Newton Polygons of Zeta Functions and L-Functions

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§0. Introduction

In this paper, we give a systematic treatment of Newton polygons of exponential sums. The Newton polygon is a nice way to describe p-adic values of the zeros or poles of zeta functions and L-functions. Our main object is the Adolphson-Sperber conjecture [2] which asserts that under a simple condition the generic Newton polygon of L-functions coincides with its lower bound. We show that the conjecture is false in its full form but true in a slightly weaker form. We also show that the full form is true in various important special cases. For example, we show that for a generic projective hypersurface of degree d, the Newton polygon of the interesting part of the zeta function coincides with its lower bound (the Hodge polygon). This gives a p-adic proof of a recent theorem of Illusie, conjectured by Dwork and Mazur. For more examples, let us consider the family of affine hypersurfaces of degree d or the family of affine hypersurfaces defined by polynomials $f(x_1, \cdots, x_n)$ of degree d_i with respect to x_i $(1 \leq i \leq n)$, where the d_i are fixed positive integers. Then for all large prime number p, the generic Newton polygon for the zeta functions of each of the two families of hypersurfaces coincides with its lower bound. Our main results are several decomposition theorems obtained using certain maximizing functions from linear programming. In particular, this suggests a possible connection between Newton polygons and resolution of singularities of toric varieties.

Let p be a prime, $q = p^a$, $\mathbf{F}_q$ the finite field of q elements, and $\mathbf{F}_{q^m}$ its extension of degree m. Fix a nontrivial additive character Ψ of $\mathbf{F}_p$. For any Laurent polynomial $f(x_1, \cdots, x_n) \in \mathbf{F}_q[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, we form the exponential sum

$$\mathrm{S}_m^*(f) = \sum \Psi(\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(f(x_1, \cdots, x_n))), \tag{0.1}$$

where the sum is over all $(x_1, \cdots, x_n) \in (\mathbf{F}_{q^m}^*)^n$. By well-known theorems of Dwork-Bombieri-Grothendieck, the following L-function is a rational function:

$$\mathrm{L}^*(f, t) = \exp\Big(\sum_{m=1}^{\infty} \mathrm{S}_m^*(f) \frac{t^m}{m}\Big) = \frac{\prod_i (1 - \alpha_i t)}{\prod_j (1 - \beta_j t)}. \tag{0.2}$$

Equivalently,

$$S_m^*(f) = \beta_1^m + \beta_2^m + \cdots - \alpha_1^m - \alpha_2^m - \cdots,$$

where the α_i and β_j are algebraic integers. Thus, finding a sharp p-adic estimate for the sums $S_m^*(f)$ is reduced to determining the p-adic values of the reciprocal roots α_i and the reciprocal poles β_j. This question can best be described in terms of Newton polygons, see [11].

The Laurent polynomial f is a sum of monomials $f = \sum_j a_j x^{V_j}$ and as such has a well defined Newton polyhedron $\Delta(f)$ at infinity. This is the convex closure in $\mathbf{R^n}$ generated by the origin and the lattice points V_j. Assuming that f is nondegenerate, Adolphson and Sperber [2] proved that $L^*(f,t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathbf{V}(f)$, where $\mathbf{V}(f)$ is the volume of $\Delta(f)$. Furthermore [2], the Newton polygon of $L^*(f,t)^{(-1)^{n-1}}$ lies above a certain explicit combinatorial lower bound $P(\Delta)$ involving lattice points in $\Delta(f)$ and its rational multiples $r\Delta(f)$. Adolphson and Sperber [2] conjectured that the Newton polygon coincides generically with its lower bound if p lies in a certain residue class.

To describe this conjecture more fully, we fix such an n-dimensional integral Newton polyhedron Δ in $\mathbf{R}^n$ containing the origin. For each face σ_i of codimension 1 not containing the origin, let $\sum_{k=1}^n \alpha_{ik} X_k = 1$ be the equation of the unique hyperplane passing through σ_i. The α_{ik} are rational numbers. Define the denominator $D(\Delta)$ of Δ to be the least common denominator of all the α_{ik}. Let $\mathcal{M}_p(\Delta)$ be the moduli space of all Laurent polynomials over $\bar{\mathbf{F}}_p$ with $\Delta(f)$ contained in Δ. It is clear that $\mathcal{M}_p(\Delta)$ is isomorphic to the affine space $(\bar{\mathbf{F}}_p)^b$ for a certain positive integer b. Hence, $\mathcal{M}_p(\Delta)$ is an irreducible quasi-projective variety. Let $\mathcal{H}_p(\Delta)$ be the moduli space of those $f \in \mathcal{M}_p(\Delta)$ such that $\Delta(f) = \Delta$, f is nondegenerate with respect to Δ and the Newton polygon of $L^*(f,t)^{(-1)^{n-1}}$ coincides with the given lower bound $P(\Delta)$. In Dwork's terminology, $\mathcal{H}_p(\Delta)$ is called the Hasse domain of the generic Laurent polynomial f defined over $\bar{\mathbf{F}}_p$ with $\Delta(f)$ contained in Δ. It is not difficult to prove that $\mathcal{H}_p(\Delta)$ is a Zariski open subset of $\mathcal{M}_p(\Delta)$ (possibly empty). Moreover (see the end of §2), the complement of $\mathcal{H}_p(\Delta)$ in $\mathcal{M}_p(\Delta)$ is an affine variety defined by a polynomial in the variables a_j, called the Hasse polynomial and denoted by $h_p(\Delta)$. A basic question is whether or not $\mathcal{H}_p(\Delta)$ is Zariski dense in $\mathcal{M}_p(\Delta)$, in other words, whether or not the Hasse polynomial $h_p(\Delta)$ is non-zero. Based on earlier results of Sperber [12-14] and Carpentier [4] about p-adic estimates of exponential sums, Adolphson and Sperber [2] made the following

Conjecture 1. *For all large prime p satisfying $p \equiv 1 \pmod{D(\Delta)}$, the Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ coincides generically with its lower bound, i.e., for such p, $\mathcal{H}_p(\Delta)$ is Zariski dense in the moduli space $\mathcal{M}_p(\Delta)$.*

It can be shown that Conjecture 1 generalizes a conjecture of Dwork [6, p.40] and Mazur [11, p.661] (for large p) which asserts that for a general hypersurface X of degree d, the Newton polygon of the interesting part of the zeta function $\mathrm{Z}(X,t)$ coincides with its lower bound (the Hodge polygon). Katz (unpublished, but see [11]) proved that the conjecture is true for the first slope. Miller (see [9]) and Koblitz [9] proved that the conjecture is true for the horizontal side. In his second paper on the Weil conjectures, Deligne [5] announced an ℓ-adic proof of the Dwork-Mazur conjecture. However, Deligne's proof has not been published. Recently, Illusie [10] found a new proof using crystalline cohomology. Similar results on other special families of varieties (curves of genus g, polarized abelian varieties of dimension g, K_3 surfaces) were obtained by Miller, Koblitz, Mumford, Norman-Oort and Ogus, see [10] and [11]. One consequence of this paper is a p-adic proof of the Dwork-Mazur conjecture.

In this paper, we shall prove

Theorem 2. *Conjecture 1 is false for all $n \geq 5$.*

Our counterexample of Conjecture 1 is an n-dimensional simplex ($n \geq 5$) on which there are no lattice points other than the vertices. Thus, the general Laurent polynomial in this case is simply the "general diagonal form". One can then compute the L-function explicitly and use Stickelberger's theorem.

One of the main results of this paper is

Theorem 3. *There is an integer $D^*(\Delta)$ depending only on Δ such that if p is a large prime and $p \equiv 1 \pmod{D^*(\Delta)}$, then the Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ coincides generically with its lower bound, i.e., for such p, $\mathcal{H}_p(\Delta)$ is Zariski dense in $\mathcal{M}_p(\Delta)$.*

This theorem shows that the Adolphson-Sperber conjecture is not too far from being true. Their conjecture is equivalent to the statement that one can take $D^*(\Delta)$ to be $D(\Delta)$ (the ideal case). Our main results are several decomposition theorems. These decomposition theorems imply

Theorem 3. They can also be used to prove that the Adolphson-Sperber conjecture is true in various special cases. In this direction, we have the following results.

Theorem 4. *Conjecture 1 is true for* $n = 2$.

For applications to zeta functions, we have

Theorem 5. *For a general projective hypersurface* X *of degree* d*, the Newton polygon of the interesting part of the zeta function* $\mathrm{Z}(X,t)$ *coincides with its lower bound (this gives a p-adic proof of the Dwork-Mazur conjecture).*

Theorem 6. *For a general affine hypersurface* X *of degree* d*, the Newton polygon of the interesting part (if it is a polynomial) of the zeta function* $\mathrm{Z}(X,t)$ *coincides with its lower bound.*

Theorem 7. *Let* X *be a general affine hypersurface of degree* d_i *with respect to* x_i $(1 \leq i \leq n)$*. Then, the Newton polygon of the interesting part (if it is a polynomial) of the zeta function* $\mathrm{Z}(X,t)$ *coincides with its lower bound.*

A harder question is to classify all large prime numbers p for which $\mathcal{H}_p(\Delta)$ is Zariski dense in $\mathcal{M}_p(\Delta)$. In the last section, we give a conjectural answer to this question which is related to Frobenius coverings of lattice points.

We now sketch the main ideas in proving Theorem 3. In the above, it is mentioned that $\mathcal{H}_p(\Delta)$ is a Zariski open subset of $\mathcal{M}_p(\Delta)$ (possibly empty). Thus, Theorem 3 is reduced to finding a single example (for each Δ) with desired Newton polygon. If Δ is a simplex, then the "diagonal forms" provide the desired examples. Thus, Theorem 3 is true for a simplex. For a general Δ, a natural question is whether or not there are decomposition theorems which successively reduce a general Δ to a simpler Δ, and eventually to a simplex. Depending on how strong we want the decomposition theorems to be, we have different answers. Ideally, one would hope that $\mathrm{L}^*(f,t)$ decomposes as a product of the L-functions corresponding to the restrictions of f to various pieces in a decomposition of Δ. We shall give such a theorem corresponding to the trivial boundary decomposition of Δ. This theorem does not reduce Theorem 3 from a general Δ to a simpler Δ, but it is useful sometimes (for example, in reducing a projective case to an affine case).

If we relax our requirments on the decomposition theorems a little and are concerned only that the Newton polygon coincides with its lower bound, then there is a non-trivial decomposition theorem obtained in [15]. This theorem reduces Theorem 3 from a general Δ to a simpler Δ having only one codimension 1 face which does not contain the origin. To describe this more precisely, let σ be an $(n-1)$-dimensional closed face of Δ which does not contain the origin and let f_σ be the restriction of f to the closed polyhedron generated by σ and the origin. If f is nondegenerate, then f_σ is nondegenerate. Thus, the theorem of Adolphson and Sperber shows that $\mathrm{L}^*(f_\sigma, t)^{(-1)^{n-1}}$ is a polynomial, whose Newton polygon lies above its lower bound $P(\sigma)$. The main result of [15] is the following facial decomposition theorem.

Theorem 8. *Let f be non-degenerate. Then the Newton polygon of $\mathrm{L}^*(f, t)^{(-1)^{n-1}}$ coincides with its lower bound $P(\Delta)$ if and only if the Newton polygon of $\mathrm{L}^*(f_\sigma, t)^{(-1)^{n-1}}$ coincides with its lower bound $P(\sigma)$ for all $(n-1)$-dimensional closed faces σ which do not contain the origin.*

If we further relax our requirements on the decomposition theorems and are concerned only that the Newton polygon coincides generically with its lower bound, then there are further non-trivial decomposition theorems. In this paper, we establish two such theorems corresponding to two types of decompositions of Δ. The first type is the star decomposition of Δ with respect to a lattice point in Δ. In this context, Theorem 8 can be viewed as a special case, corresponding to the star decomposition with respect to the origin. The second type is the hyperplane decomposition obtained by a hyperplane cut of Δ. The main technical idea is to introduce a degree polygon which involves the degree of the leading form of the Hasse polynomial $h_p(\Delta)$ in terms of certain variables. This degree polygon is finer than the Newton polygon. We shall give an upper bound for the degree polygon and prove decomposition theorems about the coincidence of the degree polygon with its upper bound for the two types of decompositions of Δ. These decompositions are fine enough to reduce a general Δ to a simplicial Δ. This immediately implies Theorem 3. The proof of Theorems 5-7 is obtained by a suitable sequence of combinations of the star decomposition and the hyperplane decomposition so that $D^*(\Delta) = 1$. In particular, the proof of Theorem 5 uses a decomposition construction on resolution of singularities of toric varieties. One might expect a deeper and more general connection bewteen generic Newton polygons and resolution of singularities.

As in [15], we shall work on the chain level and avoid any direct use of Dwork cohomology. Only at the last step will we appeal to the theorem of Adolphson and Sperber about Dwork cohomology which asserts that $\mathrm{L}^*(f, t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathbf{V}(\Delta)$. This will make the paper more elementary and self-contained. Another advantage of working on the chain level is that we do not need any non-degeneracy assumptions. Thus, the results hold for all f and all prime numbers p. All of the above results hold for a more general L-function $\mathrm{L}(f,t)$ studied by Adolphson and Sperber [2], see §8. Adolphson and Sperber [3] have recently extended their conjecture to twisted exponential sums. Our counterexample shows that this more general conjecture is also false. We expect that many of the results in the present paper generalize to these twisted sums.

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1. Counterexamples

In this section, we prove Theorem 2 by giving counterexamples of Conjecture 1. We begin with some observations on diagonal forms. Let $0, V_1, \cdots, V_n$ be the vertices of an n-dimensional integral simplex Δ. Consider the solutions of the equation

$$\sum_{j=1}^{n} u_j V_j \equiv 0 \pmod 1, \ 0 \leq u_j < 1, \ u_j \text{ rational}. \tag{1.1}$$

There is a Frobenius action on the solutions of (1.1). If $u = (u_1, \cdots, u_n)$ is a solution of (1.1), then the Frobenius action $\{pu\} = (\{pu_1\}, \cdots, \{pu_n\})$ is also a solution of (1.1), where $\{u_i\}$ is defined to be the reduction of u_i modulo 1. In other words, $\{u_i\}$ is simply the fractional part of u_i. Define the weight of u to be $\sum_{i=1}^{n} u_i$. We say that the weight function is stable under the Frobenius action if for all solutions u of (1.1),

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} \{pu_i\}.$$

Let $f(x)$ be the generic diagonal form

$$f(x) = \sum_{j=1}^{n} a_j x^{V_j}.$$

The explicit computation in [15] and the Stickelberger relation show that the Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ coincides with its lower bound if and only if the weight function is stable under the Frobenius action. We now use this fact to construct counterexamples of Conjecture 1.

