is in terms of counting rational points of X over various finite extensions of $\mathbb{F}_q$. Let $\bar{\mathbb{F}}_q$ denote a fixed algebraic closure of $\mathbb{F}_q$. The Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ is topologically generated by the q-th power Frobenius map

$$\mathrm{Frob} : a \to a^q,\ a \in \bar{\mathbb{F}}_q.$$

For each positive integer k, $\bar{\mathbb{F}}_q$ contains a unique subfield of q^k elements, denoted by $\mathbb{F}_{q^k}$, given explicitly by

$$\mathbb{F}_{q^k} = \mathrm{Fix}(\mathrm{Frob}^k) = \{a \in \bar{\mathbb{F}}_q | a^{q^k} = a\}.$$

We have

$$\bar{\mathbb{F}}_q = \cup_{k=1}^{\infty}\mathbb{F}_{q^k}.$$

Let $X(\bar{\mathbb{F}}_q)$ be the set of geometric points of X. For each positive integer k, let $X(\mathbb{F}_{q^k})$ denote the set of $\mathbb{F}_{q^k}$-rational points. Clearly,

$$X(\mathbb{F}_{q^k}) = \mathrm{Fix}(\mathrm{Frob}^k | X(\bar{\mathbb{F}}_q)).$$

The alternative definition of $Z(X,T)$ is

$$Z(X,T) = \exp(\sum_{k=1}^{\infty}\frac{T^k}{k}\#X(\mathbb{F}_{q^k})), \tag{10}$$

where $\#X(\mathbb{F}_{q^k})$ denotes the number of $\mathbb{F}_{q^k}$-rational points of X.

The rationality of $Z(X,T)$ was conjectured by Weil (1949 [26]) and first proved by Dwork [7] using p-adic analysis. That is, we have

Theorem 4.1 *(Dwork, 1960) Let X be a scheme of finite type over $\mathbb{F}_q$. Then, $Z(X,T) \in \mathbb{Q}(T)$.*

Corollary 4.2 *Let X be a scheme of finite type over $\mathbb{F}_q$. The Hasse-Weil zeta function $\zeta(X,s) = Z(X,q^{-s})$ is a meromorphic function in $s \in \mathbb{C}$.*

Write

$$Z(X,T) = \frac{\prod_{i=1}^{s}(1-\alpha_i T)}{\prod_{j=1}^{r}(1-\beta_j T)},$$

where the α_i's and β_j's are algebraic integers. Taking logarithmic derivative, one obtains an explicit formula for rational point counting:

$$\#X(\mathbb{F}_{q^k}) = \sum_{j=1}^{r}\beta_j^k - \sum_{i=1}^{s}\alpha_i^k,\ k = 1,2,\cdots.$$

The Riemann hypothesis for $Z(X,T)$ was conjectured by Weil (1949) and proved by Deligne [5] in the general form. In terms of $\zeta(X,s)$, this says

Theorem 4.3 *(Deligne, 1980). Let X be a scheme of finite type over $\mathbb{F}_q$. If $\zeta(X,s)=Z(X,q^{-s})=0$ or ∞ for some $s\in\mathbb{C}$, then $\mathrm{Re}(s)\in\frac{1}{2}\mathbb{Z}$.*

In terms of the zeros and poles of the rational function $Z(X,T)$, Deligne's theorem says that we can write

$$|\alpha_i| = q^{u_i/2},\ |\beta_j| = q^{v_j/2},\ u_i\in\mathbb{Z}\cap[0,2n],\quad v_j\in\mathbb{Z}\cap[0,2n], \tag{11}$$

where $\mathbb{Z}\cap[0,2n]$ denotes the set of integers in the interval $[0,2n]$. Furthermore, each α_i (resp. each β_j) and its Galois conjugates over $\mathbb{Q}$ have the same complex absolute value. That is, the α_i and β_j are Weil q-integers. This result immediately gives the above Riemann hypothesis consequence for $\zeta(X,s)$.

As the α_i and β_j are algebraic integers, one can also ask for their ℓ-adic absolute values if we fix an embedding of $\bar{\mathbb{Q}}$ to $\bar{\mathbb{Q}}_\ell$ of ℓ-adic numbers, where ℓ is a prime number. If $\ell\neq p$, the α_i and the β_j are eigenvalues of ℓ-adic representations and hence they are ℓ-adic units:

$$|\alpha_i|_\ell = |\beta_j|_\ell = 1.$$

For the remaining prime p, it is easy to prove

$$|\alpha_i|_p = q^{-r_i},\ |\beta_j|_p = q^{-s_j},\ r_i\in\mathbb{Q}\cap[0,2n],\ s_j\in\mathbb{Q}\cap[0,2n],$$

