Moment L-functions, Partial L-functions and Partial Exponential Sums

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0. Introduction

Let $\mathbf{F}_q$ be a finite field of characteristic $p > 0$ with q elements. Fix an algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. Throughout this paper, for any positive integer k, $\mathbf{F}_{q^k}$ denotes the subfield of $\mathbf{F}$ of degree k over $\mathbf{F}_q$. Unless otherwise stated, schemes, morphisms and sheaves defined on the base field $\mathbf{F}_q$ are denoted by letters with subscripts 0 and we indicate the base extension from $\mathbf{F}_q$ to $\mathbf{F}$ by dropping the subscripts 0. Schemes and morphisms are separated and of finite type.

Let $f_0 : X_0 \to Y_0$ be an $\mathbf{F}_q$-morphism of schemes over $\mathbf{F}_q$. Let d be a positive integer. For any positive integer k, let

$$N_k(f_0, d) = \#\{x \in X_0(\mathbf{F}_{q^{kd}}) | f_0(x) \in Y_0(\mathbf{F}_{q^k})\} = \sum_{y \in Y_0(\mathbf{F}_{q^k})} \# f_0^{-1}(y)(\mathbf{F}_{q^{kd}}).$$

It is called the d-th moment of the morphism $f_0 \otimes \mathbf{F}_{q^k}$. We define the *$d$-th moment zeta function* of the morphism f_0 to be the formal power series

$$Z(f_0, d, t) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k(f_0, d)}{k} t^k\right).$$

The sequence $Z(f_0, d, t)$ $(d = 1, 2, \cdots)$ measures the distribution of the closed points of X_0 along the fibres of f_0. For $d = 1$, $Z(f_0, 1, t) = Z(X_0, t)$ is simply the classical zeta function of the scheme X_0. Thus questions about the numbers $N_k(f_0, d)$ $(k = 1, 2, \cdots)$ translate into questions about the d-th moment zeta function $Z(f_0, d, t)$. The function $Z(f_0, d, t)$ is a rational function whose reciprocal zeros and reciprocal poles are Weil q-integers. Natural further questions are then about the weights and slopes of the zeros and poles, the total degree of $Z(f_0, d, t)$ and its variation as d varies. The p-adic limit of $Z(f_0, d, t)$ as d grows in certain p-adic direction leads to Dwork's unit root zeta function, which is a p-adic meromorphic function ([10], [11]). Thus, for finite d, the moment zeta function $Z(f_0, d, t)$ can be viewed as an algebraic approximation to Dwork's transcendental unit root zeta function, which is some sort of infinite moment zeta function. It is hoped that a good understanding on $Z(f_0, d, t)$ for finite d would lead to improved information on Dwork's unit root zeta function.

In the present paper, we study the moment zeta function $Z(f_0, d, t)$ for finite d, its dependence on d and its various generalization. The variation of $Z(f_0, d, t)$ as the new arithmetic parameter d varies provides a new dimension of arithmetical problems to study.

To give an example of our results, we consider the case of Artin-Schreier hypersurfaces. Let $g(x_1, \ldots, x_n, y_1, \ldots, y_{n'})$ be a polynomial with coefficients in $\mathbf{F}_q$, where $n, n' \geq 1$. Let X_0 be the hypersurface in $\mathbf{A}_0^{n+n'+1}$ defined by the equation

$$x_0^p - x_0 = g(x_1, \ldots, x_n, y_1, \ldots, y_{n'}),$$

and let $f_0 : X_0 \to \mathbf{A}_0^{n'}$ be the projection to the y-coordinates. Then the number $N_k(f_0, d)$ is the number of elements of the set

$$\{(x_0, \ldots, x_n, y_1, \ldots, y_{n'}) | x_0^p - x_0 = g(x_1, \ldots, x_n, y_1, \ldots, y_{n'}),\ x_i \in \mathbf{F}_{q^{dk}},\ y_j \in \mathbf{F}_{q^k}\}.$$

Heuristically (for suitable g), we expect

$$N_k(f_0, d) = q^{(dn+n')k} + O(q^{(dn+n')k/2}).$$

Deligne's estimate [2] on exponential sums implies the following result for $d = 1$.

Theorem 0.1. Given g as above, we write $g = g_m + g_{m-1} + \cdots + g_0$, where each g_i is homogeneous of degree i. Assume that the leading form g_m defines a smooth projective hypersurface in $\mathbf{P}^{n+n'-1}$, and assume that p does not divide m. Then for $d = 1$ and every positive integer k, we have the following inequality

$$|N_k(f_0, 1) - q^{(n+n')k}| \leq (p-1)(m-1)^{n+n'} q^{(n+n')k/2}.$$

What can be said about $d > 1$? To answer this question, we introduce the following terminology:

Definition 0.2. Let d be a positive integer and let g be a polynomial as above. We define the *d-th fibred sum of g* to be the following new polynomial

$$\oplus_y^d g = g(x_{11}, \ldots, x_{n1}, y_1, \ldots, y_{n'}) + \cdots + g(x_{1d}, \ldots, x_{nd}, y_1, \ldots, y_{n'}).$$

We have the following estimate on $N_k(f_0, d)$.

Theorem 0.3. Given g as above, we write $g = g_m + g_{m-1} + \cdots + g_0$, where each g_i is homogeneous of degree i. Assume that $\oplus_y^d g_m$ defines a smooth hypersurface in $\mathbf{P}^{dn+n'-1}$ and assume that p does not divide m. Then, for every positive integer k, we have the following inequality

$$|N_k(f_0, d) - q^{(dn+n')k}| \leq (p-1)(m-1)^{dn+n'} q^{(dn+n')k/2}.$$

For fixed d and large q^k, the above estimate is sharp in general. On the other hand, for fixed q^k and large d, the above estimate should be quite weak since the constant $(p-1)(m-1)^{nd+n'}$ on the right-hand side of the above inequality grows exponentially in d. Thus, an interesting problem is to obtain sharp estimate for large d as well. In this direction, we shall show that the above exponential constant $(p-1)(m-1)^{nd+n'}$ can be replaced by $c(p,g)d^{3(m+1)^n-1}$, which is a polynomial in d, for some constant $c(p,g)$ depending only on p and g. We do not have an explicit value for $c(p,g)$ yet.

Example 0.4. Consider the case that

$$g(x,y) = g_{1,m}(x_1, \ldots, x_n) + g_{2,m}(y_1, \ldots, y_{n'}) + g_{\leq m-1}(x,y),$$

where $g_{1,m}$ defines a smooth hypersurface in $\mathbf{P}^{n-1}$, $g_{2,m}$ defines a smooth hypersurface in $\mathbf{P}^{n'-1}$ and $g_{\leq m-1}$ is a polynomial of degree at most $m-1$. It is then straightforward to check that $\oplus_y^d g_m$ defines a smooth hypersurface in $\mathbf{P}^{dn+n'-1}$ if and only if d is not divisible by p. Since the condition for the fibred sum to define a smooth hypersurface is Zariski open, there exist many more examples of g to which Theorem 0.3 applies if d is not divisible by p.

The above moment zeta functions can be generalized to moment L-functions as follow. Throughout this paper, we fix a prime number l distinct from p. Let $f_0 : X_0 \to Y_0$ be an $\mathbf{F}_q$-morphism of schemes over $\mathbf{F}_q$. Let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}}_l$-sheaf on X_0, and let $F_X : X \to X$ and $F_X : F_X^*\mathcal{F} \to \mathcal{F}$ be the geometric Frobenius correspondences. Let d be a positive integer. For any positive integer k, set

$$S_k(f_0, \mathcal{F}_0, d) = \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\ f_0(x) \in Y_0(\mathbf{F}_{q^k})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}).$$

This is called the d-th moment of the morphism $f_0 \otimes \mathbf{F}_{q^k}$ associated to the sheaf $\mathcal{F}_0$. We define the *d-th moment L-function* to be

$$L(f_0, \mathcal{F}_0, d, t) = \exp\Big(\sum_{k=1}^{\infty} \frac{S_k(f_0, \mathcal{F}_0, d)}{k} t^k\Big).$$

More generally, for any object K in the triangulated category $D_c^b(X_0, \overline{\mathbf{Q}}_l)$ defined in [2] 1.1.2, we define the d-th moment L-function of K to be

$$L(f_0, K, d, t) = \prod_i L(f_0, \mathcal{H}^i(K), d, t)^{(-1)^i}.$$

Note that the d-th moment L-function of the trivial sheaf $\overline{\mathbf{Q}}_l$ is the d-th moment zeta function, and for $d = 1$, the moment L-function $L(f_0, \mathcal{F}_0, 1, t)$ coincides with the classical Grothendieck L-function $L(X_0, \mathcal{F}_0, t)$. Define the total degree of a rational function to be the sum of the number of zeros and the number of poles counted with multiplicities. We have the following result for moment L-functions:

Theorem 0.5. The moment L-function $L(X_0, \mathcal{F}_0, d, t)$ is a rational function whose total degree is bounded by the polynomial $c_1(f_0, \mathcal{F}_0)d^{c_2(f_0, \mathcal{F}_0)}$ for two positive constants $c_i(f_0, \mathcal{F}_0)$ $(i = 1, 2)$.

We can further generalize moment L-functions to the situation where more than one morphisms are involved, generalizing the partial zeta function in [12]. Let $d_1, \ldots, d_n$ be positive integers, and let d be a common multiple of them. Let $X_0, X_0^{(1)}, \ldots, X_0^{(n)}$ be schemes over $\mathbf{F}_q$ and let

$$f_i : X_0 \to X_0^{(i)} \ (i = 1, \ldots, n)$$

be $\mathbf{F}_q$-morphisms. For any positive integer k, let $N_k(X_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n)$ be the number of elements of the set

$$\{x \in X_0(\mathbf{F}_{q^{kd}}) | f_1(x) \in X_0^{(1)}(\mathbf{F}_{q^{kd_1}}), \ldots, f_n(x) \in X_0^{(n)}(\mathbf{F}_{q^{kd_n}})\}.$$

We define the *partial zeta function* to be

$$Z(X_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n, t) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k(X_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n)}{k} t^k\right).$$

Let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}}_l$-sheaf on X_0, and let $F_X : X \to X$ and $F_X : F_X^* \mathcal{F} \to \mathcal{F}$ be the geometric Frobenius correspondences. For any positive integer k, let

$$S_k(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n) = \sum_{x \in X_0(\mathbf{F}_{q^{kd}}), f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}).$$

This is some sort of partial character sums. We define the *partial L-function* of $\mathcal{F}_0$ to be

$$L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t) = \exp\left(\sum_{k=1}^{\infty} \frac{S_k(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n)}{k} t^k\right).$$

Theorem 0.6. The partial L-function $L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ is a rational function. If the sheaf $\mathcal{F}_0$ is mixed, then the reciprocal zeros and reciprocal poles of the partial L-function are Weil q-integers.

This extends the rationality result of the partial zeta function in [13]. The proof in [13] forces the partial zeta function to be rational without giving a cohomological formula which would naturally explain the rationality. Our new proof here gives such a natural cohomological formula for partial L-functions.

One of our motivations to introduce partial zeta functions and partial L-functions is to generalize moment zeta functions and moment L-functions. Another motivation (see [8] for a special case) is for possible applications in many other concrete problems in number theory, coding theory and combinatorics. It is well known that good estimates of various partial character sums play a vital role in analytic number theory. For arbitrary partial sums, the problem is too difficult. The partial sums in this paper are not arbitrary since we sum over a subset with some structure (such as subfields of a field), not over an arbitrary subset. Thus, one could expect good general result, as Theorem 0.6 confirms. In addition to its theoretical significance, we believe that a good understanding of the partial L-function would greatly increase the flexibility and applicability of the existing powerful tools in various applications.

Once we know the rationality of the partial L-function, one further basic question is about the dependence of its total degree on the parameter d. For moment L-function, Theorem 0.5

shows that the total degree can be bounded by a polynomial function in d. For general partial zeta functions and partial L-functions, we do not know if their total degrees can be bounded by a polynomial function in d. Another further basic question is about the weights of the zeros and poles of the partial L-function. This is apparently very complicated in general. We shall analyze interesting examples arising from certain partial exponential sums.

This paper is organized as follows: In §1, we give various formulas for the moment L-function and we estimate its total degree. In §2, we prove that the partial L-function is rational using a geometric construction of Faltings. In §3, we apply the general theory to study partial exponential sums and obtain result which implies Theorem 0.3 above.