Let us consider the 6-dimensional simplex Δ whose non-zero vertices are the column vectors of the following matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Denote these non-zero vertices by $V_1, \cdots, V_6$. Clearly, all of the non-zero vertices lie on the hyperplane $x_1 = 1$. It follows that $D(\Delta) = 1$. One checks that there are no lattice points on Δ except for the vertices. Thus, the generic Laurent polynomial for this simplex is simply the generic diagonal form

$$f(x) = \sum_{j=1}^{6} a_j x^{V_j},$$

where we have omitted the constant term in $f(x)$ since it does not affect the Newton polygon. The determinant of the matrix M is easily seen to be 3. Thus, $\mathrm{L}^*(f,t)^{-1}$ is a polynomial of degree 3, divisible by $(1-t)$. There are only three lattice points contained in the fundamental domain $\mathbf{R}^6/\mathbf{Z}V_1 + \cdots + \mathbf{Z}V_6$. One of them is the origin. The other two are

$$(2,1,1,1,1,1) = \sum_{i=1}^{6} \frac{1}{3} V_i, \quad (4,2,2,2,2,2) = \sum_{i=1}^{6} \frac{2}{3} V_i. \tag{1.2}$$

These two lattice points have weight 2 and 4, respectively. The Adolphson-Sperber lower bound is the polygon in the plane with vertices $(0,0)$, $(1,0)$, $(2,2)$ and $(3,6)$, whose first non-zero slope is 2. If $p \equiv 2 \pmod 3$, then one checks that the Frobenius map takes any one of the points in (1.2) to the other point. Thus, the weight function is not stable under the Frobenius action. This shows that the generic Newton polygon lies strictly above its lower bound. In fact, in this case the generic Newton polygon of $\mathrm{L}^*(f,t)^{-1}$ is the polygon in the plane with vertices $(0,0)$, $(1,0)$ and $(3,6)$, whose first non-zero slope is 3. This proves that Conjecture 1 is false if $n = 6$.

To construct counterexamples in higher dimensions, we can simply let Δ be the integral simplex generated by the origin and the column vectors of the matrix

$$M^* = \begin{pmatrix} M & B \\ 0 & I \end{pmatrix},$$

where M is the above matrix of order 6, I is the identity matrix of order $(n-6)$, the first row of B is $(1,1,\cdots,1)$ and all other rows of B are zero. Thus, the first row of M^* is $(1,1,\cdots,1)$ and $D(\Delta)=1$. One checks that there are no lattice points on Δ other than the vertices. However, if $p \equiv 2 \pmod 3$, the above argument shows that the generic Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ lies strictly above its lower bound.

A less obvious counterexample in dimension 5 is the simplex whose non-zero vertices are the column vectors V_i of the following matrix:

$$M' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Clearly, $D(\Delta)=1$. One computes that the determinant of the matrix M' has absolute value 3. A direct proof shows that there are no lattice points on Δ except for the vertices. Thus, $\mathrm{L}^*(f,t)$ is a polynomial of degree 3, divisible by $(1-t)$. There are only three lattice points contained in the fundamental domain $\mathbf{R}^5/\mathbf{Z}V_1+\cdots+\mathbf{Z}V_5$. One of them is the origin. The other two are

$$(2,1,1,1,1)=\frac{2}{3}V_1+\sum_{i=2}^{5}\frac{1}{3}V_i, \quad (3,2,2,2,2)=\frac{1}{3}V_1+\sum_{i=2}^{5}\frac{2}{3}V_i. \tag{1.3}$$

These two lattice points have weight 2 and 3, respectively. One checks that if $p \equiv 2 \pmod 3$, then the weight function is not stable under Frobenius action. Thus, the generic Newton polygon lies strictly above its lower bound. Theorem 2 is proved.

Let D^* be the smallest positive integer for which Conjecture 1 is true. We can construct examples which show that the quotient $D^*/D(\Delta)$ is unbounded as n varies. Thus, D^* depends not only on $D(\Delta)$ but also on the geometric shape of Δ.

§2. Dwork's p-Adic Theory

In this section, following Adolphson and Sperber [1] [2] (with some paraphrasing), we shall present their version of Dwork's p-adic theory describing L-functions in terms of the Fredholm determinant of a certain nuclear infinite matrix. We shall prove a new theorem (Theorem 2.4) concerning conditions under which the Newton polygon coincides with its lower bound. This result is the basis of our later discussions.

Let $\mathbf{Q}_p$ be the field of p-adic numbers, and let Ω be the completion of an algebraic closure of $\mathbf{Q}_p$. Let $\mathbf{K}$ denote the unramified extension of $\mathbf{Q}_p$ in Ω of degree a, where $q = p^a$. Let $\pi \in \Omega$ satisfy $\pi^{p-1} = -p$. Then $\Omega_1 = \mathbf{Q}_p(\pi)$ is a totally ramified extension of $\mathbf{Q}_p$ of degree $p-1$, in fact, $\Omega_1 = \mathbf{Q}_p(\zeta_p)$, where ζ_p is a primitive p-th root of unity. Let Ω_a be the compositum of Ω_1 and $\mathbf{K}$. Then Ω_a is an unramified extension of Ω_1 of degree a. The residue fields of Ω_a and $\mathbf{K}$ are both $\mathbf{F}_q$, and the residue fields of Ω_1 and $\mathbf{Q}_p$ are both $\mathbf{F}_p$. The Frobenius automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\mathrm{Gal}(\mathbf{K}/\mathbf{Q}_p)$ which is extended to Ω_a by requiring that $\tau(\pi) = \pi$. If ζ is a $(q-1)$-st root of unity in Ω_a, then $\tau(\zeta) = \zeta^p$. Denote by "ord" the additive valuation on Ω normalized by ord p=1, and denote by "ord_q" the additive valuation on Ω normalized by $\mathrm{ord}_q q$=1.

Let $E(t)$ be the Artin-Hasse exponential series:

$$E(t) = \exp(\sum_{m=0}^{\infty} \frac{t^{p^m}}{p^m}) \in (\mathbf{Z}_p \cap \mathbf{Q})[[t]]. \tag{2.1}$$

Let $\gamma \in \Omega_1$ be a root of $\sum_{m=0}^{\infty} \frac{t^{p^m}}{p^m} = 0$ satisfying $\mathrm{ord}\gamma = \mathrm{ord}\pi = \frac{1}{p-1}$. The series

$$\theta(t) = E(\gamma t) = \sum_{m=0}^{\infty} \lambda_m t^m \tag{2.2}$$

is a splitting function in Dwork's terminology, and its coefficients satisfy

$$\mathrm{ord}\lambda_m \geq \frac{m}{p-1}, \quad \lambda_m \in \Omega_1, \tag{2.3}$$

$$\mathrm{ord}\lambda_m = \frac{m}{p-1} \quad \text{for } 0 \leq m \leq p-1. \tag{2.4}$$

Write $V = (v_1, \cdots, v_n) \in \mathbf{Z}^n$, $x = (x_1, \cdots, x_n)$, $x^V = x_1^{v_1} \cdots x_n^{v_n}$. For a Laurent polynomial $f(x_1, \cdots, x_n) \in \mathbf{F}_q[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$, write

$$f = \sum_{j=1}^{J} \bar{a}_j x^{V_j}, \tag{2.5}$$

where $V_j \in \mathbf{Z}^n$ and the $\bar{a}_j$'s are non-zero elements of $\mathbf{F}_q$. Let a_j be the Teichmüller lifting of $\bar{a}_j$ in Ω, i.e., $a_j^q = a_j$. Set

$$F(f, x) = \prod_{j=1}^{J} \theta(a_j x^{V_j}) \tag{2.6}$$

$$F_a(f, x) = \prod_{i=0}^{a-1} F^{\tau^i}(f, x^{p^i}). \tag{2.7}$$

Note that (2.3) implies that $F(f,x)$ and $F_a(f,x)$ are well defined as formal Laurent series in $x_1, \cdots, x_n$ with cofficients in Ω_a.

To describe the growth conditions satisfied by F, write

$$F(f,x) = \sum_{r \in \mathbf{Z}^n} F_r(f) x^r. \tag{2.8}$$

Then from (2.2) and (2.6), one checks that

$$F_r(f) = \sum_u (\prod_{j \in J} \lambda_{u_j} a_j^{u_j}), \tag{2.9}$$

where the outer sum is over all solutions of the linear system

$$\sum_{j=1}^{J} u_j V_j = r, \ u_j \geq 0, \ u_j \text{ integral}. \tag{2.10}$$

Thus, $F_r(f) = 0$ if (2.10) has no solutions. Otherwise, (2.3) implies that

$$\mathrm{ord}(F_r(f)) \geq \frac{1}{p-1} \inf_u \{\sum_{j=1}^{J} u_j\}, \tag{2.11}$$

where the inf is taken over all solutions of (2.10).

For a rational point $r \in \mathbf{Z}^n$, Adolphson and Sperber [1] defined a weight function $w(r)$ by

$$w(r) = \inf_u \{\sum_{j=1}^{J} u_j\}, \tag{2.12}$$

where now the inf is taken over all nonnegative rational solutions of (2.10). Then (2.11) implies that

$$\mathrm{ord}(F_r(f)) \geq \frac{w(r)}{p-1}, \tag{2.13}$$

with the obvious convention that $F_r(f) = 0$ if $w(r) = +\infty$.

The weight function can be described geometrically. Let $\Delta(f)$ be the convex hull in $\mathbf{R^n}$ generated by the V_j and the origin. This is called the Newton polyhedron of f. Let $C(\Delta)$ denote the cone generated by Δ. Then $C(\Delta)$ is the union of all rays emanating from the origin and passing through $\Delta(f)$. Equation (2.10) has a solution u whose components are nonnegative rational numbers if and only if $r \in C(\Delta)$. Thus, $w(r) = +\infty$ if and only if $r \notin C(\Delta)$. If $r \in C(\Delta)$, the ray emanating from the origin and passing through r intersects $\Delta(f)$ in a face that does not contain the origin. Let $\sum_{i=1}^n \alpha_i X_i = 1$ be the equation of a hyperplane passing through this face (this hyperplane is not uniquely determined unless the face has dimension $n-1$). Then, by standard arguments in linear programming,

$$w(r) = \sum_{i=1}^{n} \alpha_i r_i. \tag{2.14}$$

Let $D(\Delta)$ be the least common denominator of all the α_i's over all the $n-1$-dimensional faces of $\Delta(f)$ which do not contain the origin. Then

$$w(\mathbf{Z}^n) \subseteq \frac{1}{D(\Delta)} \mathbf{Z}_{\geq 0} \cup \{+\infty\}, \tag{2.15}$$

where $\mathbf{Z}_{\geq 0}$ denotes the set of nonnegative integers. From the above description of $w(r)$, it is immediately seen that

Lemma 2.1. *(a) For any nonnegative rational number k, $w(kr) = kw(r)$.*

(b) $w(r+r') \leq w(r) + w(r')$, with equality holding if and only if r and r' are cofacial, i.e., $w(r)^{-1} r$ and $w(r')^{-1} r'$ lie on the same closed face (of dimension at most $n-1$) of Δ.

(c) Let $u = (u_1, \cdots, u_J)$ be a rational solution of (2.10). Let u_{j_i} be its non-zero coordinates. Then $w(r) = \sum_j u_j$ if and only if the non-zero lattice points among the V_{j_i} lie on the same closed face of codimension 1 not containing the origin.

Let $M(f)$ be the set of lattice points in the cone $C(\Delta)$. For real numbers b and c with $0 < b \leq p/(p-1)$, define the following spaces of p-adic functions:

$$\mathcal{L}(b,c) = \{ \sum_{r \in M(f)} C_r x^r \mid C_r \in \Omega_a, \ \mathrm{ord}_p C_r \geq bw(r) + c\}$$

$$\mathcal{L}(b) = \cup_{c \in \mathbf{R}} \mathcal{L}(b,c).$$

One checks that

$$F(f,x) \in \mathcal{L}(\frac{1}{p-1}, 0), \quad F_a(f,x) \in \mathcal{L}(\frac{p}{q(p-1)}, 0).$$

Define an operator ψ on formal Laurent series by

$$\psi(\sum_{r \in M(f)} C_r x^r) = \sum_{r \in M(f)} C_{pr} x^r.$$

It is immediate from Lemma 2.1(a) that $\psi(\mathcal{L}(b,c)) \subset \mathcal{L}(pb, c)$. It follows that the operator $\alpha_a = \psi^a \circ F_a(f,x)$ is an Ω_a-linear endomorphsim of the space $\mathcal{L}(b)$ and the operator $\alpha_1 = \tau^{-1} \circ \psi \circ F(f,x)$

is an Ω_a-semilinear (τ^{-1}-linear) endomorphsim of the space $\mathcal{L}(b)$. The operators α_a^m and α_1^m have well defined traces. The Dwork trace formula asserts that

$$\mathrm{S}_m^*(f) = (q^m - 1)^n \mathrm{Tr}(\alpha_a^m).$$

One checks that

$$\begin{aligned}
\alpha_1^a &= \alpha_1^{a-2}\tau^{-1} \circ \psi \circ F(f,x) \circ \tau^{-1} \circ \psi \circ F(f,x) \\
&= \alpha_1^{a-2}\tau^{-1} \circ \psi \circ \tau^{-1} \circ F^{\tau}(f,x) \circ \psi \circ F(f,x) \\
&= \alpha_1^{a-2}(\tau^{-1})^2 \circ \psi^2 \circ F^{\tau}(f,x^p) \circ F(f,x) \\
&= \cdots = \psi^a \circ F_a(f,x) = \alpha_a.
\end{aligned}$$

We now describe the matrix form of the operators α_1 and α_a. Fix a choice $\pi^{1/D}$ of D-th root of π in Ω. Define a space of functions

$$\mathcal{B} = \{ \sum_{r \in M(f)} C_r \pi^{w(r)} x^r \mid C_r \in \Omega_a(\pi^{1/D}),\ C_r \to 0 \text{ as } r \to \infty \}.$$

Then, the monomials $\pi^{w(r)}x^r$ form an orthonormal basis of the p-adic Banach space $\mathcal{B}$. Furthermore, if $b > 1/(p-1)$, then $\mathcal{L}(b) \subseteq \mathcal{B}$. The operator α_a (resp. α_1) is an Ω_a-linear (resp. Ω_a-semilinear) nuclear endomorphsim of the space $\mathcal{B}$. Let Γ be the orthonormal basis $\{\pi^{w(r)}x^r\}_{r \in M(f)}$ written as a column vector. The operator α_1 is given by $\alpha_1 \Gamma = A_1(f)\Gamma$, where $A_1(f)$ is the infinite matrix whose rows are indexed by s and columns are indexed by r.

$$A_1(f) = (a_{s,r}(f)) = (F_{ps-r}(f)\pi^{(w(r)-w(s))}). \tag{2.16}$$

Since $\alpha_a = \alpha_1^a$ and α_1 is τ^{-1}-linear, the operator α_a is given by

$$\alpha_a \Gamma = \alpha_1^a \Gamma = \alpha_1^{a-1} A_1 \Gamma = \alpha_1^{a-2} A_1^{\tau^{-1}} A_1 \Gamma = A_1^{(\tau^{-1})^{a-1}} \cdots A_1^{\tau^{-1}} A_1 \Gamma.$$

To simplify notation, let

$$A_a(f) = \prod_{i=0}^{a-1} A_1(f)^{\tau^i} = A_1 A_1^{\tau^1} \cdots A_1^{\tau^{a-1}}. \tag{2.17}$$

Then, the matrix of α_a under the basis Γ is $A_a^{\tau^{-1}}(f)$. Since we will work exclusively with matrices later on, for simplicity we call $A_1(f) = (a_{s,r}(f))$ the infinite semilinear Frobenius matrix and $A_a(f)$ the infinite linear Frobenius matrix.