where we have normalized the p-adic absolute value by $|q|_p = q^{-1}$. Deligne's integrality theorem [6] implies the following improved information:

$$r_i\in\mathbb{Q}\cap[0,n],\ s_j\in\mathbb{Q}\cap[0,n].$$

Note that each α_i (resp. each β_j) and its Galois conjugates over $\mathbb{Q}$ may have different p-adic absolute values. The rational number r_i (resp. s_j) is called the slope of α_i (resp. β_j). Its denominator can be greater than 2, but can be effectively bounded depending on X. The nature of the slopes r_i and s_j can be viewed as the p-adic Riemann hypothesis for $Z(X,T)$. It is a very interesting subject with many open problems. A classical example is the Stickelberger theorem on Gauss sum. We shall not discuss it further here, see [25] for a systematic introduction.

An immediate consequence of the rationality, to be used later, is

Corollary 4.4 *For each positive integer k, we have*

$$\begin{aligned} Z(X\otimes\mathbb{F}_{q^k},T) &= \exp\Big(\sum_{d=1}^{\infty}\frac{T^d}{d}\#X(\mathbb{F}_{q^{kd}})\Big) \\ &= \frac{\prod_{i=1}^{s}(1-\alpha_i^kT)}{\prod_{j=1}^{r}(1-\beta_j^kT)}. \end{aligned} \tag{12}$$

The next question we would like to study is how $Z(X,T)$ varies when X moves in an algebraic family. This is called the geometric variation problem. We will discuss it in next section and introduce a moment zeta function.

To finish this section, let us briefly comment on the general X case, not necessarily of characteristic p. Let X be a scheme of finite type over $\mathbb{Z}$. Each closed point $x \in |X|$ contains a unique prime p, where $N(x) = p^{\deg(x)}$. The prime number p is called the residue characteristic of x. The subset of all closed points of X of a fixed residue characteristic p coincide with the set of closed point of the reduction $X \otimes \mathbb{F}_p$. Thus, the Hasse-Weil zeta function can be decomposed as the following infinite Euler product

$$\zeta(X,s) = \prod_{p \text{ prime}} Z(X \otimes \mathbb{F}_p, p^{-s}). \tag{13}$$

This product is a finite product, if the characteristic h of X is positive. Let $h = p_1^{k_1} \cdots p_m^{k_m}$ be the standard factorization of h. Then,

$$\zeta(X,s) = \prod_{i=1}^{m} Z(X \otimes \mathbb{F}_{p_i}, p_i^{-s}). \tag{14}$$

We obtain

Corollary 4.5 *Let X be a scheme of finite type over $\mathbb{Z}$. Assume that the characteristic h of X is positive. Then the Hasse-Weil zeta function $\zeta(X,s)$ is a meromorphic function in $s \in \mathbb{C}$. Furthermore, the real parts of its zeros and poles are in $\frac{1}{2}\mathbb{Z}$.*

For the Hasse-Weil conjecture and GRH, it remains to study the case when X has characteristic zero. The Hasse-Weil zeta function contains critical information about the arithmetic variation of the Euler factors $Z(X \otimes \mathbb{F}_p, p^{-s})$ as the arithmetic parameter p varies. We shall return to this question at the end of the paper.

5 Moment zeta functions: geometric case

Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. We may view f as a family of algebraic varieties over finite fields parametrized by Y. If $y \in |Y|$, the fibre $X_y = f^{-1}(y)$ is an algebraic variety defined over the finite residue field $\mathbb{F}_{q^{\deg(y)}}$ of y. By the results in the previous section, we know that each zeta function $Z(X_y, T)$ is a rational function and satisfies the Riemann hypothesis. We would like to understand

Question 5.1 *How $Z(X_y, T)$ varies when y varies in $|Y|$? How the zeros and poles of $Z(X_y, T)$ vary when y varies in $|Y|$?*

To make this variation question more precise, a classical procedure is to understand all the high moments of this family of rational functions. The k-th moment of each rational function $Z(X_y, T)$ is obtained by raising each reciprocal zeros (resp. reciprocal poles) to its k-th power. By Corollary 4.4, this is simply given by $Z(X_y \otimes \mathbb{F}_{q^{\deg(y)k}}, T)$. Thus, we have

Definition 5.2 *For each positive integer k, the k-th moment zeta function of the family f is defined to be the product*

$$Z(k, f, T) = \prod_{y \in |Y|} Z(X_y \otimes \mathbb{F}_{q^{\deg(y)k}}, T^{\deg(y)}) \in 1 + T\mathbb{Z}[[T]].$$