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1. Moment L-functions

Let $f_0 : X_0 \to Y_0$ be an $\mathbf{F}_q$-morphism of schemes over $\mathbf{F}_q$. Let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}}_l$-sheaf on X_0, and let $F_X : X \to X$ and $F_X : F_X^*\mathcal{F} \to \mathcal{F}$ be the geometric Frobenius correspondences. Let d be a positive integer. For any positive integer k, recall that

$$S_k(f_0, \mathcal{F}_0, d) = \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\ f_0(x) \in Y_0(\mathbf{F}_{q^k})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}),$$

and the d-th moment L-function is defined to be

$$L(f_0, \mathcal{F}_0, d, t) = \exp\left(\sum_{k=1}^{\infty} \frac{S_k(f_0, \mathcal{F}_0, d)}{k} t^k\right).$$

For any finite dimensional $\overline{\mathbf{Q}}_l$-representation $G \to \mathrm{GL}(V)$ of a group G, define the d-th Adam operation $[V]^d$ of V to be the virtual representation given by

$$[V]^d = \sum_{j=1}^{d} (-1)^{j-1} j (\mathrm{Sym}^{d-j} V \otimes \wedge^j V),$$

where on the right-hand side, the sum is taken in the Grothendieck group for the category of finite dimensional $\overline{\mathbf{Q}}_l$-representations of G. For any $g \in G$, we have

$$\begin{aligned} \mathrm{Tr}(g, [V]^d) &= \mathrm{Tr}(g^d, V), \\ \det(1 - gt, [V]^d) &= \det(1 - g^d t, V). \end{aligned}$$

For any constructible $\overline{\mathbf{Q}}_l$-sheaf $\mathcal{F}_0$ on X_0, define the d-th Adam's operation $[\mathcal{F}_0]^d$ of $\mathcal{F}_0$ to be the virtual sheaf given by

$$[\mathcal{F}_0]^d = \sum_{j=1}^{d} (-1)^{j-1} j (\mathrm{Sym}^{d-j} \mathcal{F}_0 \otimes \wedge^j \mathcal{F}_0),$$

where on the right-hand side, the sum is taken in the Grothendieck group for the category of constructible $\overline{\mathbf{Q}}_l$-sheaves. Note that $\mathrm{Sym}^n \mathcal{F}_0$ (resp. $\wedge^n \mathcal{F}_0$) can be defined to be the direct factor of $\otimes^n \mathcal{F}_0$ using the projection $\frac{1}{n!} \sum_{\tau} \tau$ (resp. $\frac{1}{n!} \sum_{\tau} \mathrm{sgn}(\tau)\tau$), where the sum is taken over all permutations τ of $\{1, \ldots, n\}$ and each permutation τ acts on $\otimes^n \mathcal{F}_0$ by permuting the factors in the tensor product.

There are many different formulas for the d-th Adams operation in terms of various symmetric powers and exterior powers. The one that is typically used is Newton's formula expressing a power symmetric function in terms of elementary symmetric functions. Newton's formula has a lot of redundancy and is deficient for some applications. For instance, it contains a term which is the d-th tensor power. This term is too big for some applications such as in the proof of Dwork's conjecture. Here, it would give an exponential bound in Theorem 0.5 instead of the polynomial bound. The above formula we use for the d-th Adams operation has certain minimality property which is important for us here and which is necessary for the proof of Dwork's conjecture, see section 4 in [9] for an explanation. It is also the one used in Katz's paper [6].

Given a virtual sheaf $\mathcal{F}_0 = \mathcal{G}_0 - \mathcal{H}_0$, where $\mathcal{G}_0$ and $\mathcal{H}_0$ are (true) $\overline{\mathbf{Q}}_l$-sheaves, we define the cohomology of $\mathcal{F}_0$ to be the virtual representation

$$\sum_i (-1)^i H_c^i(X, \mathcal{G}) - \sum_i (-1)^i H_c^i(X, \mathcal{H})$$

of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, and we can talk about the trace and the characteristic polynomial of the geometric Frobenius correspondence F_X on this space. By abuse of notations, we write the above virtual $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$-representation as $\sum_i (-1)^i H_c^i(X, \mathcal{F})$ and write the trace and the characteristic polynomial of F_X on this space by $\sum_i (-1)^i \mathrm{Tr}(F_X, H_c^i(X, \mathcal{F}))$ and $\prod_i \det(1 - F_X t, H_c^i(X, \mathcal{F}))^{(-1)^i}$, respectively.

Theorem 1.1.

(i) We have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^j \mathrm{Tr}(F_Y^{kd}, (R^j f_! \mathcal{F})_{\bar{y}}) \\ &= \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^j \mathrm{Tr}(F_Y^k, ([R^j f_! \mathcal{F}]^d)_{\bar{y}}) \\ &= \sum_{i=0}^{2\mathrm{dim}Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-)^{i+j} \mathrm{Tr}(F_Y^k, H_c^i(Y, [R^j f_! \mathcal{F}]^d)), \end{aligned}$$

where rel.dim(f) is the relative dimension of the morphism f and F_Y is the geometric Frobenius correspondence.

(ii) We have

$$\begin{aligned} L(f_0, \mathcal{F}_0, d, t) &= \prod_{i=0}^{2\mathrm{dim}Y} \prod_{j=0}^{2\mathrm{rel.dim}(f)} \det(1 - F_Y t, H_c^i(Y, [R^j f_! \mathcal{F}]^d))^{(-1)^{i+j+1}} \\ &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} L(Y_0, [R^j f_{0!} \mathcal{F}_0]^d, t)^{(-1)^j} \\ &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{y \in |Y_0|} \det(1 - F_y^d t^{\deg(y)}, (R^j f_! \mathcal{F})_{\bar{y}})^{(-1)^{j+1}} \\ &= \prod_{y \in |Y_0|} L(f_0^{-1}(y) \otimes_{k(y)} k(y)_d, \mathcal{F}_0, t^{\deg(y)}), \end{aligned}$$

where $L(Y_0, [R^j f_{0!} \mathcal{F}_0]^d, t)$ is the Grothendieck L-function for the virtual sheaf $[R^j f_{0!} \mathcal{F}_0]^d$ on Y_0, $|Y_0|$ is the set of Zariski closed points in Y_0, and for any $y \in |Y_0|$, F_y is the geometric Frobenius at y, $k(y)_d$ is the field extension of the residue field $k(y)$ of degree d, and $L(f_0^{-1}(y) \otimes_{k(y)} k(y)_d, \mathcal{F}_0, t)$ is the Grothendieck L-function for the scheme $f_0^{-1}(y) \otimes_{k(y)} k(y)_d = X_0 \otimes_{Y_0} k(y)_d$ over $k(y)_d$ and the restriction of the sheaf $\mathcal{F}_0$ on this scheme. In particular, $L(f_0, \mathcal{F}_0, d, t)$ is rational.

Proof. (i) We have

$$S_k(f_0, \mathcal{F}_0, d) = \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{x \in f_0^{-1}(y)(\mathbf{F}_{q^{kd}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}).$$

By the Grothendieck trace formula, for each $y \in Y_0(\mathbf{F}_{q^k})$, we have

$$\sum_{x \in f_0^{-1}(y)(\mathbf{F}_{q^{kd}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) = \sum_{j=0}^{2\dim(f^{-1}(\bar{y}))} (-1)^j \mathrm{Tr}(F_Y^{kd}, (R^j f_! \mathcal{F})_{\bar{y}}).$$

So we have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^j \mathrm{Tr}(F_Y^{kd}, (R^j f_! \mathcal{F})_{\bar{y}}) \\ &= \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^j \mathrm{Tr}(F_Y^{k}, ([R^j f_! \mathcal{F}]^d)_{\bar{y}}). \end{aligned}$$

Again by the Grothendieck trace formula, we have

$$\sum_{y \in Y_0(\mathbf{F}_{q^k})} \mathrm{Tr}(F_Y^{k}, ([R^j f_! \mathcal{F}]^d)_{\bar{y}}) = \sum_{i=0}^{2\dim Y} (-1)^i \mathrm{Tr}(F_Y^k, H_c^i(Y, [R^j f_! \mathcal{F}]^d)).$$

So we have

$$S_k(f_0, \mathcal{F}_0, d) = \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^{i+j} \mathrm{Tr}(F_Y^k, H_c^i(Y, [R^j f_! \mathcal{F}]^d)).$$

(ii) By the definition of moment L-functions, we have

$$t\frac{d}{dt} \ln L(f_0, \mathcal{F}_0, d, t) = \sum_{k=1}^{\infty} S_k(f_0, \mathcal{F}_0, d) t^k.$$

By the last equality of (i), we have

$$\begin{aligned} \sum_{k=1}^{\infty} S_k(f_0, \mathcal{F}_0, d) t^k &= \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} \sum_{k=1}^{\infty} (-1)^{i+j} \mathrm{Tr}(F_Y^k, H_c^i(Y, [R^j f_! \mathcal{F}]^d)) t^k \\ &= \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^{i+j+1} t\frac{d}{dt} \ln \det(1 - F_Y t, H_c^i(Y, [R^j f_! \mathcal{F}]^d)). \end{aligned}$$

So we have

$$t\frac{d}{dt} \ln L(f_0, \mathcal{F}_0, d, t) = \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^{i+j+1} t\frac{d}{dt} \ln \det(1 - F_Y t, H_c^i(Y, [R^j f_! \mathcal{F}]^d)).$$

So the first equality of (ii) holds. The second equality follows from the Grothendieck formula for L-functions:

$$L(Y_0, [R^j f_{0!} \mathcal{F}_0]^d, t) = \prod_{i=0}^{2\dim(Y)} \det(1 - Ft, H_c^i(Y, [R^j f_! \mathcal{F}]^d))^{(-1)^{i+1}}.$$

The third equality follows from the definition of the Grothendieck L-functions:

$$\begin{aligned} L(Y_0, [R^j f_{0!}\mathcal{F}_0]^d, t) &= \prod_{y\in|Y_0|} \det(1 - F_y t^{\deg(y)}, ([R^j f_! \mathcal{F}]^d)_{\bar{y}})^{-1} \\ &= \prod_{y\in|Y_0|} \det(1 - F_y^d t^{\deg(y)}, (R^j f_! \mathcal{F})_{\bar{y}})^{-1}. \end{aligned}$$

The fourth equality again follows from the Grothendieck formula for L-functions:

$$L(f_0^{-1}(y) \otimes_{k(y)} k(y)_d, \mathcal{F}_0, t) = \prod_{j=0}^{2\dim(f^{-1}(\bar{y}))} \det(1 - F_y^d t, (R^j f_! \mathcal{F})_{\bar{y}})^{(-1)^{j+1}}.$$

Lemma 1.2. Let

$$V_d \xrightarrow{\phi_d} V_{d-1} \xrightarrow{\phi_{d-1}} V_{d-2} \xrightarrow{\phi_{d-2}} \cdots \xrightarrow{\phi_2} V_1 \xrightarrow{\phi_1} V_d$$

be a sequence of isomorphisms of vector spaces and let

$$\sigma : V_d \otimes V_1 \otimes \cdots \otimes V_{d-1} \to V_1 \otimes \cdots \otimes V_d$$

be the homomorphism defined by

$$\sigma(s_d \otimes s_1 \otimes \cdots \otimes s_{d-1}) = s_1 \otimes \cdots \otimes s_d.$$

Then we have

$$\mathrm{Tr}(\phi_1 \cdots \phi_d, V_d) = \mathrm{Tr}((\phi_1 \otimes \cdots \otimes \phi_d) \circ \sigma, V_d \otimes V_1 \otimes \cdots \otimes V_{d-1}).$$

Proof. Fix a basis $\{e_1, \ldots, e_m\}$ for V_d. Then the set $\{\phi_d(e_1), \ldots, \phi_d(e_m)\}$ is a basis for V_{d-1}, the set $\{\phi_{d-1}\phi_d(e_1), \ldots, \phi_{d-1}\phi_d(e_m)\}$ is a basis for V_{d-2}, ..., and the set $\{\phi_2 \cdots \phi_d(e_1), \ldots, \phi_2 \cdots \phi_d(e_m)\}$ is a basis for V_1. The homomorphism

$$(\phi_1 \otimes \cdots \otimes \phi_d) \circ \sigma : V_d \otimes V_1 \otimes \cdots \otimes V_{d-1} \to V_d \otimes V_1 \otimes \cdots \otimes V_{d-1}$$

can be described by

$$\begin{aligned} & e_{i_1} \otimes (\phi_2 \cdots \phi_d(e_{i_2})) \otimes (\phi_3 \cdots \phi_d(e_{i_3})) \otimes \cdots \otimes \phi_d(e_{i_d}) \\ \mapsto \quad & (\phi_1 \cdots \phi_d(e_{i_2})) \otimes (\phi_2 \cdots \phi_d(e_{i_3})) \otimes (\phi_3 \cdots \phi_d(e_{i_4})) \otimes \cdots \otimes \phi_d(e_{i_1}). \end{aligned}$$

Assume $\phi_1 \cdots \phi_d(e_i) = \sum\limits_{i'=1}^{m} a_{ii'} e_{i'}$. Then $(\phi_1 \otimes \cdots \otimes \phi_d) \circ \sigma$ can be described by

$$\begin{aligned} & e_{i_1} \otimes (\phi_2 \cdots \phi_d(e_{i_2})) \otimes (\phi_3 \cdots \phi_d(e_{i_3})) \otimes \cdots \otimes \phi_d(e_{i_d}) \\ \mapsto & \sum_{i'=1}^{m} a_{i_2 i'} e_{i'} \otimes (\phi_2 \cdots \phi_d(e_{i_3})) \otimes (\phi_3 \cdots \phi_d(e_{i_4})) \otimes \cdots \otimes \phi_d(e_{i_1}). \end{aligned}$$

From this description, one get

$$\mathrm{Tr}((\phi_1 \otimes \cdots \otimes \phi_d) \circ \sigma, V_d \otimes V_1 \otimes \cdots \otimes V_{d-1}) = \sum_{i=1}^{m} a_{ii} = \mathrm{Tr}(\phi_1 \cdots \phi_d, V_d).$$