By (2.13), (2.16) and Lemma 2.1, we obtain the estimate

$$\mathrm{ord} a_{s,r}(f) \geq \frac{w(ps-r)+w(r)-w(s)}{p-1} \geq w(s). \tag{2.18}$$

Thus, $A_1(f)$ and $A_a(f)$ are nuclear matrices. They all have well defined traces. In addition, the Fredholm determinants $\det(I - tA_a)$ and $\det(I - tA_1)$ are well defined and p-adically entire (i.e., convergent for all $t \in \Omega$). The Dwork trace formula asserts that

$$\mathrm{S}_m^*(f) = (q^m - 1)^n \mathrm{Tr}^{\tau^{-1}}(A_a(f)^m). \tag{2.19}$$

This form is slightly different from the usual form because we want to work with $A_a(f)$ instead of the Frobenius matrix $A_a^{\tau^{-1}}(f)$ in order to simplify notations later on. The nontrivial additive character Ψ implicit in the left side of (2.19) (see (0.1)) is given by

$$\Psi(t) = \theta(1)^t \tag{2.20}$$

for $t \in \mathbf{F}_p$ ($\theta(1)$ is a primitive p-th root of unity). An equivalent form of (2.19) is

$$\mathrm{L}^*(f,t)^{(-1)^{n-1}} = \prod_{i=0}^{n} \det{}^{\tau^{-1}}(I - tq^i A_a(f))^{(-1)^i \binom{n}{i}}, \tag{2.21}$$

where

$$\det(I - tA_a(f)) = \exp\Big(-\sum_{m=1}^{\infty} \frac{\mathrm{Tr}(A_a(f)^m) t^m}{m}\Big). \tag{2.22}$$

If $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ is a polynomial, then (2.21) shows that the determination of the Newton polygon of $\det(I - tA_a(f))$ will enable us to determine the Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$. Define a weight function $W(k)$ on non-negative integers as follows:

$$W(k) = \mathrm{card}\{r \in \mathbf{Z}^n \mid w(r) = \frac{k}{D(\Delta)}\}. \tag{2.23}$$

Let $P(\Delta)$ be the Newton polygon with vertices $(0,0)$ and

$$\Big(\sum_{k=0}^{m} W(k), \frac{1}{D(\Delta)} \sum_{k=0}^{m} kW(k)\Big), \quad m = 0, 1, 2, \cdots. \tag{2.24}$$

The lower bound of Adolphson and Sperber [1] is

Proposition 2.2. *The Newton polygon of* $\det(I - tA_a(f))$ *computed with respect to* q *lies above* $P(\Delta)$.

Let $\xi \in \Omega$ be such that $\xi^D = \pi^{p-1} = -p$; then $\mathrm{ord}\xi = 1/D$. By (2.18), $A_1(f)$ has the block form

$$A_1(f) = \begin{pmatrix} A_{00} & A_{01} & \dots & A_{0i} & \dots \\ \xi^1 A_{10} & \xi^1 A_{11} & \dots & \xi^1 A_{1i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \xi^i A_{i0} & \xi^i A_{i1} & \dots & \xi^i A_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}, \tag{2.25}$$

where the A_{ij}'s are matrices of $W(i)$ rows and $W(j)$ columns whose entries are p-adic integers in Ω. This block form implies

Proposition 2.3. *The Newton polygon of det$(I - tA_1(f))$ computed with respect to p lies above $P(\Delta)$.*

In general, we cannot expect that the Newton polygons of $\det(I - tA_a(f))$ and $\det(I - tA_1(f))$ are the same (even though they have the same lower bound). For a counterexample in the case of F-crystals, see Katz [7]. However, if we are only interested in the question of whether or not the Newton polygon of $\det(I - tA_a(f))$ coincides with its lower bound, the following theorem shows that we can restrict ourselves to $\det(I - tA_1(f))$. This result is used in [15] without proof.

Theorem 2.4. *The Newton polygon of det$(I - tA_a(f))$ computed with respect to q coincides with $P(\Delta)$ if and only if the Newton polygon of det$(I - tA_1(f))$ computed with respect to p coincides with $P(\Delta)$.*

Proof. We prove the theorem by induction on the sides. The first side of the Newton polygon of $\det(I - tA_1(f))$ coincides with the first side of $P(\Delta)$ if and only if $\det A_{00} \not\equiv 0 \pmod{\xi}$. To simplify notations, we use the convention that mod ξ (or mod p) means the reduction modulo the maximal ideal in the ring of integers in Ω. We can write

$$A_a(f) \equiv \begin{pmatrix} A_{00} \cdots A_{00}^{\tau^{a-1}} & * \\ 0 & 0 \end{pmatrix} (\mathrm{mod}\xi). \tag{2.26}$$

It follows that the first side of the Newton polygon of $\det(I - tA_a(f))$ coincides with the first side of $P(\Delta)$ if and only if $\det A_{00} \not\equiv 0 \pmod{\xi}$. Thus, the theorem is true for the first side.

Assume that $\det A_{00} \not\equiv 0 \pmod{\xi}$. We now use a triangulation procedure to prove the theorem for the first two sides. Form an elementary nuclear matrix B (to be determined) as follows:

$$B = \begin{pmatrix} I_{00} & 0 & \dots & 0 & \dots \\ \xi^1 B_{10} & I_{11} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \xi^i B_{i0} & 0 & \dots & I_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}. \tag{2.27}$$

Its inverse is

$$B^{-1} = \begin{pmatrix} I_{00} & 0 & \dots & 0 & \dots \\ -\xi^1 B_{10} & I_{11} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ -\xi^i B_{i0} & 0 & \dots & I_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}, \tag{2.28}$$

where B_{i0} is a matrix of $W(i)$ rows and $W(0)$ columns, and I_{ii} is the identity matrix of order $W(i)$. Now the matrix $BA_1B^{-\tau}$ has the form

$$BA_1B^{-\tau} = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0i} & \dots \\ \xi^1 C_{10} & \xi^1 C_{11} & \dots & \xi^1 C_{1i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \xi^i C_{i0} & \xi^i C_{i1} & \dots & \xi^i C_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}, \tag{2.29}$$

where

$$\begin{aligned} &C_{00} = A_{00} - \xi^1 A_{01} B_{10}^\tau - \dots - \xi^k A_{0k} B_{k0}^\tau - \cdots, \\ &C_{i0} = B_{i0}A_{00} + A_{i0} - \xi^1\{B_{i0}A_{01} + A_{i1}\}B_{10}^\tau - \dots - \xi^k\{B_{i0}A_{0k} + A_{ik}\}B_{k0}^\tau - \cdots \ (i > 0), \\ &C_{0j} = A_{0j},\ C_{ij} = B_{i0}A_{00} + A_{ij}\ (i \geq 1, j \geq 1). \end{aligned} \tag{2.30}$$

We want to choose suitable matrices B_{i0} such that

$$C_{i0} = 0 \quad \text{for all } i \geq 1. \tag{2.31}$$

Since A_{00} is invertible, (2.31) can be rewritten as

$$\begin{aligned} B_{i0} = -A_{i0}A_{00}^{-1} + \xi^1\{B_{i0}A_{01} + A_{i1}\}B_{10}^\tau A_{00}^{-1} + \cdots \\ +\xi^k\{B_{i0}A_{0k} + A_{ik}\}B_{k0}^\tau A_{00}^{-1} + \cdots. \end{aligned} \tag{2.32}$$

Thus, by successive iteration of (2.32) or by Hensel's lifting lemma or by the fixed point theorem for a contraction map in a Banach space, we deduce that there are solution matrices B_{i0} whose elements are p-adic integers invariant under the action of τ^a.

If we go through the above triangulation procedure again but ignore the action of τ in (2.30) and (2.32), we conclude that there is a nuclear matrix B_1 of the form (2.27), whose entries are p-adic integers, such that

$$B_1 A_1(f) B_1^{-1} = \begin{pmatrix} D_{00} & D_{01} & \dots & D_{0i} & \dots \\ 0 & \xi^1 D_{11} & \dots & \xi^1 D_{1i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & \xi^i D_{i1} & \dots & \xi^i D_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}, \tag{2.33}$$

where D_{ij} is a matrix of the same size as C_{ij}. It follows from (2.30) that $C_{ij} - D_{ij}$ is divisible by ξ. Comparing (2.33) with (2.29), we conclude that the Newton polygon of $\det(I - tA_1(f))$ coincides with $P(\Delta)$ if and only if the Newton polygon of $\det(I - tBA_1(f)B^{-\tau})$ coincides with $P(\Delta)$. Since

$$BA_a(f)B^{-1} = BA_a(f)B^{-\tau^a} = (BA_1(f)B^{-\tau}) \cdots (BA_1(f)B^{-\tau})^{\tau^{a-1}},$$

using the triangular form obtained above, we deduce that the first two sides of the Newton polygon of $\det(I - tA_1(f))$ computed with respect to p coincide with $P(\Delta)$ if and only if the first two sides of the Newton polygon of $\det(I - tA_a(f))$ computed with respect to q coincide with $P(\Delta)$. Repeating the above argument, by induction we see that the theorem is true. Actually, the rationality of $\mathrm{L}^*(f, t)$ and the trivial archimedean estimate on its zeros and poles imply that we only need to prove the theorem for the first nD sides.

We shall use the convention that the first m sides of $P(\Delta)$ mean the sides in $P(\Delta)$ with slopes at most m/D. Similarly, the m^{th} side in $P(\Delta)$ means the side with slope equal to m/D (even though this side may not exist). The above proof implies

Corollary 2.5. *The Newton polygon of det$(I - tA_a(f))$ computed with respect to q coincides with $P(\Delta)$ for the first m sides if and only if the Newton polygon of det$(I - tA_1(f))$ computed with respect to p coincides with $P(\Delta)$ for the first m sides.*

Let $\mathcal{H}_p(\Delta, m)$ (resp. $\mathcal{H}_p(\Delta, \leq m)$) be the moduli space of $f \in \mathcal{M}_p(\Delta)$ for which the m^{th} side (resp. the first m sides) of the Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound. Let $A(m)$ be the matrix of the first $\sum_{k \leq m} W(k/D)$ rows and columns in $A_1(f)$. The space $\mathcal{H}_p(\Delta, m)$ is simply the compliment of the hypersurface defined by the determinant equation

$$h_p(\Delta, m) = \frac{\det(A(m))}{\xi^{\sum_{k \leq m} \frac{k}{D} W(\frac{k}{D})}} \equiv 0 \pmod{\xi}.$$

Thus, a defining equation for $\mathcal{H}_p(\Delta, m)$ can be computed in a finite amount of time. We shall call $h_p(\Delta, m)$ the Hasse polynomial of the first m^{th} side. Similarly, the moduli space $\mathcal{H}_p(\Delta, \leq m)$ is simply the intersection of the moduli spaces $\mathcal{H}_p(\Delta, k)$ $(0 \leq k \leq m)$. Its compliment is defined by the product polynomial $h_p(\Delta, \leq m) = \prod_{k \leq m} h_p(\Delta, k)$. Very little is known about these polynomials or the Hasse variety they define. Whether these polynomials are non-zero or not is the content of Conjecture 1. In order to prove that these polynomials are non-vanishing in certain cases, we shall study the stronger question of whether or not the leading form in certain variables vanishes. This is accomplished by introducing certain maximizing functions from linear programming and investigating certain degree polygons defined by these maximizing functions.

§3. Maximizing Functions on Rational Points

In this section, we assume that the origin 0 is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin (for a more general Δ, Theorem 8 in §0 reduces it to the present case).

Let $V_1, \cdots, V_J$ be J fixed lattice points on the face Δ_1 such that the V_j include all the vertices of Δ_1. Then Δ is generated by 0 and the V_j. Let $C(\Delta)$ be the cone generated by Δ.

Definition 3.1. For a rational point $r \in C(\Delta)$, define

$$m(V_1, \Delta; r) = \sup\{u_1 \mid \sum_{j=1}^{J} u_j V_j = r, u_j \geq 0, u_j \text{ rational}\}.$$

This function is defined with respect to V_1 (not necessarily a vertex). One can similarly define $m(V_j, \Delta; r)$ with respect to any other lattice point V_j. In the following, $V_1 r$ denotes the line passing through the points V_1 and r. If r lies on the line $V_1 0$, then $V_1 r = V_1 0$. The object of this section is to determine when a solution of the equation

$$\sum_{j=1}^{J} u_j V_j = r, u_j \geq 0, u_j \text{ rational}, \tag{3.1}$$

makes u_1 maximal. Let $\Delta^* = \Delta^*(V_1, r)$ be the face of largest codimension of Δ_1 such that Δ^* contains the point of intersection of the ray joining V_1 and $\frac{1}{w(r)} r$ and the boundary of Cone(Δ). If r lies on the line $V_1 0$, we simply define $\Delta^*(V_1, r)$ to be the lattice point V_1.