Arithmetically, the k-th moment zeta function can be viewed as the zeta function associated to a partial rational point counting problem. For a positive integer d, let

$$M_d(k, f) = \#\{x \in X(\mathbb{F}_{q^{dk}}) | f(x) \in Y(\mathbb{F}_{q^d})\}.$$

Then, it is easy to check that

$$Z(k, f, T) = \exp(\sum_{d=1}^{\infty} \frac{T^d}{d} M_d(k, f)).$$

According to an observation of Faltings (see [24]), the number $M_d(k, f)$ is the number of fixed points of a certain twisted Frobenius. Let $f^{\otimes k}$ be the k-th fibre product of X over Y. Let σ be the right shifting map on the coordinates of $f^{\otimes k}$. That is, for a point $(x^{(1)}, \cdots, x^{(k)}) \in f^{\otimes k}$,

$$\sigma(x^{(1)}, \cdots, x^{(k)}) = (x^{(k)}, x^{(1)}, \cdots, x^{(k-1)}).$$

The map σ is an automorphism of $f^{\otimes k}$ of order k. It commutes with the Frobenius map. One checks that

$$M_d(k, f) = \#\mathrm{Fix}(\sigma \circ \mathrm{Frob}^d | f^{\otimes k}(\bar{\mathbb{F}}_q)).$$

This formula implies that $Z(k, f, T)$ is nearly rational in the sense of [24], as noted by Faltings.

The sequence of moment zeta functions $Z(k, f, T)$ $(k = 1, 2, \cdots,)$ contains critical information about the variation of the family $Z(X_y, T)$ parametrized by y. Our standard questions are then the meromorphic continuation and Riemann hypothesis for these moment zeta functions. The meromorphic continuation implies the existence of good structural distribution law. The Riemann hypothesis gives sharp information about the distribution law. As in the previous section, we have the following general result which is a special case from [24] on partial zeta functions. It can be proved quickly from the results of Grothnedieck-Deligne and the theory of virtual representations.

Theorem 5.3 *(Wan[24]). Let $f : X \longrightarrow Y$ be a morphism of scheme of finite type over $\mathbb{F}_q$. For each positive integer k, the k-th moment zeta function $Z(k, f, T)$ is a rational function in $\mathbb{Q}(T)$, whose reciprocal zeros and reciprocal poles are Weil q-integers. In particular, the k-th moment Hasse-Weil zeta function $Z(k, f, q^{-s})$ is a meromorphic function. The real parts of the zeros and poles of $Z(k, f, q^{-s})$ are in $\frac{1}{2}\mathbb{Z}$.*

A very interesting question is to understand the slopes of the zeros and poles of $Z(k, f, T)$, corresponding to the p-adic Riemann hypothesis for $Z(k, f, T)$. This is in general quite mysterious, already so in the special case of the unversal family of elliptic curves where the slopes reflect crucial arithmetic information about modular forms [19].

We can further ask how the sequence $Z(k, f, T)$ varies when the integer k varies. This is treated in next section. It is related to Dwork's conjecture.

6 Limiting moment zeta functions

Let $f : X \longrightarrow Y$ be a morphism of schemes of finite types over $\mathbb{F}_q$. For each positive integer k, we have the k-th moment zeta function $Z(k, f, T)$ which is a rational function satisfying the Riemann hypothesis. We would like to understand how the moment zeta function $Z(k, f, T)$ varies as the integer k varies arithmetically.

To be more specific, we fix a prime number ℓ (which may be equal to the characteristic p). We let k vary ℓ-adically and want to understand how $Z(k, f, T)$ varies ℓ-adically. An initial property would be possible ℓ-adic continuity.

Let $K_{f,\ell} = \mathbb{Q}_\ell(\alpha_i(y), \beta_j(y))$, the extension field of $\mathbb{Q}_\ell$ obtained by adjoining all ℓ-adic *unit* zeros $\alpha_i(y)$ and all ℓ-adic *unit* poles $\beta_j(y)$ of $Z(X_y, T)$ for all $y \in |Y|$. The uniform upper bound [1] for the total degree of $Z(X_y, T)$ together with a standard algebraic number theory argument implies

Proposition 6.1 *The extension $K_{f,\ell}$ over $\mathbb{Q}_\ell$ is finite.*

Denote the uniformizer of $K_{f,\ell}$ by π. Let $d_{f,\ell}$ be the unramified degree of $K_{f,\ell}$ over $\mathbb{Q}_\ell$. Let

$$D_{f,\ell} = \ell^{d_{f,\ell}} - 1. \tag{15}$$

This is simply the order of the multiplicative group of the residue field of $K_{f,\ell}$.