Let $Z_0 = X_0 \times_{Y_0} \times \cdots \times_{Y_0} X_0$ be the d-fold fibred product of X_0 over Y_0 and let $\mathcal{G}_0$ be the sheaf on Z_0 defined by

$$\mathcal{G}_0 = p_1^* \mathcal{F}_0 \otimes \cdots \otimes p_d^* \mathcal{F}_0,$$

where $p_i : X_0 \to Y_0$ $(i = 1, \ldots, d)$ are the projections. Let $\sigma_0 : Z_0 \to Z_0$ be the automorphism defined by the shifting

$$\begin{aligned} \sigma_0 : X_0 \times_{Y_0} \cdots \times_{Y_0} X_0 &\to X_0 \times_{Y_0} \cdots \times_{Y_0} X_0, \\ (x_1, \ldots, x_d) &\mapsto (x_d, x_1, \ldots, x_{d-1}), \end{aligned}$$

and let $\sigma_0 : \sigma_0^* \mathcal{G}_0 \to \mathcal{G}_0$ be the morphism of sheaves defined by the shifting

$$\begin{aligned} \sigma_0 : p_d^* \mathcal{F}_0 \otimes p_1^* \mathcal{F}_0 \otimes \cdots \otimes p_{d-1}^* \mathcal{F}_0 &\to p_1^* \mathcal{F}_0 \otimes \cdots \otimes p_d^* \mathcal{F}_0, \\ s_d \otimes s_1 \otimes \cdots \otimes s_{d-1} &\mapsto s_1 \otimes \cdots \otimes s_d. \end{aligned}$$

Lemma 1.3. We have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\, f_0(x) \in Y_0(\mathbf{F}_{q^k})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) \\ &= \sum_{z \in Z(\mathbf{F}),\, F_Z^k \sigma(z) = z} \mathrm{Tr}(F_Z^k \sigma, \mathcal{G}_{\bar{z}}). \end{aligned}$$

Proof. For any point $x \in X_0(\mathbf{F}_{q^{kd}})$ with the property $f_0(x) \in Y_0(\mathbf{F}_{q^k})$, the point

$$z = (x, F_X^k(x), \cdots, F_X^{k(d-1)}(x))$$

is a point in $Z = X \times_Y \cdots \times_Y X$ and it has the property that $F_Z^k\sigma(z) = z$. Conversely let $z = (x_1, \ldots, x_d)$ be a fixed point $F_Z^k\sigma$ in $Z = X \times_Y \cdots \times_Y X$, where $x_i \in X(\mathbf{F})$ $(i = 1, \ldots, d)$. Then we have

$$(F_X^k(x_d), F_X^k(x_1), \ldots, F_X^k(x_{d-1})) = (x_1, \ldots, x_d),$$

that is,

$$F_X^k(x_1) = x_2, \ldots, F_X^k(x_{d-1}) = x_d, F_X^k(x_d) = x_1.$$

This implies that $F_X^{kd}(x_1) = x_1$, that is, $x_1 \in X_0(\mathbf{F}_{q^{kd}})$. On the other hand, since z is a point in the fibred product of X_0 over Y_0, we have $f(x_1) = \cdots = f(x_d)$. Set $y = f(x_i)$. Applying f to the equation $F_X^k(x_1) = x_2$, we get $F_X^k(y) = y$, that is, $y \in Y_0(\mathbf{F}_{q^k})$. Therefore x_1 is a point in $X_0(\mathbf{F}_{q^{kd}})$ with the property $f_0(x_1) \in Y_0(\mathbf{F}_{q^k})$. This shows that

$$x \mapsto (x, F_X^k(x), \cdots, F_X^{k(d-1)}(x))$$

defines a one-to-one correspondence between the set of points $x \in X_0(\mathbf{F}_{q^{kd}})$ with the property $f_0(x) \in Y_0(\mathbf{F}_{q^k})$ and the set of fixed points of $F_Z^k\sigma$.

Let x be a point in $X_0(\mathbf{F}_{q^{kd}})$ with the property that $f_0(x) \in Y_0(\mathbf{F}_{q^k})$ and let

$$z = (x, F_X^k(x), \ldots, F_X^{k(d-1)}(x))$$

be the corresponding fixed point of $F_Z^k\sigma$ in Z. Note that the linear map $F_X^{kd} : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ is the composition

$$\mathcal{F}_{\bar{x}} = \mathcal{F}_{F_X^{kd}(\bar{x})} \stackrel{F_X^k}{\to} \mathcal{F}_{F_X^{k(d-1)}(\bar{x})} \stackrel{F_X^k}{\to} \cdots \stackrel{F_X^k}{\to} \mathcal{F}_{F_X^k(\bar{x})} \stackrel{F_X^k}{\to} \mathcal{F}_{\bar{x}}.$$

Using Lemma 1.2, one can show

$$\mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) = \mathrm{Tr}(F_Z^k\sigma, \mathcal{G}_{\bar{z}}).$$

Lemma 1.3 follows.

Recall the following well-known trace formula:

Lemma 1.4. Let Z_0 be a scheme over $\mathbf{F}_q$, let $\sigma_0 : Z_0 \to Z_0$ be an $\mathbf{F}_q$-morphism of finite order, let $\mathcal{G}_0$ be a sheaf on Z_0, and let $\sigma_0 : \sigma_0^*\mathcal{G}_0 \to \mathcal{G}_0$ be a morphism of sheaves over σ_0. For any positive integer k, we have

$$\sum_{z \in Z(\mathbf{F}), F^k\sigma(z) = z} \mathrm{Tr}(F_Z^k\sigma, \mathcal{G}_{\bar{z}}) = \sum_{i=0}^{2\dim Z} (-1)^i \mathrm{Tr}(F_Z^k\sigma, H_c^i(Z, \mathcal{G})).$$

From Lemmas 1.3 and 1.4, we get the following:

Corollary 1.5. Notation as above. We have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\ f(x) \in Y_0(\mathbf{F}_{q^{kd}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) \\ &= \sum_{i=0}^{2\dim Z} (-1)^i \mathrm{Tr}(F_Z^k \sigma, H_c^i(Z, \mathcal{G})). \end{aligned}$$

Theorem 1.6. Notation as above. For each d-th root of unity μ, let $H_c^i(Z, \mathcal{G})_\mu$ be the eigenvector space of σ acting on $H_c^i(Z, \mathcal{G})$ with eigenvalue μ. Then we have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{i=0}^{2\dim Z} \sum_{\mu^d=1} (-1)^i \mu \mathrm{Tr}(F_Z^k, H_c^i(Z, \mathcal{G})_\mu), \\ L(f_0, \mathcal{F}_0, d, t) &= \prod_{i=0}^{2\dim Z} \prod_{\mu^d=1} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1}\mu}. \end{aligned}$$

Proof. Obviously σ has order d. So all the eigenvalues of σ on $H_c^i(Z, \mathcal{G})$ are d-th roots of unity, σ acts on each $H_c^i(Z, \mathcal{G})_\mu$ by scalar multiplication by μ, and

$$H_c^i(Z, \mathcal{G}) = \bigoplus_{\mu^d=1} H_c^i(Z, \mathcal{G})_\mu.$$

Since σ is defined over $\mathbf{F}_q$, F_Z commutes with σ. Hence each $H_c^i(Z, \mathcal{G})_\mu$ is invariant under the action of F_Z. Let $\lambda_{i\mu 1}, \ldots, \lambda_{i\mu k_{i\mu}}$ be all the eigenvalues of F_Z on $H_c^i(Z, \mathcal{G})_\mu$, where $k_{i\mu} = \dim H_c^i(Z, \mathcal{G})_\mu$. We have

$$\mathrm{Tr}(F_Z^k \sigma, H_c^i(Z, \mathcal{G})) = \sum_{\mu^d=1} \sum_{j=1}^{k_{i\mu}} \lambda_{i\mu j}^k \mu.$$

So by Corollary 1.5, we have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{i=0}^{2\dim Z} \sum_{\mu^d=1} \sum_{j=1}^{k_{i\mu}} (-1)^i \lambda_{i\mu j}^k \mu \\ &= \sum_{i=0}^{2\dim Z} \sum_{\mu^d=1} (-1)^i \mu \mathrm{Tr}(F_Z^k, H_c^i(Z, \mathcal{G})_\mu). \end{aligned}$$

By the definition of the moment L-function, we have

$$t\frac{d}{dt} \ln L(f_0, \mathcal{F}_0, d, t) = \sum_{k=1}^{\infty} S_k(f_0, \mathcal{F}_0, d) t^k.$$

So we have

$$\begin{aligned} t\frac{d}{dt}\ln L(f_0,\mathcal{F}_0,d,t) &= \sum_{k=1}^{\infty}\sum_{i=0}^{2\dim Z}\sum_{\mu^d=1}\sum_{j=1}^{k_{i\mu}}(-1)^i\lambda_{i\mu j}^k\mu t^k \\ &= \sum_{i=0}^{2\dim Z}\sum_{\mu^d=1}\sum_{j=1}^{k_{i\mu}}(-1)^{i+1}\mu t\frac{d}{dt}\ln(1-\lambda_{i\mu j}t) \\ &= \sum_{i=0}^{2\dim Z}\sum_{\mu^d=1}(-1)^{i+1}\mu t\frac{d}{dt}\ln\det(1-F_Z t, H^i(Z,\mathcal{G})_\mu), \end{aligned}$$

that is,

$$t\frac{d}{dt}\ln L(f_0,\mathcal{F}_0,d,t) = \sum_{i=0}^{2\dim Z}\sum_{\mu^d=1}(-1)^{i+1}\mu t\frac{d}{dt}\ln\det(1-F_Z t, H^i(Z,\mathcal{G})_\mu).$$

Our assertion follows.

For any finite dimensional representation $G \to \mathrm{GL}(V)$ of a group G, consider the d-fold tensor product representation $V^{\otimes d}$. The shifting operator

$$\sigma : V\otimes\cdots\otimes V \to V\otimes\cdots\otimes V,\ s_d\otimes s_1\otimes\cdots\otimes s_{d-1}\to s_1\otimes\cdots\otimes s_d$$

is G-equivariant. For any $g\in G$, denote the action of g on $V^{\otimes d}$ also by g. Applying Lemma 1.2 to the sequence of homomorphism

$$V\stackrel{g}{\to}V\stackrel{g}{\to}\cdots\stackrel{g}{\to}V,$$

we get

$$\mathrm{Tr}(g^d,V) = \mathrm{Tr}(g\circ\sigma, V^{\otimes d}).$$

For any d-th root of unity μ, let $V_\mu^{\otimes d}$ be the eigenvector space of σ on $V^{\otimes d}$ corresponding to the eigenvalue μ. Then $V_\mu^{\otimes d}$ is invariant under G, σ acts on $V_\mu^{\otimes d}$ by multiplication by μ, and

$$V^{\otimes d} = \bigoplus_{\mu^d=1} V_\mu^{\otimes d}.$$

So we have

$$\mathrm{Tr}(g\circ\sigma, V^{\otimes d}) = \sum_{\mu^d=1}\mu\mathrm{Tr}(g, V_\mu^{\otimes d}).$$

Hence

$$\mathrm{Tr}(g^d,V) = \sum_{\mu^d=1}\mu\mathrm{Tr}(g, V_\mu^{\otimes d}).$$

Similarly, for any constructible $\mathbf{Q}_l$-sheaf $\mathcal{G}_0$ on Y_0, we have a shifting operator σ on $\mathcal{G}_0^{\otimes d}$. For any d-th root of unity, define $(\mathcal{G}_0^{\otimes d})_\mu$ to the eigensheaf of σ corresponding to the eigenvalue μ.

Theorem 1.7.