Lemma 3.2. *(i). Let $(u_1, \cdots, u_J)$ be a solution of (3.1). Let $u_{j_1}, \cdots, u_{j_k}$ (all $j_i \geq 2$) be its non-zero coordinates (thus, u_1 is not included). Then $m(V_1, \Delta; r) = u_1$ if and only if the lattice points $V_{j_1}, \cdots, V_{j_k}$ belong to $\Delta^*(V_1, r)$, (i.e., the lattice points $V_{j_1}, \cdots, V_{j_k}$ are cofacial with respect to V_1, which means that they lie on the same closed face of Δ not containing V_1).*

(ii). Let r be a rational point in the cone $C(\Delta)$. Let $\Delta'(r)$ be the polyhedron generated by V_1, the origin and $\Delta^*(V_1, r)$. Then, the following identity is true:

$$m(V_1, \Delta; r) = m(V_1, \Delta'(r); r).$$

Proof. We may assume that $r \neq 0$. Since the V_j lie on the unique face Δ_1 of codimension 1 not containing the origin, Lemma 2.1 shows that the sum $\sum_j u_j = w(r)$ depends only on r. Dividing the u_j by $w(r)$, we may assume that $w(r) = 1$. Then r lies on Δ_1. Under these assumptions, equation (3.1) can be rewritten as

$$r - V_1 = \sum_{j=2}^{J} u_j (V_j - V_1).$$

Since $u_1 + \cdots + u_J = 1$, maximizing u_1 is equivalent to minimizing the sum $\sum_{j=2}^{J} u_j$. The latter is simply the weight function of Adolphson and Sperber defined with respect to the $(n-1)$-dimensional space spanned by the $V_j - V_1$ (all $j \geq 2$). Part (c) of Lemma 2.1 implies that the sum $\sum_{j=2}^{J} u_j$ is minimal if and only if the $V_{j_i} - V_1$ are cofacial with respect to the origin. This is equivalent to saying that the V_{j_i} are cofacial with respect to V_1. This proves (i). It is easy to see that (ii) is a restatement of (i). The lemma is proved.

Lemma 3.3. (i). Let r be a rational point in the cone $C(\Delta)$. Then, $m(V_1, \Delta; pr) \geq pm(V_1, \Delta; r)$ for all positive rational numbers p. (ii). Let r_1 and r_2 be two rational points in the cone $C(\Delta)$. Then $m(V_1, \Delta; r_1 + r_2) \geq m(V_1, \Delta; r_1) + m(V_1, \Delta; r_2)$, with equality holding if and only if r_1 and r_2 are cofacial with respect to V_1, i.e., the rays emanating from V_1 and passing through r_1 and r_2 hit the same closed face of $C(\Delta)$ not containing V_1.

Proof. One can easily check (i). We now prove (ii). Let

$$r_1 = u_1 V_1 + \cdots + u_J V_J, \; u_1 = m(V_1, \Delta; r_1),$$

$$r_2 = w_1V_1 + \cdots + w_JV_J, \ w_1 = m(V_1, \Delta; r_2).$$

Then,

$$r_1 + r_2 = (u_1 + w_1)V_1 + \cdots + (u_J + w_J)V_J. \tag{3.2}$$

$$m(V_1, \Delta; r_1) + m(V_1, \Delta; r_2) = u_1 + w_1 \le m(V_1, \Delta; r_1 + r_2). \tag{3.3}$$

Let V_{j_i} (all $j_i \ge 2$) be the lattice points with non-zero coefficients in (3.2). By Lemma 3.2, the inequality in (3.3) is an equality if and only if the V_{j_i} are cofacial with respect to V_1. This happens if and only if r_1 and r_2 are cofacial with respect to V_1. The lemma is proved.

More generally, we can define a maximizing function on rational points with respect to several lattice points.

Definition 3.4. Let T be a subset of the V_j. For a rational point $r \in C(\Delta)$, define

$$m(T, \Delta; r) = \sup\{\sum_{V_j \in T} u_j \mid \sum_{j=1}^{J} u_jV_j = r, u_j \ge 0, u_j \text{ rational}\}.$$

This is a standard maximizing function in linear programming. For given r, there are various algorithms to compute $m(T, \Delta; r)$. The simplex method in linear programming shows that the set of optimal solutions is a "face" formed by some of the lattice points V_j. For our purpose, we need to describe the optimal face in various special cases. A trivial case is when T contains all the V_j. In this case, $m(T, \Delta; r) = w(r)$ (the weight function of Adolphson-Sperber). In the above, we treated the case $T = V_1$. For latter applications, we now study another case in which T is obtained by a hyperplane cut of Δ.

Let H be a hyperplane passing through the origin such that the intersection of H with Δ is an integral polyhedron of dimension $n-1$ whose vertices are contained in the V_j. Then H cuts Δ into two pieces $\Delta(1)$ and $\Delta(2)$. We may suppose that both $\Delta(1)$ and $\Delta(2)$ are closed. Then Δ is the union of $\Delta(1)$ and $\Delta(2)$. Let T be the set of the V_j which are contained in $\Delta(1)$, and let T_H be the intersection of T and H. The following lemma tells when a solution of (3.1) attains the sup in Definition 3.4.

Lemma 3.5. *Let $(u_1, \cdots, u_J)$ be a solution of (3.1). Suppose that $u_{j_1}, \cdots, u_{j_k}$ are its non-zero coordinates. (i). If $r \in C(\Delta(1))$, then $m(T, \Delta; r) = \sum_{V_j \in T} u_j$ if and only if $\Delta(1)$ contains all the* *lattice points V_{j_i}. (ii). If $r \in C(\Delta(2))$ and $m(T,\Delta;r) = \sum_{V_j \in T} u_j$, then $\Delta(2)$ contains all the lattice points V_{j_i}. (iii). If $r \in H$ and $m(T,\Delta;r) = \sum_{V_j \in T} u_j$, then H contains all the lattice points V_{j_i}.*

Proof. (i). Let $r \in C(\Delta(1))$. Since $\sum_j u_j V_j = r$ and $w(r) = \sum_j u_j$, we have

$$\sum_{V_j \in T} u_j \leq \sum_j u_j = w(r).$$

The inequality is an equality if and only if all $V_{j_i} \in T$. This is true if and only if $\Delta(1)$ contains all the lattice points V_{j_i}.

(ii). Let

$$r_1 = \sum_{V_j \in \Delta(1)} u_j V_j, \ r_2 = \sum_{V_j \in \Delta(2) - H} u_j V_j.$$

Then, $r = r_1 + r_2$ and $w(r_1) = \sum_{V_j \in T} u_j$. If $\Delta(2)$ does not contain all of the lattice points V_{j_i}, then $w(r_1) > 0$ and r_1 lies on the other side of $\Delta(2)$ with respect to H. This implies that the line segment from r_1 to r intersects the hyperplane H at a unique point r_1' ($\neq r_1$). That is, there is a unique positive number $0 < \lambda < 1$ such that $\lambda r_1 + (1-\lambda) r = r_1' \in H$. Thus,

$$r = \lambda r + (1-\lambda) r = \lambda(r_1 + r_2) + (1-\lambda) r = (\lambda r_1 + (1-\lambda) r) + \lambda r_2 = r_1' + \lambda r_2.$$

Since $w(r) = w(r_1) + w(r_2) = w(r_1') + \lambda w(r_2)$, where $0 < \lambda < 1$, we deduce that $w(r_1') > w(r_1)$. This shows that $w(r_1) = \sum_{V_j \in T} u_j$ is not the maximun value $m(T,\Delta;r)$. This proves (ii). One checks that (iii) follows from (i) and (ii). The lemma is proved.

By analogy with Lemma 3.3, it follows from Lemma 3.5 that we have

Lemma 3.6. *(i). Let r be a rational point in the cone $C(\Delta)$. Then, $m(T,\Delta;pr) \geq pm(T,\Delta;r)$ for all positive rational numbers p. (ii). Let r_1 and r_2 be two rational points in the cone $C(\Delta)$. Then $m(T,\Delta;r_1+r_2) \geq m(T,\Delta;r_1) + m(T,\Delta;r_2)$. If the equality holds, then both r_1 and r_2 lie on $C(\Delta(1))$ or $C(\Delta(2))$.*

Proof. One can easily checks (i). We now prove (ii). Let

$$r_1 = u_1 V_1 + \cdots + u_J V_J, \ \sum_{V_j \in T} u_j = m(T,\Delta;r_1),$$

$$r_2 = w_1 V_1 + \cdots + w_J V_J, \ \sum_{V_j \in T} w_j = m(T,\Delta;r_2),$$

Then,

$$r_1 + r_2 = (u_1 + w_1)V_1 + \cdots + (u_J + w_J)V_J. \tag{3.4}$$

$$m(T, \Delta; r_1) + m(T, \Delta; r_2) = \sum_{V_j \in T} (u_j + w_j) \leq m(T, \Delta; r_1 + r_2). \tag{3.5}$$

Let V_{j_i} be the lattice points with non-zero coefficients in (3.4). If $r_1 + r_2 \in C(\Delta(1))$, Lemma 3.5 shows that the inequality in (3.5) is an equality if and only if the V_{j_i} are contained in $\Delta(1)$. This implies that both r_1 and r_2 are contained in the cone $C(\Delta(1))$. If $r_1 + r_2 \in C(\Delta(2))$ and the inequality in (3.5) is an equality, then Lemma 3.5 shows that the V_{j_i} are contained in $\Delta(2)$. This implies that both r_1 and r_2 are contained in the cone $C(\Delta(2))$. The lemma is proved.

§4. Newton Polygons and Degree Polygons

The purpose of this section is to define the Newton polygon and the degree polygon of a subcone of $C(\Delta)$. First, we define the Newton polygon. Let Δ be an arbitrary n-dimensional integral convex polyhedron in $\mathbf{R}^n$ containing the origin. Let Σ be a cone contained in $C(\Delta)$ (not necessarily open or closed). Define a function on non-negative integers as follows:

$$W_\Sigma(k) = \mathrm{card}\{r \in \mathbf{Z}^n \cap \Sigma \mid w(r) = \frac{k}{D(\Delta)}\}, \tag{4.1}$$

Let $P(\Sigma)$ be the polygon in $\mathbf{R}^2$ with vertices $(0,0)$ and

$$\Big(\sum_{k=0}^{m} W_\Sigma(k), \frac{1}{D(\Delta)} \sum_{k=0}^{m} k W_\Sigma(k)\Big), \quad m = 0, 1, 2, \cdots. \tag{4.2}$$

For convenience, we shall call the vertex in (4.2) the m^{th} vertex in $P(\Sigma)$. Note that the m^{th} vertex may be equal to the $(m+1)^{\mathrm{th}}$ vertex, because it may happen that $W_\Sigma(m) = 0$. Recall that $A_1(f) = (a_{s,r}(f))$ is the semilinear Frobenius matrix defined in (2.16). We define $A_1(\Sigma)$ to be the submatrix $(a_{s,r}(f))$ with r and s running through the cone Σ. From the block form (2.25), it follows that

Proposition 4.1. *The Newton polygon of det*$(I - tA_1(\Sigma))$ *lies above* $P(\Sigma)$.

Next, we define the degree polygon. We assume that the origin 0 is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin. Let the V_j be lattice points on Δ_1 which include all vertices of Δ_1. Let T be a subset of the set $\{V_j\}$. Define a function on non-negative integers as follows:

$$Q_\Sigma(T,k) = (p-1)\sum_{w(r)=k, r\in \mathbf{Z}^n\cap\Sigma} m(T,\Delta;r). \tag{4.3}$$

Let $Q(T,\Sigma)$ be the polygon in $\mathbf{R}^2$ which is the graph of the piecewise linear function with vertices $(0,0)$ and

$$\Big(\sum_{k=0}^{m} W_\Sigma(k), \sum_{k=0}^{m} Q_\Sigma(T,k)\Big), \quad m = 0,1,2,\cdots. \tag{4.4}$$

Similarly, we shall call the vertex in (4.4) the m^{th} vertex in $Q(T,\Sigma)$. Even though $P(\Sigma)$ is a convex polygon, we do not know whether or not $Q(T,\Sigma)$ is convex.

Recall that $f(x) = \sum_{j=1}^{J} a_j x^{V_j}$ and T is a subset of the V_j. For any polynomial F in the variables a_j, we define $m(T,F)$ to be the total degree of F in those variables a_j with $V_j \in T$. If $F=0$, define $m(T,F)=-\infty$. Let F_r be the polynomial defined by (2.9) and (2.16). Then, one checks that

$$m(T,F_r) \le m(T,\Delta;r).$$

Definition 4.2. Let $P(\Sigma,x)$ (resp. $Q(T,\Sigma,x)$) be the piecewise linear function whose graph is $P(\Sigma)$ (resp. $Q(T,\Sigma)$). By Proposition 4.1, we can write

$$\det(I - tA_1(\Sigma)) = \sum_{k=0}^{\infty} p^{P(\Sigma,k)} G(\Sigma,k) t^k,$$

where $G(\Sigma,k)$ is a power series in the a_j with p-adic integral coefficients. The reduction $\bar{G}(\Sigma,k) = G(\Sigma,k) \pmod{\pi}$ is a polynomial in the a_j defined over $\bar{\mathbf{F}}_p$. The degree polygon of $\det(I - tA_1(\Sigma))$ with respect to the Newton polygon $P(\Sigma)$ is defined to be the graph of the piecewise linear function with the vertices

$$\Big(\sum_{k=0}^{m} W_\Sigma(k), m(T, \bar{G}(\Sigma, \sum_{k=0}^{m} W_\Sigma(k)))\Big) \ (m = 0,1,\cdots,\infty),$$

where the second coordinate is the degree of $\bar{G}(\Sigma, \sum_{k=0}^{m} W_\Sigma(k))$ in terms of T.