Then, for each ℓ-adic unit α in $K_{f,\ell}$ (such as those ℓ-adic unit $\alpha_i(y)$ and those ℓ-adic unit $\beta_j(y)$), the power $\alpha^{D_{f,\ell}}$ is an ℓ-adic 1-unit in $K_{f,\ell}$. In particular, we have the limiting formula

$$\lim_{m\to\infty} \alpha^{D_{f,\ell}\ell^m} = 1$$

and the congruences

$$\alpha^{k_1} \equiv \alpha^{k_2} \pmod{\pi^m}$$

for any two integers k_1 and k_2 satisfying $k_1 \equiv k_2\ (\mathrm{mod} D_{f,\ell}\ell^{m-1})$, where m is any positive integer.

This ℓ-adic continuity result and the Euler product definition of $Z(k, f, T)$, show that the limit in the following definition exists as a ℓ-adic formal power series.

Definition 6.2 *Given a morphism $f : X \longrightarrow Y$ over $\mathbb{F}_q$, a prime number ℓ, an integer k, we define the limiting moment zeta function by*

$$Z_\ell(k, f, T) = \lim_{m\to\infty} Z(k + D_{f,\ell}\ell^m, f, T) \in 1 + T\mathbb{Z}_\ell[[T]]. \tag{16}$$

Note that $Z_\ell(k,f,T)$ is very different from $Z(k,f,T)$ in general. Let

$$M_{d,\ell}(k,f) = \lim_{m\to\infty} \#\{x \in X(\mathbb{F}_{q^{d(k+D_{f,\ell}\ell^m)}}) | f(x) \in Y(\mathbb{F}_{q^d})\}.$$

This is an ℓ-adic integer. It can be viewed as the number of $\mathbb{F}_{q^d}$-rational points on a certain infinite dimensional variety defined over $\mathbb{F}_q$. The additive definition of the limiting moment zeta function can be written as

$$Z_\ell(k,f,T) = \exp(\sum_{d=1}^{\infty} \frac{T^d}{d} M_{d,\ell}(k,f)).$$

The series $Z_\ell(k,f,T)$ is clearly ℓ-adic analytic in the open unit disk $|T|_\ell < 1$. It can be viewed in certain sense as the zeta function of a certain infinite dimensional variety over $\mathbb{F}_q$ counted in certain direction. Our first fundamental question is

Question 6.3 *For $k \in \mathbb{Z}$, $Z_\ell(k,f,T)$ is an ℓ-adic meromorphic function on the whole ℓ-adic plane $|T|_\ell < \infty$.*

If $Z_\ell(k,f,T)$ is indeed ℓ-adic meromorphic everywhere, then there are ℓ-adic numbers α_i and β_j approaching to zero such that

$$Z_\ell(k,f,T) = \frac{\prod_{i=1}^{\infty}(1-\alpha_i T)}{\prod_{j=1}^{\infty}(1-\beta_j T)}.$$

In terms of the ℓ-adic sequence $M_{d,\ell}(d,f)$ parametrized by d, this means that for each integer $d > 0$, we have the formula

$$M_{d,\ell}(k,f) = \sum_{j=1}^{\infty} \beta_j^d - \sum_{i=1}^{\infty} \alpha_i^d.$$

Conversely, the existence of such a formula is equivalent to the ℓ-adic meromorphic continuation of $Z_\ell(k,f,T)$ to the whole ℓ-adic plane.

If $\ell \neq p$, the $\alpha_i(y), \beta_j(y)$ are always ℓ-adic units. In this case, we deduce that if k is a positive integer, then

$$Z_\ell(k,f,T) = Z(k,f,T),$$

which is a rational function in $\mathbb{Q}(T)$ whose reciprocal zeros and reciprocal poles are Weil q-integers, by the results in the previous section. If k is negative, it can be proved in a similar way using contragradient representations.

If $\ell = p$, the situation is more complicated. Assume that k is a positive integer. Then, $Z_p(k,f,T)$ is exactly the k-th power unit root zeta function of Dwork attached to the family f. Denote this k-th power unit root zeta function by $Z_{p-\mathrm{unit}}(k,f,T)$. This is a p-adic power series, whose coefficients are not contained in a fixed number field and hence not a rational function any more in general. Dwork [9] conjectured that $Z_{p-\mathrm{unit}}(k,f,T)$ is p-adic meromorphic in the whole p-adic plane $|T|_p < \infty$. This was proved recently in [20][21][22]. Note that here we only consider the slope zero (unit root) part. Similar results hold for higher slopes, see [23] for a simple introduction.

The possible p-adic Riemann hypothesis for $Z_{p-\mathrm{unit}}(k, f, T)$ is completely mysterious at the moment. One can further ask how $Z_{p-\mathrm{unit}}(k, f, T)$ varies p-adically as k varies p-adically. This is not well understood at all, but in the so-called rank one case it is much better understood [22], see also next section.