(i) We have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{j=0}^{2\mathrm{rel.dim}(f)} \sum_{\mu^d=1} \sum_{y \in Y_0(\mathbf{F}_{q^k})} (-1)^j \mu \mathrm{Tr}(F_Y^k, (R^j f_! \mathcal{F})^{\otimes d}_{\mu, \bar{y}}) \\ &= \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\mathrm{rel.dim}(f)} \sum_{\mu^d=1} (-1)^{i+j} \mu \mathrm{Tr}(F_Y^k, H_c^i(Y, (R^j f_! \mathcal{F})^{\otimes d}_\mu)). \end{aligned}$$

(ii) We have

$$\begin{aligned} L(f_0, \mathcal{F}_0, d, t) &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{\mu^d=1} L(Y_0, (R^j f_! \mathcal{F}_0)^{\otimes d}_\mu, t)^{(-1)^j \mu} \\ &= \prod_{i=0}^{2\dim Y} \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{\mu^d=1} \det(1 - F_Y t, H_c^i(Y, (R^j f_! \mathcal{F})^{\otimes d}_\mu))^{(-1)^{i+j+1} \mu}. \end{aligned}$$

Proof. By Theorem 1.1 (i) and the discussion above, we have

$$\begin{aligned} S_k(f_0, \mathcal{F}_0, d) &= \sum_{y \in Y_0(\mathbf{F}_{q^k})} \sum_{j=0}^{2\mathrm{rel.dim}(f)} (-1)^j \mathrm{Tr}(F_Y^{kd}, (R^j f_! \mathcal{F})_{\bar{y}}) \\ &= \sum_{j=0}^{2\mathrm{rel.dim}(f)} \sum_{\mu^d=1} \sum_{y \in Y_0(\mathbf{F}_{q^k})} (-1)^j \mu \mathrm{Tr}(F_Y^k, (R^j f_! \mathcal{F})^{\otimes d}_{\mu, \bar{y}}) \\ &= \sum_{j=0}^{2\mathrm{rel.dim}(f)} \sum_{\mu^d=1} \sum_{i=0}^{2\dim Y} (-1)^{i+j} \mu \mathrm{Tr}(F_Y^k, H_c^i(Y, (R^j f_! \mathcal{F})^{\otimes d}_\mu)). \end{aligned}$$

So

$$\begin{aligned} L(f_0, \mathcal{F}_0, d, t) &= \exp\left(\sum_{k=1}^{\infty} \frac{S_k(f_0, \mathcal{F}_0, d)}{k} t^k \right) \\ &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{\mu^d=1} \left(\sum_{k=1}^{\infty} \frac{\sum_{y \in Y_0(\mathbf{F}_{q^k})} \mathrm{Tr}(F_Y^k, (R^j f_! \mathcal{F})^{\otimes d}_{\mu, \bar{y}})}{k} t^k \right)^{(-1)^j \mu} \\ &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{\mu^d=1} L(Y_0, (R^j f_! \mathcal{F})^{\otimes d}_\mu, t)^{(-1)^j \mu} \\ &= \prod_{j=0}^{2\mathrm{rel.dim}(f)} \prod_{\mu^d=1} \prod_{i=0}^{2\dim Y} \det(1 - F_Y t, H_c^i(Y, (R^j f_! \mathcal{F})^{\otimes d}_\mu))^{(-1)^{i+j+1} \mu}. \end{aligned}$$

Define the total degree of a rational function to be the sum of the number of its zeros and the number of its poles counted with multiplicities. In the following, we estimate the total degree of $L(f_0, \mathcal{F}_0, d, t)$. We need the following lemma, which was already proved in Katz-Sarnak [7] for smooth affine X and lisse sheaf $\mathcal{F}$. The general case can be reduced to that case by decomposing X into a disjoint union of smooth affine subschemes of X so that the restriction of $\mathcal{F}$ to each piece is lisse. For the reader's convenience, we include a related proof.

Lemma 1.8. Let k be a separably closed field and let X be a scheme of finite type over k. Suppose l is a prime number distinct from the characteristic of k. Let E be a finite extension of $\mathbf{Q}_l$, let R_E be the integral closure of $\mathbf{Z}_l$ in E, and let κ_E be the residue field of R_E.

(i) For any torsion free constructible R_E-sheaf $\mathcal{F}$ on X, we have

$$\begin{aligned} \dim_E(H_c^i(X, \mathcal{F}) \otimes_{R_E} E) &\leq \dim_{\kappa_E} H_c^i(X, \mathcal{F} \otimes_{R_E} \kappa_E), \\ \chi_c(X, \mathcal{F} \otimes_{R_E} E) &= \chi_c(X, \mathcal{F} \otimes_{R_E} \kappa_E), \end{aligned}$$

where

$$\begin{aligned} \chi_c(X, \mathcal{F} \otimes_{R_E} E) &= \sum_{i=0}^{2\dim X} (-1)^i \dim_E(H_c^i(X, \mathcal{F}) \otimes_{R_E} E), \\ \chi_c(X, \mathcal{F} \otimes_{R_E} \kappa_E) &= \sum_{i=0}^{2\dim X} (-1)^i \dim_{\kappa_E} H_c^i(X, \mathcal{F} \otimes_{R_E} \kappa_E) \end{aligned}$$

are the Euler characteristics.

(ii) Let $X' \to X$ be a finite Galois étale covering. Then there exists a constant C depending only on $X' \to X$ such that for any i and for any locally constant constructible κ_E-sheaf $\mathcal{F}$ whose restriction to X' is a constant sheaf, we have

$$\dim_{\kappa_E} H_c^i(X, \mathcal{F}) \leq C \cdot \mathrm{rank}(\mathcal{F}).$$

Proof. (i) Let π be a generator of the maximal ideal of R_E. Assume $\mathcal{F}$ corresponds to the projective system $(\mathcal{F}_n)_{n\in\mathbf{N}}$, where for each n, $\mathcal{F}_n$ is a sheaf of (R_E/π^n)-modules and the projection $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism $\mathcal{F}_{n+1}/\pi^n\mathcal{F}_{n+1} \cong \mathcal{F}_n$. Since $\mathcal{F}$ has no torsion, for each n, we have an exact sequence

$$0 \to \mathcal{F}_n \xrightarrow{\pi} \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0.$$

It gives rise to the following long exact sequence of cohomology groups:

$$\cdots \to H_c^i(X, \mathcal{F}_n) \xrightarrow{\pi} H_c^i(X, \mathcal{F}_{n+1}) \to H_c^i(X, \mathcal{F}_1) \to \cdots.$$

Taking projective limits, we get the following long exact sequence:

$$\cdots \to H_c^i(X, \mathcal{F}) \xrightarrow{\pi} H_c^i(X, \mathcal{F}) \to H_c^i(X, \mathcal{F}_1) \to \cdots.$$

So we have an exact sequence

$$0 \to H_c^i(X, \mathcal{F})/\pi \to H_c^i(X, \mathcal{F}_1) \to H_c^{i+1}(X, \mathcal{F})_\pi \to 0,$$

where $H_c^i(X, \mathcal{F})/\pi$ denotes $\mathrm{coker}(\pi : H_c^i(X, \mathcal{F}) \to H_c^i(X, \mathcal{F}))$, and $H_c^{i+1}(X, \mathcal{F})_\pi$ denotes $\ker(\pi : H_c^{i+1}(X, \mathcal{F}) \to H_c^{i+1}(X, \mathcal{F}))$. Hence

$$\dim_{\kappa_E}(H_c^i(X, \mathcal{F})/\pi) + \dim_{\kappa_E}(H_c^{i+1}(X, \mathcal{F})_\pi) = \dim_{\kappa_E} H_c^i(X, \mathcal{F}_1).$$

Using the fact that the R_E-module $H_c^i(X, \mathcal{F})$ is isomorphic to a direct sum of finitely many copies of R_E and R_E/π^j $(j \in \mathbf{N})$, one can show

$$\dim_E(H_c^i(X, \mathcal{F}) \otimes_{R_E} E) = \dim_{\kappa_E}(H_c^i(X, \mathcal{F})/\pi) - \dim_{\kappa_E}(H_c^i(X, \mathcal{F})_\pi).$$

So we have

$$\dim_E(H_c^i(X, \mathcal{F}) \otimes_{R_E} E) \leq \dim_{\kappa_E}(H_c^i(X, \mathcal{F})/\pi) \leq \dim_{\kappa_E} H^i(X, \mathcal{F}_1),$$

and

$$\begin{aligned}
\chi_c(X, \mathcal{F} \otimes_{R_E} E) &= \sum_i (-1)^i \dim_E(H_c^i(X, \mathcal{F}) \otimes_{R_E} E) \\
&= \sum_i (-1)^i (\dim_{\kappa_E}(H_c^i(X, \mathcal{F})/\pi) - \dim_{\kappa_E}(H_c^i(X, \mathcal{F})_\pi)) \\
&= \sum_i (-1)^i (\dim_{\kappa_E}(H_c^i(X, \mathcal{F})/\pi) + \dim_{\kappa_E}(H_c^{i+1}(X, \mathcal{F})_\pi)) \\
&= \sum_i (-1)^i \dim_{\kappa_E} H_c^i(X, \mathcal{F}_1) \\
&= \chi_c(X, \mathcal{F}_1).
\end{aligned}$$

Note that $\mathcal{F}_1$ is nothing but $\mathcal{F} \otimes_{R_E} \kappa_E$. This proves (i).

In the case where $(X, \mathcal{F})$ over $k = \mathbf{F}$ is obtained by base change from $(X_0, \mathcal{F}_0)$ over $\mathbf{F}_q$, we can also prove the equality of Euler characteristics using Grothendieck's formula for $L(X_0, \mathcal{F}_0, t)$ and the fact that degree of $L(X_0, \mathcal{F}_0, t)$ doesn't change under reduction modulo π.

(ii) Since $\mathrm{Gal}(X'/X)$ is a finite group, there are only finitely many irreducible finite dimensional κ_E-representations of $\mathrm{Gal}(X'/X)$. Each such representation defines an irreducible locally constant constructible κ_E-sheaf on X. Let C be a constant such that

$$\dim_{\kappa_E} H^i(X, \mathcal{G}) \leq C$$

for every i and every $\mathcal{G}$ coming from an irreducible finite dimensional κ_E-representation of $\mathrm{Gal}(X'/X)$. Suppose $\mathcal{F}$ is a locally constant constructible κ_E-sheaf whose restriction to X' is a constant sheaf. Then $\mathcal{F}$ corresponds to a representation of $\mathrm{Gal}(X'/X)$. We can find a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$$

such that each $\mathcal{F}_j/\mathcal{F}_{j-1}$ is irreducible. Note that we have $m \leq \mathrm{rank}(\mathcal{F})$. Moreover, we have

$$\dim_{\kappa_E} H^i_c(X, \mathcal{F}) \leq \sum_{j=1}^{m} \dim_{\kappa_E} H^i_c(X, \mathcal{F}_j/\mathcal{F}_{j-1}) \leq Cm \leq C\mathrm{rank}(\mathcal{F}).$$

Theorem 1.9. Let d be a positive integer, let $f_0 : X_0 \to Y_0$ be an $\mathbf{F}_q$-morphism, and let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}}_l$-sheaf on X_0. Then there are two constants $c_1(f_0, \mathcal{F}_0)$ and $c_2(f_0, \mathcal{F}_0)$ depending only on f_0 and $\mathcal{F}_0$ such that for all $d \geq 1$, we have the following bound for the total degree of $L(f_0, \mathcal{F}_0, d, t)$:

$$\mathrm{tot.deg} L(f_0, \mathcal{F}_0, d, t) \leq c_1(f_0, \mathcal{F}_0) d^{c_2(f_0, \mathcal{F}_0) - 1}.$$

Furthermore, we can take

$$c_2(f_0, \mathcal{F}_0) = \max\{\dim(R^j f_! \mathcal{F})_{\bar{y}} | j \geq 0, y \in Y\}.$$

Proof. By Theorem 1.1 (ii), we have

$$L(f_0, \mathcal{F}_0, d, t) = \prod_{i=0}^{2\dim Y} \prod_{j=0}^{2\mathrm{rel.dim}(f)} \det(1 - F_Y t, H^i_c(Y, [R^j f_! \mathcal{F}]^d))^{(-1)^{i+j+1}}.$$

Taking $c_2 = \max\{\dim(R^j f_! \mathcal{F})_{\bar{y}} | j \geq 0, y \in Y\}$. We have

$$\begin{aligned} [R^j f_! \mathcal{F}]^d &= \sum_{k=1}^{d} (-1)^{k-1} k (\mathrm{Sym}^{d-k}(R^j f_! \mathcal{F}) \otimes \wedge^k (R^j f_! \mathcal{F})) \\ &= \sum_{k=1}^{\min(d, c_2)} (-1)^{k-1} k (\mathrm{Sym}^{d-k}(R^j f_! \mathcal{F}) \otimes \wedge^k (R^j f_! \mathcal{F})) \end{aligned}$$

since $\wedge^k(R^jf_!\mathcal{F}) = 0$ for $k > c_2$. So we have

$$\begin{aligned} &L(f_0, \mathcal{F}_0, d, t) \\ = &\prod_{i=0}^{2\dim Y} \prod_{j=0}^{2\text{rel.dim}(f)} \prod_{k=1}^{\min(c_2,d)} \det(1 - F_Y t, H_c^i(Y, \text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})))^{k(-1)^{i+j+k}}. \end{aligned}$$

Hence we have the following bound for the total degree:

$$\text{tot.deg}L(f_0, \mathcal{F}_0, d, t) \leq \sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\text{rel.dim}(f)} \sum_{k=1}^{\min(c_2,d)} k\dim H_c^i(Y, \text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})).$$

Let E be a finite extension of $\mathbf{Q}_l$ and let R_E be the integral closure of $\mathbf{Z}_l$ in E so that $R^jf_!\mathcal{F}_0$ $(0 \leq j \leq 2\text{rel.dim}(f))$ define torsion free R_E-sheaves. Let κ_E be the residue field of R_E. Choose a decomposition

$$Y = \coprod_{m=1}^{M} Y_m$$

of Y into a finite disjoint union of locally closed subschemes and choose a finite Galois étale covering $Y'_m \to Y_m$ for each Y_m so that for any $0 \leq j \leq 2\text{rel.dim}(f)$ and any m, the restriction of $(R^jf_!\mathcal{F}) \otimes_{R_E} \kappa_E$ to Y'_m is constant. Then for any k, the restriction of $(\text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})) \otimes_{R_E} \kappa_E$ to each Y'_m is also constant. By Lemma 1.8, there exists a constant C depending only on $Y'_m \to Y_m$ $(1 \leq m \leq M)$ such that

$$\begin{aligned} &\dim H_c^i(Y, \text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})) \\ \leq &\sum_{m=1}^{M} \dim H_c^i(Y_m, \text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})) \\ \leq &\sum_{m=1}^{M} \dim_{\kappa_E} H_c^i(Y_m, (\text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F})) \otimes_{R_E} \kappa_E) \\ \leq &C\sum_{m=1}^{M} \text{rank}\Big((\text{Sym}^{d-k}(R^jf_!\mathcal{F}) \otimes \wedge^k(R^jf_!\mathcal{F}))|_{Y_m}\Big) \\ \leq &CM \begin{pmatrix} c_2 + d - k - 1 \\ c_2 - 1 \end{pmatrix} \begin{pmatrix} c_2 \\ k \end{pmatrix}. \end{aligned}$$

So we have

$$\begin{aligned} &\text{tot.deg}L(f_0, \mathcal{F}_0, d, t) \\ \leq &\sum_{i=0}^{2\dim Y} \sum_{j=0}^{2\text{rel.dim}(f)} \sum_{k=1}^{\min(c_2,d)} kCM \begin{pmatrix} c_2 + d - k - 1 \\ c_2 - 1 \end{pmatrix} \begin{pmatrix} c_2 \\ k \end{pmatrix} \\ \leq &(2\text{rel.dim}(f) + 1)(2\dim Y + 1)CM \sum_{k=1}^{c_2} k \begin{pmatrix} c_2 + d - k - 1 \\ c_2 - 1 \end{pmatrix} \begin{pmatrix} c_2 \\ k \end{pmatrix}. \end{aligned}$$

It is easy to see that the last expression is a polynomial in d with degree at most $c_2 - 1$. This proves our assertion.