Proposition 4.3. *(i). The degree polygon of det$(I - tA_1(\Sigma))$ lies below $Q(T,\Sigma)$. (ii). If for all rational points r in the cone Σ and all non-negative solutions of the equation $\sum_{j=1}^{J} u_j V_j = r$, we always have $\sum_{V_j\in T} u_j = m(T,\Delta;r)$, then the degree polygon of det$(I - tA_1(\Sigma))$ coincides with* *its upper bound $Q(T,\Sigma)$ if and only if the generic Newton polygon (associated with the V_j) of det$(I-tA_1(\Sigma))$ coincides with its lower bound $P(\Sigma)$.*

Proof. Let m be a non-negative integer. By the block form of the matrix $A_1(\Sigma)$, the determinant expansion of a matrix and Lemma 3.6, we deduce that

$$\begin{aligned} m\Big(T,\bar{G}\big(\Sigma,\sum_{k=0}^{m}W_\Sigma(k)\big)\Big) &\le \max_\sigma \sum_{w(r)\le m} m(T,F_{pr-\sigma(r)}) \\ &\le \max_\sigma \sum_{w(r)\le m} m(T,\Delta;pr-\sigma(r)) \\ &\le \max_\sigma \sum_{w(r)\le m} (pm(T,\Delta;r)-m(T,\Delta;\sigma(r))) \\ &\le (p-1)\sum_{w(r)\le m} m(T,\Delta;r) \\ &= \sum_{k=0}^{m} Q_\Sigma(T,k), \end{aligned}$$

where $r\in\mathbf{Z}^n\cap\Sigma$ and σ runs through the permutations on $\sum_{k=0}^{m}W_\Sigma(k)$ letters. This proves (i).

Under the assumption of (ii), the polynomial F_r is homogeneous of degree $m(T,\Delta;r)$ in terms of the variables corresponding to T. Furthermore, the identity

$$m(T,\Delta;r_1+r_2)=m(T,\Delta;r_1)+m(T,\Delta;r_2)$$

is true for all rational points r_1 and r_2 of the cone Σ. This identity implies that the polynomial $\bar{G}\big(\Sigma,\sum_{k=0}^{m}W_\Sigma(k)\big)$ is also homogeneous in terms of the variables corresponding to T. Thus, this polynomial either vanishes or has the desired degree $\sum_{k=0}^{m}Q_\Sigma(T,k)$. The lemma is proved.

Proposition 4.4. *If the degree polygon of det$(I-tA_1(\Sigma))$ coincides with $Q(T,\Sigma)$ at the m^{th} vertex, then the Newton polygon of det$(I-tA_1(\Sigma))$ coincides with $P(\Sigma)$ at the m^{th} vertex. If T contains all the V_j, then the converse is also true.*

Proposition 4.4 follows directly from the definition of the degree polygon and (ii) of Proposition 4.3. Thus, the degree polygon is finer than the Newton polygon. Grothendieck's specialization theorem says that the Newton polygon goes up under specializations. The following is an analogue. This proposition follows from the fact that the degree of a polynomial decreases under specializations.

Proposition 4.5. *The degree polygon of det($I - tA_1(\Sigma)$) defined with respect to the Newton polygon $P(\Sigma)$ goes down under specializations, where a specialization means a specialization of the affine space whose coordinates are the a_j with $V_j \notin T$.*

§5. Boundary Decompositions and Facial Decompositions

Let Δ be an n-dimensional integral convex polyhedron in $\mathbf{R}^n$ containing the origin. In this section, we give two preliminary decomposition theorems which follow from the properties of the weight function $w(r)$ that were described in Lemma 2.1.

First, we consider the boundary (or interior) decomposition theorem. Let $B(\Delta)$ be the unique interior decomposition of the cone $C(\Delta)$ into a union of disjoint relatively open cones. Its elements are the interiors of those closed faces in $C(\Delta)$ which contain the origin. Thus, the interior cone of $C(\Delta)$ is the unique element in $B(\Delta)$ of dimension n. The origin itself (if it is a vertex) is the unique element in $B(\Delta)$ of dimension 0. For $\Sigma \in B(\Delta)$, recall that $A_1(\Sigma)$ is the "Σ" piece $(a_{s,r}(f))$ in $A_1(f)$ such that r and s run through the cone Σ. Let $A_1(\Sigma, f_\Sigma)$ be the "interior piece" of the Frobenius matrix $A_1(f_\Sigma)$, where f_Σ is the restriction of f to the closure of Σ. The boundary decomposition theorem is the following

Theorem 5.1. *We have*

$$\det(I - tA_1(f)) = \prod_{\Sigma \in B(\Delta)} \det(I - tA_1(\Sigma, f_\Sigma)). \tag{5.1}$$

In particular, the Newton polygon of det($I - tA_1(f)$) coincides with its lower bound $P(\Delta)$ if and only if the Newton polygon of det($I - tA_1(\Sigma, f_\Sigma)$) coincides with its lower bound $P(\Sigma)$ for all $\Sigma \in B(\Delta)$.

Proof. First, we claim that $A_1(\Sigma)$ coincides with the matrix $A_1(\Sigma, f_\Sigma)$. To prove the claim, let r and s be two lattice points of Σ. We need to prove that $F_{ps-r}(f) = F_{ps-r}(f_\Sigma)$. Let u_j be non-negative integers satisfying $\sum_j u_j V_j = ps - r$. Without loss of generality, we assume that $u_j > 0$ for all $j \leq k$ and $u_j = 0$ for all $j > k$. From the equation $ps = r + \sum_j u_j V_j$, we deduce that s is in the interior of the cone spanned by the V_j (all $j \leq k$) and r. Because of the nature of the interior decomposition, we conclude that the V_j (all $j \leq k$) and r are in the closure of the cone Σ. This implies that $F_{ps-r}(f) = F_{ps-r}(f_\Sigma)$. The claim is proved.

Now, fix an ordering of $B(\Delta) = \{\Sigma_0, \cdots, \Sigma_h\}$ such that $\dim(\Sigma_i) \le \dim(\Sigma_{i+1})$. Let $i < j$, $s \in \Sigma_i$ and $r \in \Sigma_j$. We prove that $ps - r$ lies outside of $C(\Delta)$, that is, $w(ps - r) = +\infty$ and $F_{ps-r}(f) = 0$. Otherwise, suppose that $ps - r$ lies in $C(\Delta)$. Let $ps - r \in \Sigma_k$. From the identity $ps = r + (ps - r)$, we deduce that s lies in the interior of the cone spanned by Σ_j and Σ_k. This means that $\dim\Sigma_i \ge \dim\Sigma_j$ with equality holding only if $ps - r \in \bar{\Sigma}_j$. Since $\dim\Sigma_i \le \dim\Sigma_j$, we must have $ps - r \in \bar{\Sigma}_j$. From the identity $ps = r + (ps - r)$ again, we conclude that $s \in \Sigma_j$. This shows that Σ_i and Σ_j are not disjoint, a contradiction. Thus, $ps - r$ lies outside of $C(\Delta)$.

Let B_{ij} $(0 \le i, j \le h)$ be the nuclear submatrix of $A_1(f)$ consisting of all $(a_{s,r}(f))$ with $s \in \Sigma_i$ and $r \in \Sigma_j$. Then, under a permutation of the orthonormal basis, we have the block form

$$A_1(f) = \begin{pmatrix} B_{00} & B_{01} & \ldots & B_{0h} \\ B_{10} & B_{11} & \ldots & B_{1h} \\ \vdots & \vdots & \ddots & \vdots \\ B_{h0} & B_{h1} & \ldots & B_{hh} \end{pmatrix}. \tag{5.2}$$

In previous paragraph, we showed that $B_{ij} = 0$ for $i < j$. Thus, the block form (5.2) is lower triangular. The theorem follows because the claim shows that $B_{ii} = A_1(\Sigma_i) = A_1(\Sigma_i, f_{\Sigma_i})$.

Corollary 5.2. *Let $\Sigma \in B(\Delta)$. If the Newton polygon of $\det(I - tA_1(f))$ coincides with $P(\Delta)$, then the Newton polygon of $\det(I - tA_1(f_\Sigma))$ coincides with its lower bound $P(\bar{\Sigma})$.*

Proof. Applying (5.1) to f_Σ, we see that $\det(I - tA_1(f_\Sigma))$ is a product of certain factors in (5.1). The corollary follows.

Corollary 5.3. *Let $d_1, \cdots, d_n$ be positive integers. Let D be the least common multiple of the d_i. Let f be the general polynomial whose Newton polyhedron is the simplex generated by the origin and the vertices $(d_1, 0, \cdots, 0), \cdots, (0. \cdots, d_n)$. Then the generic Newton polygon of the L-function $\mathrm{L}^*(f, t)^{(-1)^{n-1}}$ coincides with its lower bound if and only if $p \equiv 1$ $(\mathrm{mod} D)$.*

Proof. The sufficiency of this corollary follows from the computation in [15]. Because for the diagonal form $f(x) = \sum_{i=1}^n x_i^{d_i}$, the Newton polygon of $\mathrm{L}^*(f, t)^{(-1)^{n-1}}$ coincides with its lower bound for all $p \equiv 1$ (mod D). If $n = 1$, the Stickelberger theorem implies that the Newton polygon for $f(x) = x^{d_1}$ coincides with its lower bound if and only if $p \equiv 1$ (mod d_1). Thus, the necessity is true for $n = 1$. If $n > 1$, the necessity follows from Corollary 5.2 and the case $n = 1$.

Remark. The referee pointed out that the necessity in general is also implied (at least in many cases) by ramification.

To describe the facial decomposition theorem obtained in [15], let $F(\Delta)$ be the set of relatively open cones generated by the relatively open faces in Δ of dimension at most $(n-1)$ such that the closure of these faces does not contain the origin. The origin itself is considered as an element of $F(\Delta)$. Clearly, $C(\Delta)$ is the disjoint union of the relatively open cones in $F(\Delta)$. For simplicity, we call $F(\Delta)$ the open facial decomposition of the cone $C(\Delta)$ defined with respect to the origin. The intersection $S(\Delta)$ of $F(\Delta)$ with Δ gives a decomposition of Δ. This decomposition $S(\Delta)$ is called the open facial decomposition of Δ with respect to the origin. If V is any lattice point other than the origin, we can similarly define an open facial decomposition $S(\Delta, V)$ of Δ with respect to V. We shall use this construction in next section.

For each $\Sigma \in F(\Delta)$, recall that $A_1(\Sigma, f_\Sigma)$ is the "interior piece" of the Frobenius matrix $A_1(f_\Sigma)$, where f_Σ is the restriction of f to the closure of Σ. Note that there is a difference between $A_1(\Sigma, f_\Sigma)$ and $A_1(\Sigma)$ defined in the last section. The latter matrix is defined in terms of f, while the former is defined in terms of f_Σ. We first prove the following open facial decomposition theorem. The closed facial decomposition is proved in Theorem 5.5.

Theorem 5.4. *For a non-negative integer m, the Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound at the m^{th} vertex if and only if the Newton polygon of $\det(I - tA_1(\Sigma, f_\Sigma))$ coincides with its lower bound at the m^{th} vertex for all $\Sigma \in F(\Delta)$.*

Proof. The proof is similar to the proof of Theorem 5.1. First, we claim that if $\Sigma \in F(\Delta)$ and r belongs to the closure of Σ, then

$$\operatorname{ord}_p(F_r(f) - F_r(f_\Sigma)) > \frac{w(r)}{p-1}, \tag{5.3}$$

where the right side is the trivial lower bound of $\operatorname{ord}_p(F_r(f))$ (see (2.13)). To prove the claim, let u_j be non-negative integers satisfying $\sum_j u_j V_j = r$. Without loss of generality, we assume that $u_j > 0$ for all $j \leq k$ and $u_j = 0$ for all $j > k$. If now $\sum_j u_j = w(r)$, then part (c) of Lemma 2.1 shows that the V_j (all $j \leq k$) are cofacial with respect to the origin. Since the V_j (all $j \leq k$) also span r, it follows that the V_j (all $j \leq k$) lies in the closure of Σ. This shows that if a monomial of p-adic order $w(r)/(p-1)$ appears in $F_r(f)$, then this monomial also appears in $F_r(\Sigma)$. The claim is proved.

Now, fix an ordering of $F(\Delta) = \{\Sigma_0, \cdots, \Sigma_h\}$ such that $\dim(\Sigma_i) \leq \dim(\Sigma_{i+1})$. Let $i < j$, $s \in \Sigma_i$ and $r \in \Sigma_j$. We prove that r and $ps - r$ are not cofacial with respect to the origin, that is, $w(r) + w(ps - r) > pw(s)$ (by Lemma 2.1). Otherwise, suppose that r and $ps - r$ are cofacial with respect to the origin. From the identity $ps = r + (ps - r)$, we deduce that r, s and $ps - r$ are cofacial with respect to the origin. This shows that Σ_i and Σ_j are two elements in the boundary decomposition of certain piece of $C(\Delta)$. Using the identity $ps = r + (ps - r)$ and the same argument as in the second paragraph of the proof of Theorem 5.1, we get a contradiction. This proves that r and $ps - r$ are not cofacial with respect to the origin and $w(r) + w(ps - r) > pw(s)$.

Let B_{ij} $(0 \leq i, j \leq h)$ be the nuclear submatrix of $A_1(f)$ consisting of all $(a_{s,r}(f))$ with $s \in \Sigma_i$ and $r \in \Sigma_j$. Then we also have the block form (5.2) for $A_1(f)$. In previous paragraph, we showed that this block form is in some sense lower triangular. Since $B_{ii} = A_1(\Sigma_i)$, the triangular property shows that the Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound at the m^{th} vertex if and only if the Newton polygon of $\det(I - tA_1(\Sigma))$ coincides with its lower bound at the m^{th} vertex for all $\Sigma \in F(\Delta)$. Although the matrix $A_1(\Sigma)$ is very different from the desired matrix $A_1(\Sigma, f_\Sigma)$, the Newton polygon of $\det(I - tA_1(\Sigma))$ and the Newton polygon of $\det(I - tA_1(\Sigma, f_\Sigma))$ have the same lower bound expressed in terms of $W_\Sigma(k)$.

To finish the proof, we need to show that if we replace the matrix $A_1(\Sigma)$ defined in terms of f by the matrix $A_1(\Sigma, f_\Sigma)$ defined in terms of f_Σ, we will not change the property of the coincidence of the Newton polygon with its lower bound. Let r, $s \in \Sigma$. If $ps - r$ belongs to the closure of Σ, then (5.3) shows that we can replace $F_{ps-r}(f)$ by $F_{ps-r}(f_\Sigma)$. If $ps - r$ does not belong to the closure of Σ, then $ps - r$ and r are not cofacial (one can apply the argument of the second paragraph). Then, the p-order of $a_{s,r}(f) = \pi^{w(r)-w(s)} F_{ps-r}$ is strictly larger than the expected number $w(s)$, while $a_{s,r}(f_\Sigma) = 0$. In this case, we can also replace $F_{ps-r}(f)$ by $F_{ps-r}(f_\Sigma)$. The theorem is proved.