We can extend the above results from positive integers to all integers. Thus, the above question has a positive answer and we have

Theorem 6.4 *(Wan [21][22]). Let ℓ be a prime. Let k be an integer. Then, the limiting moment zeta function $Z_\ell(k, f, T)$ is ℓ-adic meromorphic everywhere. If $\ell \neq p$, it is rational over $\mathbb{Q}$, whose reciprocal zeros and reciprocal poles are Weil q-integers.*

The integer domain of the variable k in the function $Z_\ell(k, f, T)$ can be extended to a larger ℓ-adic domain as follows. Write $k = k_1 + D_{f,\ell}k_2$. Then, it is easy to check that the formal power series $Z_\ell(k_1 + D_{f,\ell}k_2, f, T)$ is ℓ-adically continuous in k_2. This continuity implies that we can define

Definition 6.5 *Let k_1 be a fixed integer. Let k_2 be any ℓ-adic integer. We define*

$$Z_\ell(k_1, k_2, f, T) = \lim_{m \to \infty} Z_\ell(k_1 + D_{f,\ell}k_2(m), f, T),$$

where $k_2(m)$ is a sequence of strictly increasing positive integers which converges ℓ-adically to k_2.

This is a well defined ℓ-adic power series for any integer $k_1 \in \mathbb{Z}$ and any ℓ-adic integer $k_2 \in \mathbb{Z}_\ell$. It is independent of the choice of the sequence $k_2(m)$ converging ℓ-adically to k_2.

The function $Z_\ell(k_1, k_2, f, T)$ in the two variables (k_2, T) should be viewed as the ℓ-adic zeta function attached to the morphism f. It is then natural to ask

Question 6.6 *For $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_\ell$, is $Z_\ell(k_1, k_2, f, T)$ an ℓ-adic meromorphic function on the closed unit disk $|T|_\ell \leq 1$? or even on the whole ℓ-adic plane $|T|_\ell < \infty$?*

In general, the ℓ-adic meromorphic continuation of $Z_\ell(k_1, k_2, f, T)$ to the closed unit disk $|T|_\ell \leq 1$ is already not clear, even for the universal family of elliptic curves if $\ell \neq p$. It would be premature to conjecture the ℓ-adic meromorphic continuation to the entire ℓ-adic plane $|T|_\ell < \infty$. For this reason, we simply state the above problem as a question instead of a conjecture as the answer could be negative in general (there seems to have a little negative feeling in the general case). It is however interesting to find out when the answer is positive.

The previous theorem shows that the answer is positive if $k_2 \in \mathbb{Z}$. Another positive result in this direction is the following theorem which says that the answer is aslo positive for some special $k_2 \in \mathbb{Z}_\ell \cap \mathbb{Q}$.

For each integer $n > m$ for some fixed large integer m depending on $D_{f,\ell}$, the integer $\ell^{n!}(\ell^{(n+1)!-n!} - 1)$ is clearly divisible by $D_{f,\ell}$. Thus, for an integer d, we can write the formal symbal $d\ell^\infty$ (the zero element in $\mathbb{Z}_\ell$) in the form

$$d\ell^\infty = (d + \sum_{n=0}^{m} d\ell^{n!}(\ell^{(n+1)!-n!} - 1)) + \sum_{n=m+1}^{\infty} d\ell^{n!}(\ell^{(n+1)!-n!} - 1) = k_1 + D_{f,\ell}k_2,$$

where $k_2 \in \mathbb{Z}_\ell \cap \mathbb{Q}$.

Theorem 6.7 *(Lenstra-Wan [15]) Let ℓ be a prime. Let d be an integer. Let k_1 and k_2 be defined as above. Then, $Z_\ell(k_1, k_2, f, T) \in \mathbb{Q}_\ell(T)$, an ℓ-adic rational function whose reciprocal zeros and reciprocal poles are Weil q-integers.*

This theorem can be proved quickly using the Grothendieck-Deligne results and Brauer's virtual lifting theorem for modulo ℓ representations.

Let us now look at an important special case.

Example 6.8 Let $n \geq 2$ be an integer. Let f be the family of Calabi-Yau projective hypersurfaces over $\mathbb{F}_q$ defined by the equation

$$X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1} + yX_0X_1\cdots X_n = 0,$$

parametrized by the affine line $y \in \mathbb{A}^1$. In the case $n = 2$, this is a family of elliptic curves. In the case $n = 3$, this is a family of $K-3$ surfaces. In the case $n = 4$, this is a family of Calabi-Yau quintic hypersurfaces.