Remark 1.10. The second constant $c_2(f_0, \mathcal{F}_0)$ is at least effective when $\mathcal{F}_0$ is the constant sheaf by Katz's bound [5] on the l-adic Betti number. But the first constant $c_1(f_0, \mathcal{F}_0)$ is not effective yet in general, even for the constant sheaf. The polynomial growth of the total degree with respect to d can already be seen in the classical example of the universal family f_E of elliptic curves. In this case, the total degree is bounded up to a constant by the dimension of the space of weight d modular forms, which grows like a linear polynomial in d by the Riemann-Roch theorem. More recently, Brock-Granville [1] and Katz [6] considered the d-th moment problem for other morphisms, such as the universal family of hyper-elliptic curves of genus g and the universal family of smooth projective hypersurfaces of fixed degree and dimension. Our bound can be used in these situations to get more precise information about the non-effective constants in their archimedean estimates.

2. Partial L-functions

Let $d_1, \ldots, d_n$ be positive integers and let d be a common multiple of them. Let $X_0, X_0^{(1)}, \ldots, X_0^{(n)}$ be schemes over $\mathbf{F}_q$, let $f_i : X_0 \to X_0^{(i)}$ $(i = 1, \ldots, n)$ be $\mathbf{F}_q$-morphisms, and let $\mathcal{F}_0$ be a constructible $\overline{\mathbf{Q}}_l$-sheaf on X_0. Denote by $F_X : X \to X$ and $F_X : F_X^* \mathcal{F} \to \mathcal{F}$ the geometric Frobenius correspondences. For any positive integer k, let

$$S_k(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n) = \sum_{x \in X_0(\mathbf{F}_{q^{kd}}), f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}).$$

The partial L-function of $\mathcal{F}_0$ is

$$L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t) = \exp\left(\sum_{k=1}^{\infty} \frac{S_k(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n)}{k} t^k \right).$$

Define the *Faltings' scheme* Z_0 associated to $(X_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n)$ to be the subscheme of the d-fold product X_0^d so that a point $(x_1, \ldots, x_d)$ in $X^d(\mathbf{F})$ lies in $Z(\mathbf{F})$ if and only if

$$f_i(x_j) = f_i(x_{j'}) \text{ whenever } j \equiv j' \bmod d_i \ (i \in \{1, \ldots, n\},\ j, j' \in \{1, \ldots, d\}).$$

Let $\sigma_0 : X_0^d \to X_0^d$ be the automorphism defined by

$$(x_1, \ldots, x_d) \to (x_d, x_1, \ldots, x_{d-1}).$$

Then Z_0 is invariant under σ_0. Let $\alpha : Z_0 \to X_0^d$ be the immersion, let $p_j : X_0^d \to X_0$ ($j \in \{1, \ldots, d\}$) be the projections, and let $\mathcal{G}_0$ be the $\overline{\mathbf{Q}}_l$-sheaf on Z_0 defined by

$$\mathcal{G}_0 = \alpha^* p_1^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_d^* \mathcal{F}_0.$$

Define a morphism of sheaves

$$\sigma_0 : \sigma_0^* \mathcal{G}_0 \to \mathcal{G}_0$$

over $\sigma_0 : Z_0 \to Z_0$ by

$$\begin{aligned} \sigma_0 : \alpha^* p_d^* \mathcal{F}_0 \otimes \alpha^* p_1^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_{d-1}^* \mathcal{F}_0 &\to \alpha^* p_1^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_d^* \mathcal{F}_0, \\ s_d \otimes s_1 \otimes \cdots \otimes s_{d-1} &\mapsto s_1 \otimes \cdots \otimes s_d. \end{aligned}$$

More generally, for any permutation τ of $\{1, \ldots, d\}$, we define $\tau_0 : X_0^d \to X_0^d$ to be the automorphism

$$(x_1, \ldots, x_n) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(n)}).$$

If τ has the property that for each i, the condition $j \equiv j' \bmod d_i$ implies the condition $\tau(j) \equiv \tau(j') \bmod d_i$, then Z_0 is invariant under τ_0. For such τ, we have a morphism of sheaves

$$\tau_0 : \tau_0^* \mathcal{G}_0 \to \mathcal{G}_0$$

over $\tau_0 : Z_0 \to Z_0$ defined by

$$\begin{aligned} \tau_0 : \alpha^* p_{\tau(1)}^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_{\tau(d)}^* \mathcal{F}_0 &\to \alpha^* p_1^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_d^* \mathcal{F}_0, \\ s_{\tau(1)} \otimes \cdots \otimes s_{\tau(d)} &\mapsto s_1 \otimes \cdots \otimes s_d. \end{aligned}$$

Finally let $F_Z : Z \to Z$ and $F_Z : F_Z^* \mathcal{G} \to \mathcal{G}$ be the geometric Frobenius correspondences.

Lemma 2.1. Notation as above. We have

$$\begin{aligned} S_k(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n) &= \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\ f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) \\ &= \sum_{z \in Z(\mathbf{F}), F^k\sigma(z) = z} \mathrm{Tr}(F_Z^k \sigma, \mathcal{G}_{\bar{z}}). \end{aligned}$$

Proof. Let $(x_1, \ldots, x_d) \in X^d(\mathbf{F})$ be a point lying in $Z(\mathbf{F})$ fixed by $F_Z^k \sigma : Z \to Z$. Then we have

$$F_X^k(x_j) = x_{j+1} \ (j \in \{1, \ldots, d\}),$$

where for $j = d$, we take $j+1$ to be 1. So

$$F^k_{X^{(i)}}(f_i(x_j)) = f_i(x_{j+1})\ (i \in \{1, \ldots, n\},\ j \in \{1, \ldots, d\}).$$

Iterating these equations d_i times and using the fact that $f_i(x_j) = f_i(x_{j'})$ whenever $j \equiv j' \bmod d_i$, we get

$$F^{kd_i}_{X^{(i)}}(f_i(x_j)) = f_i(x_j).$$

Hence $f_i(x_j) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})$. If we iterate d times the equation $F_X^k(x_j) = x_{j+1}$, we get $F_X^{kd}(x_j) = x_j$. So $x_j \in X_0(\mathbf{F}_{q^{kd}})$. In particular, x_1 is a point in $X_0(\mathbf{F}_{q^{kd}})$ with the property that $f_i(x_1) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})$ for each i. Conversely, one can show that for any point x in $X_0(\mathbf{F}_{q^{kd}})$ with the property that $f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})$ for each i, the point $(x, F_X^k(x), \ldots, F_X^{k(d-1)}(x))$ is the unique point in $Z(\mathbf{F})$ that is fixed by $F_Z^k\sigma$ and has x as the first component. So

$$x \mapsto (x, F_X^k(x), \ldots, F_X^{k(d-1)}(x))$$

defines a one-to-one correspondence between the set of points x in $X_0(\mathbf{F}_{q^{kd}})$ with the property that $f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})$ for each i and the set of fixed points of $F_Z^k\sigma$ in $Z(\mathbf{F})$.

Let x be a point in $X_0(\mathbf{F}_{q^{kd}})$ with the property that $f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})$ for each i and let $z = (x, F_X^k(x), \ldots, F_X^{k(d-1)}(x))$ be the corresponding fixed point of $F_Z^k\sigma$ in Z. Note that the linear map $F_X^{kd} : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ is the composition

$$\mathcal{F}_{\bar{x}} = \mathcal{F}_{F_X^{kd}(\bar{x})} \stackrel{F_X^k}{\to} \mathcal{F}_{F_X^{k(d-1)}(\bar{x})} \stackrel{F_X^k}{\to} \cdots \stackrel{F_X^k}{\to} \mathcal{F}_{F_X^k(\bar{x})} \stackrel{F_X^k}{\to} \mathcal{F}_{\bar{x}}.$$

Using Lemma 1.2, one can show

$$\mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) = \mathrm{Tr}(F_Z^k\sigma, \mathcal{G}_{\bar{z}}).$$

Lemma 2.1 follows.

Combining Lemma 2.1 and Lemma 1.4 together, we get the following:

Corollary 2.2. Notation as above. We have

$$\begin{aligned} S_k(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n) &= \sum_{x \in X_0(\mathbf{F}_{q^{kd}}),\ f_i(x) \in X_0^{(i)}(\mathbf{F}_{q^{kd_i}})} \mathrm{Tr}(F_X^{kd}, \mathcal{F}_{\bar{x}}) \\ &= \sum_{i=0}^{2\dim Z} (-1)^i \mathrm{Tr}(F_Z^k\sigma, H_c^i(Z, \mathcal{G})). \end{aligned}$$

Theorem 2.3. Notation as above. For each d-th root of unity μ, let $H_c^i(Z, \mathcal{G})_\mu$ be the eigenvector space of σ acting on $H_c^i(Z, \mathcal{G})$ with eigenvalue μ. Then we have

$$\begin{aligned} S_k(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n) &= \sum_{i=0}^{2\dim Z} \sum_{\mu^d=1} (-1)^i \mu \mathrm{Tr}(F_Z^k, H_c^i(Z, \mathcal{G})_\mu), \\ L(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t) &= \prod_{i=0}^{2\dim Z} \prod_{\mu^d=1} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1}\mu}. \end{aligned}$$

Proof. Using Corollary 2.2 in place of Corollary 1.5, the proof is completely the same as the proof of Theorem 1.6.

Theorem 2.4. The partial L-function $L(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ is a rational function. More explicitly, let Λ_d be a subset of the set of d-th roots of unity so that for each $d'|d$, Λ_d contains exactly one primitive d'-th root of unity. For the primitive d'-th root of unity μ in Λ_d, let

$$n_\mu = \sum_{g \in \mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q})} g\mu = \begin{cases} 0 & \text{if } d' \text{ is not squarefree}, \\ (-1)^r & \text{if } d' \text{ is a product of } r \text{ distinct prime numbers}. \end{cases}$$

Then we have

$$L(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t) = \prod_{i=0}^{2\dim Z} \prod_{\mu \in \Lambda_d} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1} n_\mu}.$$

Proof. Let e be an arbitrary integer relatively prime to d. For any integer m, we denote the unique number in $\{1, \ldots, d\}$ that is equal to m modulo d by $\overline{m}$. Then σ_0 corresponds to the permutation

$$i \mapsto \overline{i-1}$$

and σ_0^e corresponds to the permutation

$$i \mapsto \overline{i-e}.$$

Let τ be a permutation defined by

$$\tau(i) = \overline{ie}$$

for all i. Then τ induces an automorphism

$$\begin{aligned} \tau_0 : Z_0 &\to Z_0 \\ (x_1, \ldots, x_n) &\mapsto (x_{\tau(1)}, \ldots, x_{\tau(n)}). \end{aligned}$$

and we have a morphism of sheaves

$$\tau_0 : \tau_0^* \mathcal{G}_0 \to \mathcal{G}_0$$

over $\tau_0 : Z_0 \to Z_0$ defined by

$$\begin{aligned} \tau_0 : \alpha^* p_{\tau(1)}^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_{\tau(d)}^* \mathcal{F}_0 &\to \alpha^* p_1^* \mathcal{F}_0 \otimes \cdots \otimes \alpha^* p_d^* \mathcal{F}_0, \\ s_{\tau(1)} \otimes \cdots \otimes s_{\tau(d)} &\mapsto s_1 \otimes \cdots \otimes s_d. \end{aligned}$$

It is easy to verify that $\tau\sigma = \sigma^e \tau$ as permutations, $\tau_0 \sigma_0 = \sigma_0^e \tau_0$ as automorphisms of Z_0, and the following diagram of morphisms of sheaves commute:

$$\begin{array}{ccc} \tau_0^* \sigma_0^{e*} \mathcal{G}_0 = \sigma_0^* \tau_0^* \mathcal{G}_0 & \stackrel{\sigma_0^*(\tau_0)}{\to} & \sigma_0^* \mathcal{G}_0 \\ \downarrow \tau_0^*(\sigma_0^e) & & \downarrow \sigma_0 \\ \tau_0^* \mathcal{G}_0 & \stackrel{\tau_0}{\to} & \mathcal{G}_0. \end{array}$$

It follows that σ_0^e and $\tau_0 \sigma_0 \tau_0^{-1}$ induce the same homomorphism on $H_c^i(Z, \mathcal{G})$ for any i. So for each d-th root of unity μ, τ_0^{-1} induces an isomorphism

$$\tau^{-1} : H_c^i(Z, \mathcal{G})_\mu \stackrel{\cong}{\to} H_c^i(Z, \mathcal{G})_{\mu^e}.$$

Since F_Z commutes with τ, this implies that

$$\det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu) = \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_{\mu^e}).$$

This is true for any integer e relatively prime to d. So if μ and μ' are Galois conjugate roots of unity, then we have

$$\det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu) = \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_{\mu'}).$$

Note that any d-th root of unity is Galois conjugate to one and only one element in Λ_d. Combining with the formula in Theorem 2.3, we get

$$L(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t) = \prod_{i=0}^{2\dim Z} \prod_{\mu \in \Lambda_d} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1} n_\mu},$$

where n_μ is the integer $\sum_{g \in \mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q})} g\mu$. So $L(X_0; \mathcal{F}; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ is a rational function. Note that if μ is a primitive d'-th root of unity, then the integer n_μ is simply the negative of the next to the leading coefficient of the d'-th cyclotomic polynomial. Thus n_μ is just the value of the Möbius function at d'. It is zero if d' is not squarefree. It is $(-1)^r$ if d' is a product of r distinct primes.