The following is the closed facial decomposition theorem.

Theorem 5.5. *For a non-negative integer m, the Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound $P(\Delta)$ at the m^{th} vertex if and only if the Newton polygon of $\det(I - tA_1(f_\Sigma))$ coincides with its lower bound $P(\bar{\Sigma})$ at the m^{th} vertex for all n-dimensional $\Sigma \in F(\Delta)$. Note that $A_1(f_\Sigma)$ is the semilinear Frobenius matrix for the restriction f_Σ of f to the closure of Σ.*

Proof. Let $B(\bar{\Sigma})$ be the open boundary decomposition of $\bar{\Sigma}$ (or $\bar{\Sigma} \cap \Delta$). Then, one checks that

$$F(\Delta) = \bigcup_{\Sigma \in F(\Delta),\ \dim(\Sigma) = n} B(\bar{\Sigma}).$$

Theorem 5.5 then follows from Theorem 5.1 and Theorem 5.4.

Now, Theorem 8 follows immediately from Theorem 5.5 and the theorem of Adolphson and Sperber. For Kloosterman type sums, one can use Theorem 8 to completely classify all prime numbers p such that the Newton polygon coincides with its lower bound. This would improve the Newton polygon results obtained by Sperber [13] and Carpentier [4].

§6. Star Decompositions

The purpose of this section is to establish a star decomposition theorem for Δ. The results will be weaker. But they are strong enough for the generic Newton polygon. Since we are only concerned with the non-emptiness of the Hasse domain $\mathcal{H}_p(\Delta)$, Theorem 5.5 shows that we may assume that the origin 0 is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin. Let $V_1, \cdots, V_J$ be J fixed lattice points on the face Δ_1 which include all vertices of Δ_1. Then Δ is generated by 0 and the V_j. Let $C(\Delta)$ be the cone generated by Δ. Let f be the general Laurent polynomial $f = \sum_j a_j x^{V_j}$. Clearly, the Newton polyhedron of f is Δ. In this section, we take $T = V_1$ (not necessarily a vertex). Thus, the degree polygon is defined in terms of V_1.

We first describe the open star decomposition of Δ_1 with respect to V_1. Let $S(\Delta_1, V_1)$ be the open facial decomposition (see §5) of the $(n-1)$-dimensional Δ_1 defined with respect to V_1. The vertex V_1 is an element of $S(\Delta_1, V_1)$. For $\sigma \in S(\Delta_1, V_1)$, define $\Sigma(\sigma)$ to be the open cone spanned by σ and the origin. If σ is the vertex V_1, then $\Sigma(\sigma)$ is the open half line spanned by the origin and V_1. We always have

$$\dim\Sigma(\sigma) = \dim\sigma + 1.$$

Let $F(\Delta, V_1)$ be the set consisting of the **origin** and all $\Sigma(\sigma)$, where σ runs through $S(\Delta_1, V_1)$. Then the closed cone $C(\Delta)$ is the disjoint union of the open cones Σ in $F(\Delta, V_1)$. In particular, we have

$$W(k) = \sum_{\Sigma \in F(\Delta, V_1)} W_\Sigma(k).$$

We call $F(\Delta, V_1)$ the open star decomposition of $C(\Delta)$ with respect to V_1. The open facial decomposition in §5 is simply the open star decomposition with respect to the origin.

Unless otherwise stated, in this section we shall use σ to denote an element of $S(\Delta_1, V_1)$ and Σ to denote an element of $F(\Delta, V_1)$. Let f_Σ be the restriction of the Laurent polynomial f to the **closed** cone spanned by Σ. Then f_Σ is a Laurent polynomial whose Newton polyhedron is the intersection of Δ and the closure of Σ. Dwork's theory as described in §2 applies to f_Σ. Let $A_1(\Sigma, f_\Sigma)$ be the "interior piece" of the "closed" matrix $A_1(f_\Sigma) = (a_{s,r}(f_\Sigma))$ consisting of all $a_{s,r}(f_\Sigma)$ with s and r running through Σ.

The following proposition follows from the block form of $A_1(\Sigma, f_\Sigma)$.

Proposition 6.1. *The Newton polygon of det$(I - tA_1(\Sigma, f_\Sigma))$ computed with respect to p lies above $P(\Sigma)$. The degree polygon of det$(I - tA_1(\Sigma, f_\Sigma))$ defined with respect to $P(\Sigma)$ lies below $Q(V_1, \Sigma)$.*

The open star decomposition theorem is

Theorem 6.2. *For any non-negative integer m, the degree polygon of det$(I - tA_1(f))$ coincides with $Q(V_1, \Delta)$ at the m^{th} vertex if and only if the degree polygon of det$(I - tA_1(\Sigma, f_\Sigma))$ coincides with $Q(V_1, \Sigma)$ at the m^{th} vertex for all $\Sigma \in F(\Delta, V_1)$.*

To prove this result, we first prove

Lemma 6.3. *Let $\Sigma \in F(\Delta, V_1)$, $r \in \bar{\Sigma}$ and $f = \sum_{j=1}^{J} a_j x^{V_j}$. Then*

$$m(V_1, F_r(f) - F_r(f_\Sigma)) < m(V_1, \Delta; r).$$

That is, the total degree of $F_r(f) - F_r(f_\Sigma)$ in the variable a_1 is strictly smaller than the expected value $m(V_1, \Delta; r)$.

Proof. Let $u_1, \cdots, u_J$ be non-negative integers satisfying $r = \sum_j u_j V_j$. Without loss of generality, we may assume that $u_1 \geq 0$, $u_j > 0$ for $2 \leq j \leq k$ and $u_j = 0$ for $j > k$. Assume that $u_1 = m(V_1, \Delta; r)$. Lemma 3.2 shows that the V_j $(2 \leq j \leq k)$ are cofacial with respect to V_1. Since $r - u_1 V_1 = \sum_{j=2}^{k} u_j V_j$, the construction of the open star decomposition shows that the V_j $(j \leq k)$ lie in the closure of the cone Σ. This shows that if a monomial in the a_j of degree $m(V_1, \Delta; r)$ with respect to a_1 appears in $F_r(f)$, then the same monomial also appears in $F_r(f_\Sigma)$. Thus, $F_r(f)$ and $F_r(f_\Sigma)$ have the same initial terms of degree $m(V_1, \Delta; r)$ with respect to a_1. The lemma is proved.

Proof of Theorem 6.2. Fix an ordering of $F(\Delta, V_1) = \{\Sigma_0, \cdots, \Sigma_h\}$ such that $\dim(\Sigma_i) \leq \dim(\Sigma_{i+1})$. Let $i < j$, $s \in \Sigma_i$ and $r \in \Sigma_j$. We claim that r and $ps - r$ are not cofacial with respect to V_1. Otherwise, suppose that r and $ps - r$ are cofacial with respect to V_1. From the identity $ps = r + (ps - r)$, it follows that r, s and $ps - r$ are all cofacial with respect to V_1. Using the identity $ps = r + (ps - r)$ and the same argument as in the second paragraph of the proof of Theorem 5.1, we get a contradiction. The claim is proved. By Lemma 3.3, we then have

$$m(V_1, a_{s,r}(f)) = m(V_1, F_{ps-r}) \leq m(V_1, \Delta; ps - r) < pm(V_1, \Delta; s) - m(V_1, \Delta; r). \tag{6.1}$$

Let B_{ij} $(0 \leq i, j \leq h)$ be the nuclear submatrix of $A_1(f)$ consisting of all $(a_{s,r}(f))$ with $s \in \Sigma_i$ and $r \in \Sigma_j$. The Newton polygon of the entire function $\det(I - tB_{ii})$ lies above $P(\Sigma_i)$. Furthermore, under a permutation of orthonormal basis,

$$A_1(f) = \begin{pmatrix} B_{00} & B_{01} & \dots & B_{0h} \\ B_{10} & B_{11} & \dots & B_{1h} \\ \vdots & \vdots & \ddots & \vdots \\ B_{h0} & B_{h1} & \dots & B_{hh} \end{pmatrix}. \tag{6.2}$$

If $a_{s,r}(f)$ is an element in B_{ij} with $i < j$, then (6.1) shows that the degree $m(V_1, a_{s,r}(f))$ of the polynomial $a_{s,r}(f)$ in the variable a_1 is strictly smaller than the expected value $pm(V_1, \Delta; s) - m(V_1, \Delta; r)$. This means that the above block form for $A_1(f)$ is, in some sense, lower triangular with respect to the degree of a_1. By induction, we deduce that

$$\det(I - tA_1(f)) = \prod_{i=0}^{h} \det(I - tB_{ii}) + \sum_{k=0}^{\infty} p^{P(\Delta,k)} G(k) t^k, \tag{6.3}$$

where $G(k)$ is a power series in the a_j with p-adic integral coefficients such that the reduction $\bar{G}(k) = G(k) \pmod{\pi}$ is a polynomial over $\bar{\mathbf{F}}_p$ whose degree in a_1 is strictly smaller than their upper bound (i.e., the number $Q(V_1, \Delta, k)$, see the notation in Definition 4.2). Thus, the degree polygon of $\det(I - tA_1(f))$ defined with respect to $P(\Delta)$ coincides with $Q(V_1, \Delta)$ at the m^{th} vertex if and only if the degree polygon of the first term on the right side of (6.3) (defined with respect to $P(\Delta)$) coincides with $Q(V_1, \Delta)$ at the m^{th} vertex. One further shows that the latter is true if and only if the degree polygon of $\det(I - tA_1(\Sigma))$ defined with respect to $P(\Sigma)$ coincides with $Q(V_1, \Sigma)$ at the m^{th} vertex for all $\Sigma \in F(\Delta, V_1)$. Although the matrix $A_1(\Sigma)$ is very different from the desired matrix $A_1(\Sigma, f_\Sigma)$, the degree polygon of $\det(I - tA_1(\Sigma))$ and the degree polygon of $\det(I - tA_1(\Sigma, f_\Sigma))$ have the same upper bound $Q(V_1, \Sigma)$.

To finish the proof, we need to show that if we replace the matrix $A_1(\Sigma)$ defined in terms of f by the matrix $A_1(\Sigma, f_\Sigma)$ defined in terms of f_Σ, we will not change the property of the coincidence of the degree polygon with its upper bound. Let r, $s \in \Sigma$. If $ps - r$ belongs to the closure of Σ, Lemma 6.3 shows that we can replace $F_{ps-r}(f)$ by $F_{ps-r}(f_\Sigma)$. If $ps - r$ does not belong to the closure of Σ, then $ps - r$ and r are not cofacial with respect to V_1 (one can apply the argument of the first paragraph). Then, the degree in a_1 of $F_{ps-r}(f)$ is strictly smaller than the expected number $pm(V_1, \Delta; s) - m(V_1, \Delta; r)$, while $F_{ps-r}(f_\Sigma) = 0$. In this case, we can also replace $F_{ps-r}(f)$ by $F_{ps-r}(f_\Sigma)$. The theorem is proved.

Applying Theorem 6.2 and the boundary decomposition theorem (Theorem 5.1), we deduce the following closed star decomposition theorem.

Theorem 6.4. *The degree polygon of det$(I - tA_1(f))$ coincides with $Q(V_1, \Delta)$ at the m^{th} vertex if and only if the degree polygon of det$(I - tA_1(f_\Sigma))$ defined with respect to $P(\bar{\Sigma})$ coincides with $Q(V_1, \bar{\Sigma})$ at the m^{th} vertex for all n-dimensional $\Sigma \in F(\Delta, V_1)$.*

Proof. The proof is similar to the proof of Theorem 5.5. Let $\Sigma \in F(\Delta, V_1)$ be an n-dimensional cone. Let $B(\bar{\Sigma})$ be the open boundary decomposition of the closed cone $\bar{\Sigma}$ (or the closed polyhedron $\bar{\Sigma} \cap \Delta$). Then, one checks that

$$F(\Delta, V_1) = \bigcup_{\Sigma \in F(\Delta, V_1),\ \dim(\Sigma) = n} B(\bar{\Sigma}).$$

Theorem 6.4 then follows from Theorem 5.1 and Theorem 6.2.

We are now in a position to prove that the degree polygon coincides with its upper bound if p lies in a certain residue class.

Theorem 6.5. *Let Δ have only one face of codimension 1 which does not contain the origin. Let $V_1, \cdots, V_J$ be J lattice points on Δ which include all non-zero vertices of Δ. Let f be the general Laurent polynomial $\sum_j a_j x^{V_j}$. Let V_1 be a lattice point (not necessarily a vertex) on the unique face of codimension 1 not containing the origin. Let m be a fixed non-negative integer. There is an effectively computable positive integer $D^*(m)$ depending only on the V_j and m such that if $p \equiv 1 \pmod{D^*(m)}$, then the degree polygon of det$(I - tA_1(f))$ with respect to V_1 coincides with its upper bound $Q(V_1, \Delta)$ at the m^{th} vertex.*

Proof. For a general Δ, by applying Theorem 6.4 we may assume that V_1 is a vertex. The degree polygon is defined with respect to V_1. By the specialization theorem (Proposition 4.5), we may further assume that the V_j are the non-zero vertices in Δ. Note that under such specializations, the upper bound remains fixed.

We first assume that Δ is a simplex with vertices $0, V_1, \cdots, V_n$. Let D^* be the torsion order of the torsion abelian group $\mathbf{Z}^n/\mathbf{Z}V_1 + \cdots + \mathbf{Z}V_n$. For $r \in C(\Delta)$, the equation

$$\sum_{j=1}^{n} u_j V_j = r,\ u_j \geq 0,\ u_j \text{ rational}$$

has exactly one solution and this unique solution will make u_1 maximal. In the notation of (2.9), $F_r(f)$ is either zero or a monomial of degree $m(V_1, \Delta; r)$ with respect to a_1. The assumption of part (ii) of Proposition 4.3 is satisfied. We deduce that if the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound $P(\Delta)$ at the m^{th} vertex, then the degree polygon of $\det(I - tA_1(f))$ with respect to V_1 coincides with its upper bound $Q(V_1, \Delta)$ at the m^{th} vertex. Under the assumption that $p \equiv 1 \pmod{D^*}$, it is shown in [15] that the generic Newton polygon coincides with its lower bound for all sides. Thus, the theorem is true for a simplex Δ.