If $\ell \neq p$, the zeta function $Z_\ell(k_1, k_2, f, T)$ is in general not known to be ℓ-adic meromorphic on the closed unit disk $|T|_\ell \leq 1$ if $k_2 \in \mathbb{Z}_\ell$. If in addition, either $k_2 \in \mathbb{Z}$ or $k_1 + D_{f,\ell}k_2$ is equal to the formal symbal $d\ell^\infty$ as defined above, then $Z_\ell(k_1, k_2, f, T)$ is rational.

If $\ell = p$, then there is only one (or none) non-trivial p-adic unit root for the zeta function $Z(X_y, T)$ and one can show that $D_{f,p} = p - 1$. Our result implies that the limiting moment zeta function $Z_p(k_1, k_2, f, T)$ is always p-adic meromorphic everywhere for all $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_p$. This follows from the rank one case of Dwork's conjecture as given in [22]. The cases $n = 2, 3$ with $k_2 \in Z$ had been proved previously by Dwork (1971[8], 1973[9]). Such p-adic meromorphic continuation should be related to deep p-adic properties of the mirror map.

7 Moment L-functions: geometric case

To give a further and more precise exposition, we need to work with the languages of ℓ-adic representations and ℓ-adic ètale cohomology.

Let Y be a geometrically connected smooth affine scheme of finite type over $\mathbb{F}_q$ with function field $\mathbb{F}_q(Y)$. Let $\pi_1(Y)$ be the arithmetic fundamental group of Y. It is the absolute Galois group of $\mathbb{F}_q(Y)$ modulo the closed subgroup generated by the inertial subgroups at closed points of Y. Let ℓ be a prime number. Let

$$\psi_\ell : \pi_1(Y) \longrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell) \tag{17}$$

be a continuous ℓ-adic representation. Equivalently, ψ_ℓ defines a lisse ℓ-adic ètale sheaf on Y. The L-function of ψ_ℓ is defined in a standard manner as follow:

$$L(\psi_\ell, T) = \prod_{y \in |Y|} \frac{1}{\det(I - \psi_\ell(\mathrm{Frob}_y)T^{\deg(y)})} \in 1 + T\mathbb{Z}_\ell[[T]], \tag{18}$$

where Frob_y denotes the Frobenius conjugacy class of $\pi_1(Y)$ at y. If $\psi_\ell = 1$ is the trivial representation, then $L(\psi_\ell, T) = Z(Y, T)$ is rational. More generally, if ψ_ℓ is of finite order, then $L(\psi_\ell, T)$ is also rational. This follows from the following general result.

Theorem 7.1 *(Grothendieck[11]). Let ψ_ℓ be a continuous ℓ-adic representation of $\pi_1(Y)$ as above. If $\ell \neq p$, then $L(\psi_\ell, T)$ is a rational function over $\mathbb{Q}_\ell$.*

In the remaining case that $\ell = p$, the situation is more complicated and the L-function is not rational in general. Katz [12] conjectured that $L(\psi_p, T)$ is p-adic meromorphic everywhere. This turned out to be false in general [18]. Dwork's original conjecture [9] says that the L-function $L(\psi_p, T)$ is p-adic meromorphic if the p-adic representation ψ_p is geometric in some sense (coming from the relative p-adic ètale cohomology of a family of smooth proper varieties for instance). This conjecture of Dwork has been proved to be true. Thus, we have

Theorem 7.2 *(Wan [21][22]). Let ψ_p be a continuous p-adic representation of $\pi_1(Y)$. If ψ_p is geometric in the sense of [21], then $L(\psi_p, T)$ is p-adic meromorphic.*

If $\ell \neq p$, the geometric Langlands conjecture as proved by Lafforgue [14] shows that up to a constant twist, every irreducible ℓ-adic representation of $\pi_1(Y)$ comes from algebraic geometry. On the other hand, if $\ell = p$, there are a lot more non-geometric highly transcendental p-adic representations because of very wild ramifications. It is not clear how to characterize the geometric p-adic representations of $\pi_1(Y)$. This can be viewed as some sort of p-adic geometric Langlands conjecture, which has not been formulated yet.