Question 2.5. Fix X_0, $\mathcal{F}_0$, $f_1, \ldots, f_n$. Take d to be the least common multiple of $d_1, \ldots, d_n$. Can the total degree of $L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ be bounded by a polynomial in d when $d_1, \ldots, d_n$ vary ?

Theorem 1.9 says that the answer is yes in the case of moment L-functions. In [3], we show that when $\mathcal{F}_0$ is the constant sheaf, the total degree of $L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ can be bounded by an exponential function of d. By Theorem 3.3 in the next section, when $\mathcal{F}_0$ arises from exponential sums, the total degree of $L(X_0; \mathcal{F}_0; f_1, \ldots, f_n; d; d_1, \ldots, d_n; t)$ can again be bounded by an exponential function of d. By extending the ground field if necessary, we may assume that the integers d_i are relatively prime.

3. Application to Exponential Sums

In this section, we analyze the weights and the total degrees for the partial L-functions of exponential sums in greater detail. For convenience, we only consider the case where X_0 is a subscheme of the affine space $\mathbf{A}_0^n$ even though partial exponential sums can also be defined in more general context. Fix a nontrivial additive character $\psi : \mathbf{F}_p \to \overline{\mathbf{Q}}_l^*$. For each positive integer k, denote by $\psi_k : \mathbf{F}_{q^k} \to \overline{\mathbf{Q}}_l^*$ the additive character defined by

$$\psi_k(u) = \psi(\mathrm{Tr}_{\mathbf{F}_{q^k}/\mathbf{F}_p}(u)).$$

Let $d_1, \ldots, d_n$ be positive integers and let d be a common multiple of them. Let $f : X_0 \to \mathbf{A}_0^1$ be an $\mathbf{F}_q$-morphism. For each positive integer k, define the partial exponential sum by

$$S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n) = \sum_{(x_1,\ldots,x_n)\in X(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1, \ldots, x_n)).$$

When $d_1 = \cdots = d_n = d = 1$, the above partial exponential sum coincides with the classical exponential sum

$$S_{\psi,k}(X_0; f) = \sum_{x\in X_0(\mathbf{F}_{q^k})} \psi_k(f(x)).$$

The partial L-function associated to the above partial exponential sum is defined to be

$$L(X_0; f; d; d_1, \ldots, d_n; t) = \exp\left(\sum_{k=1}^{\infty} \frac{S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n)}{k} t^k\right).$$

The Artin-Schreier morphism

$$\mathbf{A}_0^1 \to \mathbf{A}_0^1,\ x \mapsto x^p - x$$

is an étale Galois covering with Galois group $\mathbf{F}_p$. Let $\mathcal{L}_\psi$ be the lisse $\overline{\mathbf{Q}}_l$-sheaf of rank 1 on $\mathbf{A}_0^1$ obtained by pushing out this covering using the character $\psi^{-1} : \mathbf{F}_p \to \overline{\mathbf{Q}}_l^*$. One can show that for any $x = (x_1, \ldots, x_n) \in X(\mathbf{F}) \subset \mathbf{A}^n(\mathbf{F})$ with $x_i \in \mathbf{F}_{q^{kd_i}}$ for each i, we have

$$\psi_{kd}(f(x_1, \ldots, x_n)) = \mathrm{Tr}(F_X^{kd}, (f^*\mathcal{L}_\psi)_{\bar{x}}).$$

So we have

$$\begin{aligned} S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n) &= \sum_{(x_1,\ldots,x_n)\in X(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1, \ldots, x_n)) \\ &= \sum_{(x_1,\ldots,x_n)\in X(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \mathrm{Tr}(F_X^{kd}, (f^*\mathcal{L}_\psi)_{\bar{x}}). \end{aligned}$$

So the partial L-function

$$L(X_0; f; d; d_1, \ldots, d_n; t) = \exp\left(\sum_{k=1}^{\infty} \frac{S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n)}{k} t^k\right)$$

associated to the partial exponential sum coincides with the partial L-function

$$L(X_0; f^*\mathcal{L}_\psi; \pi_1, \ldots, \pi_n; d; d_1, \ldots, d_n; t) = \exp\left(\sum_{k=1}^{\infty} \frac{\sum_{(x_1,\ldots,x_n)\in X(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \mathrm{Tr}(F_X^{kd}, (f^*\mathcal{L}_\psi)_{\bar{x}})}{k} t^k\right)$$

defined in §2, where $\pi_i : X_0 \hookrightarrow \mathbf{A}_0^n \to \mathbf{A}_0^1$ $(i = 1, \ldots, n)$ are the projections. By Theorem 2.4, $L(X_0; f; d; d_1, \ldots, d_n; t)$ is rational. From Theorem 2.3, we get the following:

Proposition 3.1. Let Z_0 be the Faltings' scheme associated to $(X_0; \pi_1, \ldots, \pi_n; d; d_1; \ldots, d_n)$, where $\pi_i : X_0 \hookrightarrow \mathbf{A}_0^n \to \mathbf{A}_0^1$ $(i = 1, \ldots, n)$ are the projections. Let $\mathcal{G}_0 = \alpha^* p_1^* f^* \mathcal{L}_\psi \otimes \cdots \otimes \alpha^* p_d^* f^* \mathcal{L}_\psi$, where $p_j : X_0^d \to X_0$ $(j = 1, \ldots, n)$ are the projections, and $\alpha : Z_0 \to X_0^d$ is the immersion. For each d-th root of unity μ, let $H_c^i(Z, \mathcal{G})_\mu$ be the eigenvector space of the morphism σ acting on $H_c^i(Z, \mathcal{G})$ with eigenvalue μ. Then we have

$$\begin{aligned} S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n) &= \sum_{(x_1,\ldots,x_n)\in X(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1, \ldots, x_n)) \\ &= \sum_{i=0}^{2\dim Z} \sum_{\mu^d=1} (-1)^i \mu \mathrm{Tr}(F_Z^k, H_c^i(Z, \mathcal{G})_\mu), \end{aligned}$$

$$L(X_0; f; d; d_1, \ldots, d_n; t) = \prod_{i=0}^{2\dim Z} \prod_{\mu^d=1} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1}\mu}.$$

To estimate the sum $S_{\psi,k}(X_0; f; d; d_1, \ldots, d_n)$, we need information about the cohomology groups $H_c^i(Z, \mathcal{G})$. The following result is a direct consequence of a result of Deligne ([2] 3.7.2.3):

Proposition 3.2. Let $X_0 = \mathbf{A}_0^n$ and let $f : \mathbf{A}_0^n \to \mathbf{A}_0^1$ be a morphism given by a polynomial $f(x_1, \ldots, x_n)$ with coefficients in $\mathbf{F}_q$. Consider the polynomial $g(x_{11}, \ldots, x_{1d_1}, \ldots, x_{n1}, \ldots, x_{nd_n})$ obtained from

$$f(x_{11}, \ldots, x_{n1}) + \cdots + f(x_{1d}, \ldots, x_{nd})$$

by taking $x_{ij} = x_{ij'}$ for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$, where j' is the unique number satisfying $1 \le j' \le d_i$ and $j \equiv j' \bmod d_i$. Suppose p does not divide the degree m of the polynomial g and the homogeneous degree m part of g defines a smooth hypersurface in the projective space $\mathbf{P}^{(\sum_{i=1}^n d_i - 1)}$.

(i) We have $H_c^r(Z, \mathcal{G}) = 0$ for $r \neq \sum_{i=1}^n d_i$.

(ii) When $r = \sum_{i=1}^n d_i$, we have $\dim H_c^r(Z, \mathcal{G}) = (m-1)^r$ and $H_c^r(Z, \mathcal{G})$ is pure of weight r, that is, for any eigenvalue λ of the action of the geometric Frobenius correspondence F_Z on $H_c^r(Z, \mathcal{G})$, λ is an algebraic number and all its conjugates have archimedean absolute value $q^{\frac{r}{2}}$.

Proof. The Faltings' scheme Z_0 for $X_0 = \mathbf{A}_0^n$ can be described as follows: For any point $x = (x^{(1)}, \ldots, x^{(d)}) \in (\mathbf{A}^n)^d(\mathbf{F})$, where each $x^{(j)}$ is a point in $\mathbf{A}^n(\mathbf{F})$, write $x^{(j)} = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix}$, where each $x_{ij} \in \mathbf{F}$ is the i-th coordinate of $x^{(j)}$. We say the point x corresponds to the $(n \times d)$-matrix (x_{ij}). Then Z_0 is given by the equations

$$x_{ij} = x_{ij'} \text{ whenever } j \equiv j' \bmod d_i \ (i \in \{1, \ldots, n\},\ j, j' \in \{1, \ldots, d\}).$$

Define an isomorphism $\mathbf{A}_0^{d_1 + \cdots + d_n} \cong Z_0$ by sending any point

$$(x_{11}, \ldots, x_{1d_1}, \ldots, x_{n1}, \ldots, x_{nd_n})$$

in $\mathbf{A}_0^{d_1 + \cdots + d_n}(\mathbf{F})$ to the point in $Z_0(\mathbf{F})$ corresponding to the $(n \times d)$-matrix whose i-th row is given by

$$(x_{i1}, \ldots, x_{id_i}, x_{i1}, \ldots, x_{id_i}, \ldots, x_{i1}, \ldots, x_{id_i}).$$

For any morphism $f : \mathbf{A}_0^n \to \mathbf{A}_0^1$ given by a polynomial $f(x_1, \ldots, x_n)$ with coefficients in $\mathbf{F}_q$, let $g(x_{11}, \ldots, x_{1d_1}, \ldots, x_{n1}, \ldots, x_{nd_n})$ be the polynomial defined in Proposition 3.2. One can verify that through the above isomorphism, the sheaf $\mathcal{G}_0 = \alpha^* p_1^* f^* \mathcal{L}_\psi \otimes \cdots \otimes \alpha^* p_d^* f^* \mathcal{L}_\psi$ on Z_0 is identified with the sheaf $g^*\mathcal{L}_\psi$ on $\mathbf{A}_0^{d_1+\cdots+d_n}$. Proposition 3.2 then follows from [2] 3.7.2.3.

Theorem 3.3. (i) Let $f(x_1, \ldots, x_n)$ be a polynomial of degree m with coefficients in $\mathbf{F}_q$. We have the following bound for the total degree of the partial L-function of the partial exponential sum:

$$\text{tot.deg}L(\mathbf{A}_0^n; f; d; d_1, \ldots, d_n; t) \leq 3(m+1)^{\sum_{i=1}^n d_i}.$$

(ii) Under the assumption of Proposition 3.2, we have the following bound for the total degree of the partial L-function of the partial exponential sum:

$$\text{tot.deg}L(\mathbf{A}_0^n; f; d; d_1, \ldots, d_n; t) \leq (m-1)^{\sum_{i=1}^n d_i}.$$

Proof. Keep the notation in the proof of Proposition 3.2. By Proposition 3.1, we have

$$\text{tot.deg}L(\mathbf{A}_0^n; f; d; d_1, \ldots, d_n; t) \leq \sum_{i=0}^{2(d_1+\cdots+d_n)} \dim H_c^i(\mathbf{A}^{d_1+\cdots+d_n}, g^*\mathcal{L}_\psi).$$

(i) then follows from [5] Theorem 10, and (ii) follows from Proposition 3.2.