In general, we showed that we may assume that the V_j are the vertices on the unique face of codimension 1 not containing the origin. We now use induction on the number J of V_j on Δ. Assume that the theorem is true for all Δ with fewer than J non-zero vertices V_j. We now prove that the theorem is true for any Δ with J non-zero vertices V_j. If $F(\Delta, V_1)$ has at least two elements of dimension n, then Theorem 6.4 gives a non-trivial decomposition. By induction, we conclude that the degree polygon (with respect to the vertex V_1) of $\det(I - tA_1(f))$ coincides with $Q(V_1, \Delta)$ at the m-th vertex for all p in a certain residue class $p \equiv 1 \pmod{D^*(m)}$. If $F(\Delta, V_1)$ has only one element of dimension n, then Δ has only one closed face of codimension 1 not containing V_1. In this case, all solutions of the equation

$$\sum_{j=1}^{J} u_j V_j = r,\ u_j \geq 0,\ u_j \text{ rational}$$

satisfy $w(r) = u_1 + \cdots + u_J$. Thus,

$$r - w(r)V_1 = \sum_{j=2}^{J} u_j (V_j - V_1).$$

Since all the V_j (all $j \geq 2$) are cofacial with respect to V_1, one sees that $\sum_{j=2}^{J} u_j$ is the weight of $r - w(r)V_1$ with respect to the vertices of $\{V_j - V_1\}_{j=2}^{J}$. Hence, $\sum_{j=2}^{J} u_j$ depends only on V_1 and r but not on the particular $u \in (\mathbf{Z}_{\geq 0})^J$ satisfying $\sum_{j=1}^{J} u_j V_j = r$. Thus, u_1 depends only on V_1 and r. This proves that any solution (if it exists) will make u_1 maximal. Thus, the assumption in part (ii) of Proposition 4.3 is satisfied. We deduce that that if the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound $P(\Delta)$ at the m^{th} vertex, then the degree polygon of $\det(I - tA_1(f))$ with respect to V_1 coincides with its upper bound $Q(V_1, \Delta)$ at the m^{th} vertex.

To finish the proof, choose a vertex, say V_2, such that $F(\Delta, V_2)$ has at least two elements of dimension n (if such a V_2 does not exist, then Δ is already simplicial and we are done). In this way, Theorem 6.4 gives a non-trivial decomposition with respect to V_2. By induction we conclude that for all primes in the residue class $p \equiv 1 \pmod{D^*(m)}$, the degree polygon (and hence the generic Newton polygon) with respect to V_2 of $\det(I - tA_1(f))$ coincides with $Q(V_2, \Delta)$ (resp. $P(\Delta)$) at the m^{th} vertex. By the discussion in the previous paragraph, this implies that for all such p, the degree polygon with respect to V_1 of $\det(I - tA_1(f))$ also coincides with $Q(V_1, \Delta)$ at the m^{th} vertex. The theorem is proved.

Remark. Let the V_j be the non-zero vertices in Δ. Let f be the smallest family of Laurent polynomials with $\Delta(f) = \Delta$, i.e., $f(x) = \sum_i a_j x^{V_j}$. If Δ is a simplex, one can use the Stickelberger relation to find the smallest positive integer D^* such that whenever $p \equiv 1 \pmod{D^*}$, the generic Newton polygon of $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ coincides with its lower bound (see the beginning of §1). For a general Δ, the proof of Theorem 6.5 shows that we can still find the smallest positive integer D^* such that whenever $p \equiv 1 \pmod{D^*}$, the degree polygon of $\det(I - tA_1(f))$ with respect to each of the vertices V_i coincides with its upper bound. This means that the leading form with respect to each variable a_i is non-zero for all of the Hasse polynomials $h_p(\Delta, m)$. Thus, the D^* so obtained is best possible in the stronger sense that the degree polygon with respect to each of the vertices V_i coincides with its upper bound.

Proof of Theorem 3. Let Δ be an n-dimensional integral convex polyhedron in $\mathbf{R}^n$ containing the origin. The theorem of Adolphson-Sperber shows that $\mathrm{L}^*(f,t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathbf{V}(\Delta)$ for general f and large primes p. By (2.21) and Corollary 2.5, it suffices to prove that there is a positive integer D^* such that if $p \equiv 1 \pmod{D^*}$ and p is large, then the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound $P(\Delta)$ for the first nD sides. Theorem 5.5 implies that we may assume that Δ has only one face of codimension 1 not containing the origin. Then we can apply Theorem 6.5 and take D^* to be the least common multiple of the $D^*(m)$ $(0 \leq m \leq nD)$. Theorem 3 is proved.

7. Hyperplane Decompositions

The decomposition theorem in §6 is based on a cut of Δ with respect to a lattice point V_1. One can consider other types of cuts with respect to several lattice points. In this section, we give a decomposition theorem for hyperplane cuts of Δ. This is then used to prove that Conjecture 1 is true in some special cases.

As in section 6, we assume that the origin 0 is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin. Let $V_1, \cdots, V_J$ be J fixed lattice points on the face Δ_1 which include all vertices of Δ_1. Then Δ is generated by 0 and the V_j. Let $f = \sum_j a_j x^{V_j}$ be the general Laurent polynomial associated with the V_j. Let H be a hyperplane passing through the origin such that the intersection of H with Δ is an integral polyhedron of dimension $n-1$ whose vertices are contained in the V_j. Then H cuts Δ into two pieces $\Delta(1)$ and $\Delta(2)$. We assume that both $\Delta(1)$ and $\Delta(2)$ are closed. Then, one sees that Δ is the union of the two pieces $\Delta(1)$ and $\Delta(2)$. Let T be the set of V_j which are contained in $\Delta(1)$. Let T_H be the intersection of T and H.

For $i = 1, 2$, let f_i be the restriction of the general Laurent polynomial f to $\Delta(i)$. Then f_i is the generic Laurent polynomial associated with those V_j which are contained in $\Delta(i)$. The Newton polyhedron of f_i is $\Delta(i)$. Dwork's theory as described in §2 applies to f_i. The following theorem is an analogue of the closed star decomposition theorem.

Theorem 7.1. *Let m be a non-negative integer. The degree polygon of $\det(I - tA_1(f))$ defined with respect to T coincides with $Q(T, \Delta)$ at the m^{th} vertex if and only if both the Newton polygon of $\det(I - tA_1(f_1))$ coincides with $P(C(\Delta(1)))$ at the m^{th} vertex and the degree polygon of $\det(I - tA_1(f_2))$ with respect to T_H coincides with $Q(T_H, \Delta(2))$ at the m^{th} vertex.*

Remark. Since T consists of all the V_j which are contained in $\Delta(1)$, the Newton polygon of $\det(I - tA_1(f_1))$ coincides with its lower bound $P(C(\Delta(1)))$ if and only if the degree polygon of $\det(I - tA_1(f_1))$ with respect to T coincides with its upper bound $Q(T, \Delta(1))$. Thus, Theorem 7.1

is a decomposition theorem about degree polygons.

Proof of Theorem 7.1. Let Σ_i $(i=1,2)$ be the n-dimensional cone spanned by $\Delta(i)-H$. Let Σ_0 be the $(n-1)$-dimensional cone spanned by $\Delta\cap H$. Then, the cone $C(\Delta)$ is the disjoint union of the Σ_i $(i=0,1,2)$. Let $0\le i<j\le 2$, $s\in\Sigma_i$ and $r\in\Sigma_j$. We claim that r and $ps-r$ can not both lie on the cone $C(\Delta(1))$ or $C(\Delta(2))$. If $i=0$, it is clear that $ps-r$ will not lie on the closure of Σ_j (while r does). If $i=1$, then $ps-r$ will not lie on the closure of Σ_j (while r does). The claim is proved. Lemma 3.6 shows that

$$m(T,a_{s,r}(f)) = m(T,F_{ps-r}) \le m(T,\Delta;ps-r) < pm(T,\Delta;s) - m(T,\Delta;r). \tag{7.1}$$

Let B_{ij} $(0\le i,j\le 2)$ be the nuclear submatrix of $A_1(f)$ consisting of all $(a_{s,r}(f))$ with $s\in\Sigma_i$ and $r\in\Sigma_j$. The Newton polygon of the entire function $\det(I-tB_{ii})$ lies above $P(\Sigma_i)$. Furthermore, under a permutation of orthonormal basis,

$$A_1(f) = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}. \tag{7.2}$$

If $a_{s,r}(f)$ is an element in B_{ij} with $i<j$, then (7.1) shows that the degree $m(T,a_{s,r}(f))$ of the polynomial $a_{s,r}(f)$ in the variables corresponding to T is strictly smaller than the expected value $pm(T,\Delta;s)-m(T,\Delta;r)$. This means that the above block form for $A_1(f)$ is, in some sense, lower triangular with respect to the degree defined in terms of T. We deduce that

$$\det(I-tA_1(f)) = \prod_{i=0}^{2}\det(I-tB_{ii}) + \sum_{k=0}^{\infty} p^{P(\Delta,k)}G(k)t^k, \tag{7.3}$$

where $G(k)$ is a power series in the a_j with p-adic integral coefficients such that the reduction $\bar{G}(k)=G(k) \pmod{\pi}$ is a polynomial over $\bar{\mathbf{F}}_p$ whose degree in T is strictly smaller than its upper bound (i.e., the number $Q(T,\Delta,k)$, see the notation in Definition 4.2) . Thus, the degree polygon of $\det(I-tA_1(f))$ defined with respect to $P(\Delta)$ coincides with $Q(T,\Delta)$ at the m^{th} vertex if and only if the degree polygon of the first term on the right side of (7.3) (defined with respect to $P(\Delta)$) coincides with $Q(T,\Delta)$ at the m^{th} vertex. One further shows that the latter is true if and only if the degree polygon of $\det(I-tB_{ii})$ defined with respect to $P(\Sigma_i)$ coincides with $Q(T,\Sigma_i)$ at the m^{th} vertex for all $0\le i\le 2$. Although the matrix $A_1(\Sigma_i)$ is very different from the desired matrix

$A_1(\Sigma_i, f_{\Sigma_i})$, the degree polygon of $\det(I - tA_1(\Sigma_i))$ and the degree polygon of $\det(I - tA_1(\Sigma_i, f_{\Sigma_i}))$ have the same upper bound $Q(T, \Sigma_i)$.

Now, we need to show that if we replace the matrix $A_1(\Sigma_i)$ defined in terms of f by the matrix $A_1(\Sigma_i, f_{\Sigma_i})$ defined in terms of f_{Σ_i}, we will not change the property of the coincidence of the degree polygon with its upper bound. Let r, $s \in \Sigma_i$. If $ps - r$ belongs to the closure of Σ_i, Lemma 3.5 shows that the degree of $F_{ps-r}(f) - F_{ps-r}(f_{\Sigma_i})$ in T is strictly smaller than the expected degree $m(T, \Delta; ps - r)$. Thus, we can replace $F_{ps-r}(f)$ by $F_{ps-r}(f_{\Sigma_i})$. If $ps - r$ does not belong to the closure of Σ_i, then Lemma 3.6 shows that the degree of $F_{ps-r}(f)$ in T is strictly smaller than the expected degree $pm(T, \Delta; s) - m(T, \Delta; r)$, while $F_{ps-r}(f_{\Sigma_i}) = 0$. In this case, we can also replace $F_{ps-r}(f)$ by $F_{ps-r}(f_{\Sigma_i})$. This establish the "open" decomposition that the degree polygon of $\det(I - tA_1(f))$ defined with respect to $P(\Delta)$ coincides with $Q(T, \Delta)$ at the m^{th} vertex if and only if the degree polygon of $\det(I - tA_1(\Sigma_i, f_{\Sigma_i}))$ defined with respect to $P(\Sigma_i)$ coincides with $Q(T, \Sigma_i)$ at the m^{th} vertex for all $0 \leq i \leq 2$.

To finish the proof, we apply the boundary decomposition theorem (Theorem 5.1) and deduce that

$$\det(I - tA_1(f_1)) = \det(I - tA_1(\Sigma_0, f_{\Sigma_0}))\det(I - tA_1(\Sigma_1, f_{\Sigma_1})).$$

$$\det(I - tA_1(f_2)) = \det(I - tA_1(\Sigma_0, f_{\Sigma_0}))\det(I - tA_1(\Sigma_2, f_{\Sigma_2})).$$

Thus, the degree polygon of $\det(I - tA_1(\Sigma_i, f_{\Sigma_i}))$ coincides with $Q(T, \Sigma_i)$ at the m^{th} vertex for all $0 \leq i \leq 2$ if and only if the degree polygon of $\det(I - tA_1(f_i))$ defined with respect to $P(\Delta(i))$ coincides with $Q(T, C(\Delta(i)))$ at the m^{th} vertex for $1 \leq i \leq 2$. Theorem 7.1 is proved.

By the chain level version of Conjecture 1, we mean the assertion that the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound for all primes p in the residue class $p \equiv 1 \pmod{D(\Delta)}$, where $D(\Delta)$ is the denominator of Δ. If $\mathrm{L}^*(f, t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathbf{V}(\Delta)$ for some f (this is true if f is non-degenerate), then the chain level version of Conjecture 1 implies Conjecture 1. The chain level version is stronger because it does not exclude small prime numbers. We now use the above theorems to prove that the chain level version of Conjecture 1 is true in various special cases. For simplicity, in what follows Conjecture 1 means the chain level version.

Assume now that Δ in $\mathbf{R}^{n+1}$ is the $(n+1)$-dimensional integral convex polyhedron generated by the origin and some n-dimensional face Δ_1 not containing the origin. By a suitable coordinate change (applied to Laurent polynomials) of the form

$$x_j \longrightarrow \prod_{i=1}^{n+1} x_i^{t_{ij}},\ \mathcal{T} = (t_{ij}) \in \mathrm{GL}(n+1, \mathbf{Z}),$$

we may assume that Δ_1 lies on the hyperplane $x_{n+1} = D$. Let the V_j be the set of all lattice points in Δ_1. We first treat the case when Δ_1 is an integral unit cube.