From now on, we assume that ψ_ℓ is a geometric ℓ-adic representation, where ℓ may be equal to p. From the above results, we know that the L-function $L(\psi_\ell, T)$ is ℓ-adic meromorphic everywhere (in fact, rational if $\ell \neq p$). For an integer k, we can define the k-th moment L-function of ψ_ℓ by

$$L(\psi_\ell^k, T) = \prod_{y \in |Y|} \frac{1}{\det(I - \psi_\ell(\mathrm{Frob}_y)^k T^{\deg(y)})} \in 1 + T\mathbb{Z}_\ell[[T]]. \tag{19}$$

This L-function is the k-th moment of the Euler factors of $L(\psi_\ell, T)$. It is also the L-function of the k-th Adams operation ψ_ℓ^k, which is a virtual ℓ-adic representation. Similarly, we have

Theorem 7.3 *(Wan [22][24]). Let ψ_ℓ be a geometric ℓ-adic representation of $\pi_1(Y)$. Then, for each integer k, $L(\psi_\ell^k, T)$ is ℓ-adic meromorphic everywhere (rational if $\ell \neq p$).*

A further question is to understand the ℓ-adic variation of $L(\psi_\ell^k, T)$ as the integer k varies ℓ-adically. Just as in the zeta function case, this leads to a suitable limiting moment L-function.

Let $D = D_{\psi_\ell}$ be the order of the image of the residue repsentation $\bar{\psi}_\ell$. Let k_1 and k_2 be integers. As in the previous section, it is easy to see that the power series $L(\psi_\ell^{k_1+Dk_2}, T)$ is ℓ-adically continuous in k_2. This continuity shows that $L(\psi_\ell^{k_1+Dk_2}, T)$ is a well defined ℓ-adic power series in $\mathbb{Z}_\ell[[T]]$ for all $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}_\ell$. This power series is clearly ℓ-adic analytic in the open unit disk $|T|_\ell < 1$. It can be viewed as the ℓ-adic L-function attached to the representation ψ_ℓ. We can ask if this limiting moment L-function is ℓ-adic meromorphic on the closed unit disk or even everywhere. This is unknown if the rank of ψ_ℓ is greater than 1. But, if the rank of ψ_ℓ is 1, it is true.

Theorem 7.4 *(Wan [22]) Let ψ_ℓ be a rank one geometric ℓ-adic representation of $\pi_1(Y)$, where ℓ may be equal to p. For all integers k_1 and ℓ-adic integers $k_2 \in \mathbb{Z}_\ell$, the limiting moment L-function $L(\psi_\ell^{k_1+Dk_2}, T)$ is ℓ-adic meromorphic everywhere (rational if $\ell \neq p$).*

This result is easy if $\ell \neq p$ since ψ_ℓ is then geometrically of finite order by geometric class field theory. If $\ell = p$, the result is non-trivial and follows from our work on the rank one case [22] of Dwork's conjecture. For higher rank geometric representations, just as in the previous section on limiting moment zeta functions, $L(\psi_\ell^{k_1+Dk_2}, T)$ is known to be ℓ-adic meromorphic if either k_2 is a integer (Wan [21]) or $k_1 + Dk_2$ is the formal symbol $d\ell^\infty$ (Lenstra-Wan [15]).

In the case that $\ell = p$, the above rank one theorem and its proof [22] can be pushed further as first observed by Coleman in his private notes on the author's proof of the Dwork conjecture. The full form of this refinement, which we now describe, has been carried out by Grosse-Klönne [10].

Let ψ_p be a continuous rank one geometric p-adic representation

$$\psi_p : \pi_1(Y) \longrightarrow \mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^*.$$

Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let W_p be the set of continuous p-adic homomorphisms from $\mathbb{Z}_p^*$ to $\mathbb{C}_p^*$. It is called the weight space. It is a rigid analytic space. It can be identified with a finite disjoint union of open unit disks. For $\chi \in W_p$, the composition $\chi \circ \psi_p$ is a continuous group homomorphism from $\pi_1(Y)$ to $\mathbb{C}_p^*$. For example, for each $k \in \mathbb{Z}$, the map

$a \to a^k$ is a continuous group homomorphism from $\mathbb{Z}_p^*$ to $\mathbb{C}_p^*$. One can define an L-function as above by

$$L(\chi \circ \psi_p, T) = \prod_{y \in |Y|} \frac{1}{1 - (\chi \circ \psi_p)(\mathrm{Frob}_y) T^{\deg(y)}} \in 1 + T\mathbb{C}_p[[T]]. \tag{20}$$

In this way, $L(\chi \circ \psi_p, T)$ becomes a function in the two variables (χ, T). In the special case that χ is the k-th power map, $L(\chi \circ \psi_p, T)$ becomes the k-th moment L-function $L(\psi_p^k, T)$, which is just $L(\psi_p^{\otimes k}, T)$ since ψ_p has rank one.

Theorem 7.5 *(Grosse-Klönne [10]) Let ψ_p be a rank one geometric p-adic representation of $\pi_1(Y)$. Then, $L(\chi \circ \psi_p, T)$ is a two variable meromorphic function in the domain $(\chi, T) \in W_p \times \mathbb{C}_p$.*

In the special case that ψ_p comes from the universal family of elliptic curves over $\mathbb{F}_p$, this result was proved by Coleman [3] and the zero locus of the rigid meromorphic function $L(\chi \circ \psi_p, T)$ is closely related to the eigencurve [4]. The special value $L(\chi \circ \psi_p, 1)$, which is a meromorphic function in $\chi \in W_p$, is related to Iwasawa theory and p-adic L-functions.