Note that even under the assumption of Proposition 3.2, we do not know the exact total degree of the partial L-function $L(\mathbf{A}_0^n; f; d; d_1, \ldots, d_n; t)$ if $d > 1$, due to possible cancellation of the zeros and poles. It would be interesting to know if the bound in Theorem 3.3 (ii) can be further improved for $d > 1$. For $d = 1$, it is already an equality. Theorem 3.7 below shows that if $d_1 = \cdots = d_m$, $d_{m+1} = \cdots = d_n$, and $d_m | d_{m+1}$, then the total degree can be bounded by a polynomial of d.

Theorem 3.4. Let e be the greatest common divisor of $d_1, \ldots, d_n$. Under the assumption of Proposition 3.2, we have

$$\left| \sum_{(x_1,\ldots,x_n)\in\mathbf{A}^n(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1, \ldots, x_n)) \right| \leq (m-1)^{\sum_{i=1}^n \frac{d_i}{e}} q^{\frac{k}{2}\sum_{i=1}^n d_i}.$$

Proof. Replacing d_i by $\frac{d_i}{e}$ and replacing q be q^e, we may assume that $e = 1$. Let $r = \sum_{i=1}^n d_i$. By Propositions 3.1 and 3.2 (i), we have

$$\sum_{(x_1,\ldots,x_n)\in\mathbf{A}^n(\mathbf{F}),\ x_i\in\mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1, \ldots, x_n)) = (-1)^r \sum_{\mu^d=1} \mu \text{Tr}(F_Z^k, H_c^r(Z, \mathcal{G})_\mu).$$

For each d-th root of unity μ, let $\lambda_{\mu 1}, \ldots, \lambda_{\mu b_\mu}$ be all the eigenvalues counted with multiplicities of F_Z on $H_c^i(Z, \mathcal{G})_\mu$. By Proposition 3.2 (ii), we have $|\lambda_{\mu i}| = q^{\frac{r}{2}}$ and $\sum_{\mu^d=1} b_\mu = (m-1)^r$. So we have

$$\begin{aligned}
\left| \sum_{(x_1,\ldots,x_n)\in \mathbf{A}^n(\mathbf{F}),\ x_i \in \mathbf{F}_{q^{kd_i}}} \psi_{kd}(f(x_1,\ldots,x_n)) \right| &= \left| (-1)^r \sum_{\mu^d=1} \mu \mathrm{Tr}(F_Z^k, H_c^r(Z,\mathcal{G})_\mu) \right| \\
&= \left| (-1)^r \sum_{\mu^d=1} \sum_{i=1}^{b_\mu} \lambda_{\mu i}^k \mu \right| \\
&\leq \sum_{\mu^d=1} \sum_{i=1}^{b_\mu} |\lambda_{\mu i}^k \mu| \\
&= \sum_{\mu^d=1} \sum_{i=1}^{b_\mu} q^{\frac{kr}{2}} \\
&= (m-1)^{\sum_{i=1}^n d_i} q^{\frac{k}{2} \sum_{i=1}^n d_i}.
\end{aligned}$$

Note that it is possible to use the finer formula in Theorem 2.4 to improve the constant in Theorem 3.4.

We now analyze how often the condition in Proposition 3.2 is satisfied. By extending the ground field, we may assume $d_1, \ldots, d_n$ are relatively prime. Take d to be their least common multiple. Let m be a positive integer not divisible by p, let $\mathbf{A}_p(m,n)$ be the parameter space of homogenous polynomials $f(x_1,\ldots,x_n)$ in n variables of degree m, and let $M_p(m; d_1, \ldots, d_n)$ be the parameter subspace of those polynomials $f(x_1,\ldots,x_n) \in \mathbf{A}_p(m,n)$ satisfying the condition of Proposition 3.2. It is clear that $M_p(m; d_1,\ldots,d_n)$ is Zariski open in $\mathbf{A}_p(m,n)$. It is Zariski dense in $\mathbf{A}_p(m,n)$ whenever it is nonempty.

Theorem 3.5. The scheme $M_p(m; d_1,\ldots,d_n)$ is Zariski dense in $\mathbf{A}_p(m,n)$ if and only if d is not divisible by p. Equivalently, the scheme $M_p(m; d_1,\ldots,d_n)$ is empty if and only if d is divisible by p.

Proof. First, we assume that d is not divisible by p. Taking f to be the diagonal polynomial $x_1^m + \cdots + x_n^m$, we see that the polynomial g defined in Proposition 3.2 is

$$g = \sum_{i=1}^n \sum_{j=1}^{d_i} \frac{d}{d_i} x_{ij}^m.$$

This is a diagonal polynomial in $d_1 + \cdots + d_n$ variables, which defines a smooth projective hypersurface in $\mathbf{P}^{(\sum_{i=1}^{n} d_i - 1)}$ since d is not divisible by p. Thus, $M_p(m; d_1, \ldots, d_n)$ is Zariski dense in $\mathbf{A}_p(m, n)$

Next, we assume that d is divisible by p. Since the d_i's are relatively prime, without loss of generality, we may assume that for some integer $1 \leq k < n$, each d_i $(1 \leq i \leq k)$ is divisible by p and none of the d_j $(k+1 \leq j \leq n)$ is divisible by p. Let $f \in \mathbf{A}_p(m, n)$. The system

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_k} = 0$$

defines an affine scheme W of dimension at least $n - k \geq 1$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a non-zero point in $W(\mathbf{F})$, and let g be the polynomial defined in Proposition 3.2. One checks that at the point $x_{ij} = \alpha_i$ $(1 \leq i \leq n, 1 \leq j \leq d_i)$, we have

$$\frac{\partial g}{\partial x_{ij}} = \frac{d}{d_i} \frac{\partial f}{\partial x_i}|_\alpha = 0.$$

The last equality holds for $1 \leq i \leq k$ because α is a point in $W(\mathbf{F})$, and it holds for $k+1 \leq i \leq n$ because $\frac{d}{d_i}$ is divisible by p and the ground field has characteristic p. This implies that g does not define a smooth hypersurface in the projective space $\mathbf{P}^{(\sum_{i=1}^{n} d_i - 1)}$. Thus, the scheme $M_p(m; d_1, \ldots, d_n)$ is empty. The theorem is proved.

What we would also like is to find many polynomials $f(x_1, \cdots, x_n)$ over $\mathbf{F}_q$ such that the assumption of Proposition 3.2 is simultaneously satisfied for all choices of d_i such that p does not divide d.

Question 3.6. Let m and n be positive integers and let p be a prime number not dividing m. Define

$$M_p^*(m, n) = \bigcap_{\{d_1, \ldots, d_n\}, p \nmid d} M_p(m; d_1, \ldots, d_n).$$

Does $M_p^*(m, n)$ contain a Zariski open dense subset of $\mathbf{A}_p(m, n)$?

The above diagonal example shows that $M_p^*(m, n)$ is non-empty. See Theorem 3.9 for a partial result toward a positive answer of Question 3.6.

Despite the purity result in Proposition 3.2, we know very little about the multiplicities of the reciprocal zeros and reciprocal poles of the partial L-functions due to possible cancellations. In fact, we believe that there is often a great of cancellations for large d. Thus an important problem is to improve the exponential factor $(m-1)^{\sum_{i=1}^{n}\frac{d_i}{e}}$ in Theorem 3.4 to a polynomial in terms of the d_i's. Using Theorem 1.9, one can show this is possible in the case of moment L-functions. We now make the estimate more precise in the case of moment exponential sums.

Theorem 3.7. Let $h(x_1,\ldots,x_n,y_1,\ldots,y_{n'})$ be a polynomial of degree m with coefficients in $\mathbf{F}_q$. Consider the d-th fibred sum

$$g=\oplus_y^d h=h(x_{11},\ldots,x_{n1},y_1,\ldots,y_{n'})+\cdots+h(x_{1d},\ldots,x_{nd},y_1,\ldots,y_{n'}).$$

Suppose p does not divide m and the homogeneous degree m part of g defines a smooth hypersurface in the projective space $\mathbf{P}^{nd+n'-1}$.

(i) We have

$$\left|\sum_{x_i\in\mathbf{F}_{q^{kd}},\, y_j\in\mathbf{F}_{q^k}}\psi_{kd}(h(x_1,\ldots,x_n,y_1,\ldots,y_{n'}))\right|\leq(m-1)^{nd+n'}q^{k(nd+n')/2}.$$

(ii) We can replace the constant $(m-1)^{nd+n'}$ on the right-hand side of the above inequality by $c(p,h)d^{3(m+1)^n-1}$ for some constant $c(p,h)$ depending only on p and h. Moreover, the total degree of the partial L-function

$$\exp\left(\sum_{k=1}^{\infty}\frac{\sum_{x_i\in\mathbf{F}_{q^{kd}},\, y_j\in\mathbf{F}_{q^k}}\psi_{kd}(h(x_1,\ldots,x_n,y_1,\ldots,y_{n'}))}{k}t^k\right)$$

associated to the partial exponential sum is bounded by $c(p,h)d^{3(m+1)^n-1}$.

Proof. The first part is a special case of Theorem 3.4, where n is replaced by $n+n'$, $d_1=\cdots=d_n=d$, and $d_{n+1}=\cdots=d_{n+n'}=1$. To prove the second part, let $X_0=\mathbf{A}_0^{n+n'}$, $Y_0=\mathbf{A}_0^{n'}$, $f_0: A_0^{n+n'}\to A_0^{n'}$ the projection, and $h: A_0^{n+n'}\to\mathbf{A}_0^1$ the morphism defined by the polynomial $h(x_1,\ldots,x_n,y_1,\ldots,y_{n'})$. Note that we have

$$\sum_{x_i\in\mathbf{F}_{q^{kd}},\, y_j\in\mathbf{F}_{q^k}}\psi_{kd}(h(x_1,\ldots,x_n,y_1,\ldots,y_{n'}))=\sum_{x\in X_0(\mathbf{F}_{q^{kd}}),\, f_0(x)\in Y_0(\mathbf{F}_{q^k})}\mathrm{Tr}(F_X^{kd},h^*\mathcal{L}_\psi).$$

So the partial L-function associated to the partial exponential sum coincides with the moment L-function

$$L(f_0,h^*\mathcal{L}_\psi,d,t)=\exp\left(\sum_{k=1}^{\infty}\frac{\sum_{x\in X_0(\mathbf{F}_{q^{kd}}),\, f_0(x)\in Y_0(\mathbf{F}_{q^k})}\mathrm{Tr}(F_X^{kd},h^*\mathcal{L}_\psi)}{k}t^k\right).$$

By Theorem 1.6 or by Theorem 2.3, we have

$$L(f_0, h^*\mathcal{L}_\psi, d, t) = \prod_{i=0}^{2\dim Z} \prod_{\mu^d=1} \det(1 - F_Z t, H_c^i(Z, \mathcal{G})_\mu)^{(-1)^{i+1}\mu}.$$

So by Proposition 3.2, all the reciprocal zeros and reciprocal poles of $L(f_0, h^*\mathcal{L}_\psi, d, t)$ have archimedean absolute value $q^{(nd+n')/2}$. By Theorem 1.9, the total degree of $L(f_0, h^*\mathcal{L}_\psi, d, t)$ is bounded by $c(p,h)d^{c_2-1}$ for some constant $c(p,h)$ depending only on p and h and

$$c_2 = \max\{\dim(R^j f_!(h^*\mathcal{L}_\psi))_{\bar{y}} | j \geq 0, \bar{y} \in Y(\mathbf{F})\}.$$

From the estimate of Theorem 10 in [5], we deduce that $c_2 \leq 3(m+1)^n$. So we can write

$$L(f_0, h^*\mathcal{L}_\psi, d, t) = \frac{(1-\beta_1 t)\cdots(1-\beta_v t)}{(1-\alpha_1 t)\cdots(1-\alpha_u t)}$$

with $|\alpha_i| = |\beta_j| = q^{(nd+n')/2}$ and $u + v \leq c(p,h)d^{3(m+1)^n-1}$. We have

$$\sum_{x_i \in \mathbf{F}_{q^{kd}},\, y_j \in \mathbf{F}_{q^k}} \psi_{kd}(h(x_1, \ldots, x_n, y_1, \ldots, y_{n'})) = \sum_{i=1}^{u} \alpha_i^k - \sum_{j=1}^{v} \beta_j^k.$$

So

$$\begin{aligned} \left| \sum_{x_i \in \mathbf{F}_{q^{kd}},\, y_j \in \mathbf{F}_{q^k}} \psi_{kd}(h(x_1, \ldots, x_n, y_1, \ldots, y_{n'})) \right| &\leq \sum_{i=1}^{u} |\alpha_i|^k + \sum_{j=1}^{v} |\beta_j^k| \\ &\leq c(p,h)d^{3(m+1)^n-1} q^{k(nd+n')/2}. \end{aligned}$$