8 Moment zeta functions: arithmetic case

In the arithmetic (characteristic zero) case, we can similarly define a k-th moment zeta function. We now briefly describe this. Let X be a scheme of finite type over $\mathbb{Z}$ as in section 3. We assume that the characteristic is zero. The Hasse-Weil zeta function attempts to understand the arithmetic variation of the family $Z(X \otimes \mathbb{F}_p, p^{-s})$ as the prime p varies. It corresponds to the first moment zeta function. More generally, for each positive integer k, we define the k-th moment Hasse-Weil zeta function by

$$\zeta(k, X, s) = \prod_{p \text{ prime}} Z(X \otimes \mathbb{F}_{p^k}, p^{-s}). \tag{21}$$

In the case that $k = 1$, this reduces to the original Hasse-Weil zeta function. For example, one checks that

$$\zeta(k, \mathbb{A}^n/\mathbb{Z}, s) = \zeta(\mathbb{Z}, s - nk).$$

Generalizing the conjectures in section 3, we have

Conjecture 8.1 *Let X be a scheme of finite type over $\mathbb{Z}$. For each positive integer k, the k-th moment Hasse-Weil zeta function $\zeta(k, X, s)$ extends to a meromorphic function in the whole complex plane $s \in \mathbb{C}$.*

The meaning of this conjecture is that there exists a good structural distribution law for the family $Z(X \otimes \mathbb{F}_p, p^{-s})$ as p varies.

To get further information about the conjectural distribution law, one needs to know the nature of the zeros and poles. This is given by the following further generalization of the Riemann Hypothesis.

Conjecture 8.2 *Let X be a scheme of finite type over $\mathbb{Z}$. For each positive integer k, if $\zeta(k, X, s) = 0$ or ∞ for some $s \in \mathbb{C}$, then* $\mathrm{Re}(s) \in \frac{1}{2}\mathbb{Z}$.

The meaning of this conjecture is that there is a sharp distribution law for the arithmetic family of meromorphic functions $Z(X \otimes \mathbb{F}_p, p^{-s})$ parametrized by p. By the previous theorems, the geometric analogue of the above two conjectures hold when we consider a geometric family $f : X \longrightarrow Y$ over a finite field instead of the arithmetic family $X \longrightarrow \mathrm{Spec}(\mathbb{Z})$.

Returning to the characteristic zero case. If $\dim(X) = 1$, it is easy to show that $\zeta(k, X, s)$ is meromorphic for all k by Hecke's theorem. If $\dim(X) = 2$, the meromorphic continuation of $\zeta(k, X, s)$ is unknown, even in the case that the generic fibre $X \otimes \mathbb{Q}$ is an elliptic curve without CM and $k \geq 10$. Assuming the meromorphic continuation of $\zeta(k, X, s)$, one can further ask the p-adic variation of $\zeta(k, X, s)$ (or its special value at $s = 1$ or at $s = 0$) as k varies p-adically. The resulting limiting moment zeta function (if it exists) for $X = Y \times \mathbb{A}^n$ $(n > 0)$ is presumably related to the conjectural construction of the p-adic zeta function of Y. This is easily seen to be the case for $Y = \mathrm{Spec}(\mathbb{Z})$, since for any positive integer k, we have the limiting formula

$$\zeta_p(\mathbb{Z}, 1 - 2k) = \lim_{m \to \infty} \zeta(k + (p-1)p^m, \mathbb{A}^2/\mathbb{Z}, 1),$$

where $\zeta_p(\mathbb{Z}, s)$ is the p-adic zeta function of $\mathbb{Z}$.

Instead of considering a single scheme X over $\mathbb{Z}$, we can also consider a morphism $f : X \longrightarrow Y$ of schemes of finite types over $\mathbb{Z}$. The k-th moment zeta function of the morphism f is defined to be

$$\zeta(k, f, s) = \prod_{y \in |Y|} Z(X_y \otimes \mathbb{F}_y(k), N(y)^{-s}),$$

where $\mathbb{F}_y(k)$ denotes the degree k extension field of the finite residue field $\mathbb{F}_y$ at y. The above two meromorphic continuation and GRH conjectures extend to $\zeta(k, f, s)$.

In the characteristic zero case, one can also define moment L-functions and possibly limiting moment L-functions. This would involve additional notions such as compatible systems and motives. We leave such a generalization to the reader.

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