Remark 3.8. In the proof of Theorem 3.7, we see that the reciprocal zeros and reciprocal poles of the moment L-function $L(f_0, h^*\mathcal{L}_\psi, d, t)$ associated to the exponential sum are pure of weight $nd + n'$. We get this using the formula in Theorem 2.3 and Deligne's result Proposition 3.2. If we make the extra assumption that the leading form of the polynomial $h(x_1, \ldots, x_n, 0, \ldots, 0)$ defines a smooth hypersurface in $\mathbf{P}^{n-1}$, then we can obtain the same result about the weights of the reciprocal zeros and reciprocal poles of the moment L-function $L(f_0, h^*\mathcal{L}_\psi, d, t)$ using the formula in Theorem 1.1 and Deligne's result. Indeed, for each fixed point $(y_1, \ldots, y_{n'})$ the leading form of the polynomial $h(x_1, \ldots, x_n, y_1, \ldots, y_{n'})$ in terms of x_i's is the same as the leading form of $h(x_1, \ldots, x_n, 0, \ldots, 0)$. So by [2] 3.7.2.3, $R^q f_{0!} h^*\mathcal{L}_\psi$ vanishes for $q \neq n$ and $R^n f_{0!} h^*\mathcal{L}_\psi$ is pure of weight n. Together with the Künneth formula, these imply that

$$L(\mathbf{A}_0^{n'}, \otimes^d (R^n f_{0!} h^*\mathcal{L}_\psi), t) \quad = \quad L(\mathbf{A}_0^{n'}, Rf_{0!} h^*\mathcal{L}_\psi \otimes^L \cdots \otimes^L Rf_{0!} h^*\mathcal{L}_\psi, t)$$

$$
\begin{aligned}
&= L(\mathbf{A}_0^{n'}, Rp_{0!}g^*\mathcal{L}_\psi, t) \\
&= L(\mathbf{A}_0^{nd+n'}, g^*\mathcal{L}_\psi, t),
\end{aligned}
$$

where in the second equality, p_0 is the projection $\mathbf{A}_0^{nd+n'} \to \mathbf{A}_0^{n'}$, and g is the d-th fibred sum of h. If we assume the leading form of g defines a smooth projective hypersurface, then by [2] 3.7.2.3 again, all the reciprocal zeros and reciprocal poles of $L(\mathbf{A}_0^{nd+n'}, g^*\mathcal{L}_\psi, t)$ have weights $nd+n'$. On the other hand, by Theorem 1.1, we have

$$
\begin{aligned}
L(f_0, h^*\mathcal{L}_\psi, d, t) &= L(\mathbf{A}_0^{n'}, [R^n f_{0!}h^*\mathcal{L}_\psi]^d, t)^{(-1)^n} \\
&= \prod_{j=1}^{d} L(\mathbf{A}_0^{n'}, \mathrm{Sym}^{d-j}R^n f_{0!}h^*\mathcal{L}_\psi \otimes \wedge^j R^n f_{0!}h^*\mathcal{L}_\psi, t)^{j(-1)^{n+j-1}}.
\end{aligned}
$$

Now, each sheaf $\mathrm{Sym}^{d-j}R^n f_! h^*\mathcal{L}_\psi \otimes \wedge^j R^n f_! h^*\mathcal{L}_\psi$ is a direct factor of the sheaf $\otimes^d(R^n f_! h^*\mathcal{L}_\psi)$. Thus, each $H_c^i(\mathbf{A}^n, \mathrm{Sym}^{d-j}R^n f_! h^*\mathcal{L}_\psi \otimes \wedge^j R^n f_! h^*\mathcal{L}_\psi)$ is a direct factor of $H_c^i(\mathbf{A}^n, \otimes^d(R^n f_! h^*\mathcal{L}_\psi))$ as $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$-modules. It follows that the set of zeros and poles of $L(\mathbf{A}_0^{n'}, \mathrm{Sym}^{d-j}R^n f_{0!}h^*\mathcal{L}_\psi \otimes \wedge^j R^n f_{0!}h^*\mathcal{L}_\psi, t)$ is contained in the set of zeros and poles of $L(\mathbf{A}_0^{n'}, \otimes^d(R^n f_{0!}h^*\mathcal{L}_\psi), t))$. We conclude that all the reciprocal zeros and reciprocal poles of $L(f_0, h^*\mathcal{L}_\psi, d, t)$ also have weight $nd+n'$.

We now analyze how often the condition in Theorem 3.7 is simultaneously satisfied for all positive integer d not divisible by p. Let m, n, n' be positive integers with m not divisible by p. Recall that $\mathbf{A}_p(m, n+n')$ denotes the parameter space of homogenous polynomials $f(x_1, ..., x_n, y_1, \cdots, y_{n'})$ in $n+n'$ variables of degree m. Let $M_p(m; n; n'; d)$ denote the parameter subspace of those polynomials $f \in \mathbf{A}_p(m, n+n')$ such that the d-fibred sum $\oplus_y^d f$ defines a smooth hypersurface in the projective space $\mathbf{P}^{nd+n'-1}$. By Theorem 3.5, we know that $M_p(m; n; n'; d)$ is Zariski dense open in $\mathbf{A}_p(m, n+n')$ if and only if d is not divisible by p.

Theorem 3.9. Let m, n, n' be positive integers with m not divisible by p. The intersection

$$
M_p^*(m; n; n') = \bigcap_{p \nmid d} M_p(m; n; n'; d).
$$

contains a Zariski open dense subset of $\mathbf{A}_p(m, n+n')$.

Proof. By Lemma 3.10 below, there is a Zariski open dense $U \in \mathbf{A}_p(m, n+n')$ such that if

$f(x,y) = f(x_1, \ldots, x_n, y_1, \ldots, y_{n'})$ lies in U, then for every point $y \in \mathbf{A}^{n'}(\mathbf{F})$, the system

$$\frac{\partial f}{\partial x_1}(x,y) = \cdots = \frac{\partial f}{\partial x_n}(x,y) = 0$$

defines a finite set $W(f,y)$ with at most $(m-1)^n$ points. Let $a_1(y), \cdots, a_N(y)$ be all the distinct values taken by the vector

$$\frac{\partial f}{\partial y} = (\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_{n'}})$$

at the finitely many points of $W(f,y)$. (N depends on y). We have

$$N \leq (m-1)^n.$$

Let $P = (x^{(1)}, \ldots, x^{(d)}, y)$ be a singular point on projective hypersurface $\oplus_y^d f = 0$ defined by the d-th fibred sum, where

$$x^{(j)} = (x_{1j}, \ldots, x_{nj}).$$

One checks that

$$\frac{\partial f}{\partial x_i}(x^{(j)}, y) = \frac{\partial(\oplus_y^d f)}{\partial x_{ij}}|_P = 0$$

for all $1 \leq i \leq n$ and $1 \leq j \leq d$. So each $(x^{(j)}, y)$ is a point of the finite set $W(f,y)$. On the other hand, one computes that

$$\sum_{j=1}^{d} \frac{\partial f}{\partial y}(x^{(j)}, y) = \frac{\partial(\oplus_y^d f)}{\partial y}|_P = 0.$$

For $1 \leq j \leq N$, let k_j be the multiplicity of the value $a_j(y)$ occurring in the family

$$\{\frac{\partial f}{\partial y}(x^{(1)}, y), \ldots, \frac{\partial f}{\partial y}(x^{(d)}, y)\}.$$

Then we have the relation

$$\begin{aligned} \sum_{j=1}^{N} k_j a_j &= 0, \\ \sum_{j=1}^{N} k_j &= d. \end{aligned}$$

Let k_j' be the least positive residue of k_j modulo p, we still have

$$\sum_{j=1}^{N} k_j' a_j = 0$$

since the ground field has characteristic p. This means that for $d' = \sum_{j=1}^{N} k'_j$, the projective hypersurface defined by the d'-th fibred sum $\oplus_y^{d'} f$ also has a singular point. More explicitly, letting $x'^{(j)}$ $(j = 1, \ldots, N)$ be some points in $W(f, y)$, not equal all zero, so that $a_j(y) = \frac{\partial f}{\partial y}(x'^{(j)}, y)$, and letting $Q = (x'^{(1)}, \ldots, x'^{(1)}, \ldots, x'^{(N)}, \ldots, x'^{(N)})$, where each $x'^{(j)}$ repeats k_j-times, then Q is a singular point of the projective hypersurface defined by the d'-th fibred sum $\oplus_y^{d'} f$. Now

$$d' \leq pN \leq p(m-1)^n.$$

So for any $f \in U$, if the projective hypersurface defined by the d-th fibred sum $\oplus_y^d f$ has a singular point, then the projective hypersurface defined by the d'-th fibred sum $\oplus_y^{d'} f$ has a singular point for some $d' \leq p(m-1)^n$. Thus, the set $M_p^*(m; n; n')$ contains the intersection

$$U \bigcap \Big(\bigcap_{p \not| d} M_p(m; n; n'; d) \Big) = U \bigcap \Big(\bigcap_{1 \leq d \leq p(m-1)^n, p \not| d} M_p(m; n; n'; d) \Big),$$

which is a Zariski open dense subset of $\mathbf{A}_p(m, n + n')$ by Theorem 3.5. The theorem is proved.

Lemma 3.10. Let $f(x, y) = f(x_1, \ldots, x_n, y_1, \ldots, y_{n'})$ be a homogeneous polynomial of degree m with coefficients in $\mathbf{F}$. Suppose $p \not| m$ and $f(x, 0)$ defines a smooth hypersurface in $\mathbf{P}^{n-1}$. Then for any fixed point $y = (y_1, \ldots, y_n) \in \mathbf{A}^n(\mathbf{F})$, the system of equations

$$\frac{\partial f}{\partial x_1}(x, y) = \cdots = \frac{\partial f}{\partial x_n}(x, y) = 0$$

has at most $(m-1)^n$ solutions in terms of x.

Proof. By the Euler identity, we have

$$mf(x, 0) = \sum_i \frac{\partial f}{\partial x_i}(x, 0).$$

Since $p \not| m$ and $f(x, 0)$ defines a smooth projective hypersurface, the only solution of the system of equations

$$\frac{\partial f}{\partial x_1}(x, 0) = \cdots = \frac{\partial f}{\partial x_n}(x, 0) = 0$$

is $x = 0$. So the ring $A_0 = \mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x, 0), \ldots, \frac{\partial f}{\partial x_n}(x, 0))$ is a finite dimensional $\mathbf{F}$-algebra. For any fixed point $y \in \mathbf{A}^n(\mathbf{F})$, we claim that the ring $A_y = \mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x, y), \ldots, \frac{\partial f}{\partial x_n}(x, y))$ is also a finite dimensional $\mathbf{F}$-algebra. This already implies that the system

$$\frac{\partial f}{\partial x_1}(x, y) = \cdots = \frac{\partial f}{\partial x_n}(x, y) = 0$$

has only finitely many solutions in terms of x.

Let B be a finite family of multi-indices $v=(v_1,\ldots,v_n)$ so that the images of $x^v=x_1^{v_1}\cdots x_n^{v_n}$ $(v\in B)$ in A_0 generate the $\mathbf{F}$-vector space A_0. Let's prove the images of x^v $(v\in B)$ in A_y also generate the $\mathbf{F}$-vector space A_y. For any monomial x^u, we can write

$$x^u=\sum_{v\in B}a_vx^v+\sum_{i=1}^n g_i\frac{\partial f}{\partial x_i}(x,0),$$

where $a_v\in\mathbf{F}$ and $g_i\in\mathbf{F}[x]$ are polynomials of degree at most $|u|-(m-1)$. We have

$$x^u=\sum_{v\in B}a_vx^v+\sum_{i=1}^n g_i\frac{\partial f}{\partial x_i}(x,y)+\sum_{i=1}^n g_i\left(\frac{\partial f}{\partial x_i}(x,0)-\frac{\partial f}{\partial x_i}(x,y)\right).$$

Note that as a polynomial of x, $f(x,0)-f(x,y)$ has degree $\leq m-1$. So each $\frac{\partial f}{\partial x_i}(x,0)-\frac{\partial f}{\partial x_i}(x,y)$ has degree strictly less than $m-1$ in terms of x. Thus each $g_i(\frac{\partial f}{\partial x_i}(x,0)-\frac{\partial f}{\partial x_i}(x,y))$ has degree strictly less than $|u|$. By induction on $|u|$, we see that any x^u can be written as

$$x^u=\sum_{v\in B}b_vx^v+\sum_{i=1}^n h_i\frac{\partial f}{\partial x_i}(x,y),$$

where $b_v\in\mathbf{F}$ and $h_i\in\mathbf{F}[x]$. So the images of x^v $(v\in B)$ in A_y generate the $\mathbf{F}$-vector space A_y.

We have

$$\begin{aligned}
\dim\mathbf{F}[x] &= n,\\
\dim\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y)) &\geq \dim\mathbf{F}[x]-1,\\
\dim\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y),\frac{\partial f}{\partial x_2}(x,y)) &\geq \dim\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y))-1,\\
&\vdots\\
\dim\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y),\ldots,\frac{\partial f}{\partial x_n}(x,y)) &\geq \dim\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y),\ldots,\frac{\partial f}{\partial x_{n-1}}(x,y))-1.
\end{aligned}$$

Since $A_y=\mathbf{F}[x]/(\frac{\partial f}{\partial x_1}(x,y),\ldots,\frac{\partial f}{\partial x_n}(x,y))$ is a finite dimensional $\mathbf{F}$-algebra, we have $\dim A_y=0$. This implies that all the inequalities above are equalities. Our assertion then follows from the Bézout theorem ([4] Theorem I.7.7).

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