Lemma 7.2. *Let Δ be generated by the origin and $V + \sum_{i=1}^{n} \epsilon_i e_i$, where V is a lattice point on the hyperplane $x_{n+1} = D$, the e_i are an integral basis for the lattice points in $\mathbf{R}^n$ and the ϵ_i take values among $\{0, 1\}$ (i.e., Δ_1 is an integral unit cube of dimension n). Then Conjecture 1 is true.*

Proof. Since Δ_1 is a unit cube, there are no lattice points on Δ_1 other than the vertices. We use Theorem 6.4 to decompose Δ with respect to the vertex V. In this way, we get n integral polyhedra $\Delta(i)$ $(i = 1, \cdots, n)$, where the non-zero vertices of $\Delta(i)$ are V and $V + e_i + \sum_{j \neq i}^{n} \epsilon_j e_j$. Since each $\Delta(i)$ is simplicial with respect to V and there are no lattice points on $\Delta(i) \cap \Delta_1$ other than the vertices, one checks that the assumption of part (ii) of Proposition 4.3 is satisfied. We deduce that the degree polygon for $\Delta(i)$ with respect to V coincides with its upper bound if and only if the Newton polygon for $\Delta(i)$ coincides with its lower bound. Thus, Conjecture 1 for Δ is reduced to the $\Delta(i)$. Now, for each i we apply Theorem 6.4 to $\Delta(i)$ with respect to the vertex $V + e_i$. We obtain $(n-1)$ integral polyhedra $\Delta(ij)$ $(j \neq i)$, where the non-zero vertices of $\Delta(ij)$ are V, $V + e_i$ and $V + e_i + e_j + \sum_{k \neq j,i}^{n} \epsilon_k e_k$. Since each $\Delta(ij)$ is simplicial with respect to $V + e_i$ and there are no lattice points on $\Delta(ij) \cap \Delta_1$ other than the vertices, the above argument shows that the degree polygon for $\Delta(ij)$ with respect to $V + e_i$ coincides with its upper bound if and only if the Newton polygon for $\Delta(ij)$ coincides with its lower bound. Thus, Conjecture 1 for Δ is reduced to the $\Delta(ij)$. Repeating the above procedure and using induction, we see that Conjecture 1 for Δ is reduced to the simplicial $\Delta(i_1 i_2 \cdots i_n)$, where $i_1 \cdots i_n$ runs over the permutations of $\{1, 2, \cdots, n\}$ and the non-zero vertices of $\Delta(i_1 i_2 \cdots i_n)$ are the column vectors of the matrix

$$M = (V, V + e_{i_1}, V + e_{i_1} + e_{i_2}, \cdots, V + e_{i_1} + \cdots + e_{i_n}).$$

Clearly, $\det(M) = D \det(e_{i_1}, \cdots, e_{i_n})$. Since the e_i are an integral basis for the lattice points in

$\mathbf{R}^n$, the determinant of M has absolute value D. This proves that Conjecture 1 is true for each $\Delta(i_1 i_2 \cdots i_n)$, and hence also for Δ.

Next, we treat the case when Δ is a general integral box.

Theorem 7.3. *Let Δ be generated by the origin and $V + \sum_{i=1}^n \epsilon_i d_i e_i$, where the d_i are positive integers (thus, Δ_1 is a box of dimension n). Then Conjecture 1 is true.*

Proof. The idea is to use Theorem 7.1 to reduce the theorem to the case treated in Lemma 7.2. We use induction on the number $d = \sum_i d_i$. Lemma 7.2 shows that the theorem is true for $d = n$ (all $d_i = 1$). Assume now that $d > n$. Then one of the d_i, say d_1, is greater than 1. Let H be the hyperplane generated by the origin and the vertices $V + (d_1 - 1)e_1 + \sum_{j=2}^n \epsilon_j d_j e_j$. Similarly, let K be the hyperplane generated by the origin and the vertices $V + d_1 e_1 + \sum_{j=2}^n \epsilon_j d_j e_j$ (these vertices are the vertices of a boundary face of Δ_1). Then H cuts Δ into two pieces $\Delta(1)$ and $\Delta(2)$, where the non-zero vertices of $\Delta(1)$ are $V + \epsilon_1(d_1 - 1)e_1 + \sum_{i=2}^n \epsilon_i d_i e_i$ and the non-zero vertices of $\Delta(2)$ are $V + (d_1 - 1 + \epsilon_1)e_1 + \sum_{i=2}^n \epsilon_i d_i e_i$. To apply Theorem 7.1, we need to prove that the degree polygon for $\Delta(2)$ with respect to the lattice points in H coincides with its upper bound. Let $V_1, \cdots, V_h$ be the lattice points on the unique face of $\Delta(2)$ not containing the origin. For each $j \leq h$, V_j lies on one of the hyperplanes H or K. If V_j lies on H, then the e_1 coordinate of $V_j - V$ is $d_1 - 1$. If V_j lies on K, then the e_1 coordinate of $V_j - V$ is d_1. Without loss of generality, we may assume that $V_1, \cdots, V_k$ lie on H and the rest of the V_j (all $j \leq h$) lie on K. Thus, $T_H = \{V_1, \cdots, V_k\}$.

Let r be a rational point in the cone generated by $\Delta(2)$. We may assume that $w(r) = 1$. Then r lies on the unique face of $\Delta(2)$ not containing the origin. We can write

$$r = V + (d_1 - 1 + c_1)e_1 + \sum_{j=2}^n c_j d_j e_j, \ 0 \leq c_j \leq 1,$$

where c_1 is uniquely determined by r because $d_1 - 1 + c_1$ is the e_1 coordinate of $r - V$. If $(u_1, \cdots, u_n)$ is a non-negative rational solution of the equation $\sum_{j=1}^h u_j V_j = r$, then $w(r) = \sum_{j=1}^h u_j = 1$ implies that

$$\sum_{j=1}^h u_j (V_j - V) = r - V.$$

Comparing the e_1 coordinate, we see that

$$(\sum_{j=1}^k u_j)(d_1 - 1) + (\sum_{j=k+1}^h u_j)d_1 = (d_1 - 1 + c_1).$$

$$w(r) = \sum_{j=1}^{k} u_j + \sum_{j=k+1}^{h} u_j = 1.$$

This shows that $(u_1 + \cdots + u_k)$ depends only on c_1, and hence only on r. Thus, any solution will make $\sum_{i=1}^{k} u_i$ maximal and the assumption of part (ii) of Proposition 4.3 is satisfied. We deduce that if the generic Newton polygon for $\Delta(2)$ coincides with its lower bound, then the degree polygon for $\Delta(2)$ with respect to the lattice points in T_H coincides with its upper bound. By induction, Conjecture 1 is true for $\Delta(2)$. Hence, the degree polygon for $\Delta(2)$ coincides with its upper bound. By induction, Conjecture 1 is also true for $\Delta(1)$. We can then apply Theorem 7.1 to conclude that Conjecture 1 is true for Δ. The theorem is proved.

To treat more general cases, we define an incomplete integral unit cube to be an integral convex polyhehron contained in an integral unit cube. If we intersect an arbitrary Δ_1 with an integral unit cube, we obtain an incomplete cube but in general it might not be integral (i.e., the vertices of the incomplete cube may not be lattice points).

Theorem 7.4. *Let Δ_1 be an n-dimensional integral convex polyhedron lying on the hyperplane $x_{n+1} = D$. (i). Assume that the intersections of Δ_1 with all n-dimensional integral unit cubes in $\mathbf{R}^n$ (lying on the hyperplane $x_{n+1} = D$) are integral and assume that Conjecture 1 is true for all of these intersections. Then Conjecture 1 is true for Δ, where Δ is generated by the origin and Δ_1. (ii). Assume that $\{H_j\}$ $(1 \leq j \leq k)$ is a finite sequence of successive parrallel hyperplanes in the n-dimensional space $x_{n+1} = D$ (thus, there are no lattice points lying strictly between H_j and H_{j+1}). Assume that H_1 and H_k do not intersect the interior of Δ_1. Assume that this sequence of hyperplances $\{H_j\}$ cut Δ_1 into finitely many integral polyhedra and assume that Conjecture 1 is true for each of these polyhedra, then Conjecture 1 is true for Δ.*

Proof. (i). It is clear that Δ_1 is contained in some n-dimensional integral box. We can then repeat the proof of Theorem 7.3 and use induction. In this case, we also obtain some incomplete integral unit cubes for which we can use our assumptions.

(ii). The proof is to use Theorem 7.1 and induction on the number k. If $k \leq 2$, it is trivial. If $k > 2$, let H be the space spanned by H_2 and the origin. Then, H is a hyperplane in the $(n+1)$-dimensional space. One sees that H non-trivially cuts Δ into two pieces $\Delta(1)$ and $\Delta(2)$. Here $\Delta(2)$ denotes the piece intesecting H_1. Thus, T is the set of lattice points in $\Delta(1) \cap \Delta_1$ and

T_H is the the set of lattice points in $H \cap \Delta_1$. By induction and our assumptions, Conjecture 1 is true for both $\Delta(1)$ and $\Delta(2)$. Since H_1 does not intersect the interior of $\Delta(2)$ and H_1 is successive and parrallel to H_2, the proof of Theorem 7.3 shows that the degree polygon for $\Delta(2)$ with respect to T_H coincides with its upper bound if and only if the Newton polygon for $\Delta(2)$ coincides with its lower bound. The latter is true by our assumption. By induction, the Newton polygon for $\Delta(1)$ also coincides with its lower bound. The theorem then follows from Theorem 7.1.

For applications to the zeta functions of affine hypersurfaces of degree d, we need to consider the case that Δ is the $(n+1)$-dimensional simplex generated by the origin, V and $V + de_i$. Thus, Δ_1 is the n-dimensional simplex with vertices V and $V + de_i$.

Theorem 7.5. *Let Δ be generated by the origin, V and $V + de_i$, where d is a positive integer and V is a lattice point on the hyperplane $x_{n+1} = D$. Then Conjecture 1 is true.*

Proof. To simplify notations, we may assume that Δ_1 is the n-dimensional simplex in $\mathbf{R}^n$ generated by the origin and the vertices de_i. We may also assume that the e_i are the standard basis. Now, we use the regular subdivision of a simplex as described in [8, p117-119]. Define the cumulative coordinates as follows:

$$\begin{aligned} y_1 =& x_1, \\ y_2 =& x_1 + x_2, \\ & \cdots\cdots \\ y_n =& x_1 + x_2 + \cdots + x_n. \end{aligned}$$

One sees that Δ_1 is identified with the set of n-tuples of real numbers $y_1, \cdots, y_n$ such that

$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq d.$$

Consider the following hyperplanes in $\mathbf{R}^n$:

$$H(i,j;k): \ y_i - y_j = k, \text{ where } 0 \leq j < i \leq n, \ 0 \leq k \leq d.$$

We claim that these hyperplanes cut Δ_1 into integral simplexes with determinant 1 (equivalently, with volume $1/n!$). To prove the claim, let $y_1, \cdots, y_n$ be a point in $\Delta_1' = \Delta_1 - \cup_{i,j,k} H(i,j;k)$. If we put $t_i = y_i - [y_i]$, it is clear that $t_i \neq t_j$ whenever $i > j$ (if $t_i = t_j$, then $y_i - y_j$ is an integer k

in the interval $[0, d]$. This contradicts with the assumption that $y_1, \cdots, y_n \in \Delta_1'$). Hence there is a unique permutation π such that

$$0 < t_{\pi(1)} < t_{\pi(2)} < \cdots < t_{\pi(n)} < 1.$$

It is also clear that the connected component of Δ_1' determined by the point $y_1, \cdots, y_n$ is the set of points $y_1', \cdots, y_n'$ such that

$$0 < y'_{\pi(1)} - [y'_{\pi(1)}] < \cdots < y'_{\pi(n)} - [y'_{\pi(n)}] \leq 1.$$

This is clearly the interior of a simplex with integral vertices and with determinant 1. The claim is proved.

For each fixed pair (i, j) with $i > j$, the hyperplanes $H(i, j; k)$ $(0 \leq k \leq d)$ form a sequence of successive parrallel hyperplances. Using the claim together with part (ii) of Theorem 7.4, by induction we deduce that Conjecture 1 is true for Δ. Theorem 7.5 is proved.

For applications to the zeta functions of projective hypersurfaces of degree d, we need to consider the case that Δ is the n-dimensional simplex in $\mathbf{R}^{n+1}$ generated by the origin and $V + de_i$. This Δ is a codimension 1 face of the polyhedron treated in Theorem 7.5. The boundary decomposition theorem (Corollary 5.2) shows that Conjecture 1 is also true for this Δ. Thus, we have

Corollary 7.6. *Let Δ be generated by the origin and $V + de_i$, where d is a positive integer and V is a lattice point on the hyperplane $x_{n+1} = D$. Then Conjecture 1 is true.*

As a final application, we prove

Corollary 7.7. *Conjecture 1 is true if $n \leq 2$.*

Proof. If $n = 1$, it is trivial. Assume now that $n = 2$. By Theorem 5.5, we may assume that the origin is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin. Let D be the denominator of Δ. Since $n = 2$, one sees that Δ must be a simplex. By a suitable coordinate change, we may then assume that Δ_1 is the line segment connecting $V_1 = (D, k)$ and $V_2 = (D, k + l)$, where l is a positive integer. It is easy to see that the intersection of Δ_1 with any integral unit line segment is integral. The corollary follows from part (i) of Theorem 7.4. One can also use part (ii) of Theorem 7.4 to get the conclusion.

§8. The Function $\mathrm{L}(f,t)$

Let $0 \leq h \leq n$, and let $f \in \mathbf{F}_q[x_1, \cdots, x_n, (x_1 \cdots x_h)^{-1}]$ be a Laurent polynomial. One can define the exponential sum

$$\mathrm{S}_m(f) = \sum \Psi(\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(f(x_1, \cdots, x_n))), \tag{8.1}$$

where the sum is over all $(x_1, \cdots, x_n) \in (\mathbf{F}_{q^m})^n$ such that $x_j \neq 0$ $(1 \leq j \leq h)$. If $h = n$, $\mathrm{S}_m(f)$ reduces to $\mathrm{S}^*_m(f)$. The associated L-function is

$$\mathrm{L}(f,t) = \exp\Big(\sum_{m=1}^{\infty} \mathrm{S}_m(f)\frac{t^m}{m}\Big). \tag{8.2}$$

In this section, we explain how the results in §5 − 7 carry over to the function $\mathrm{L}(f,t)^{(-1)^{n-1}}$ and give several applications to zeta functions.

Let U be a subset of $\{1, 2, \cdots, n\}$, $S_1 = \{1, 2, \cdots, h\}$ and $S_2 = \{h+1, \cdots, n\}$. Let $C(U) = \{r \in C(\Delta) \mid r_i > 0 \text{ if } i \in U \cap S_2\}$. Let $A_1(U)$ denote the semilinear nuclear matrix $(a_{s,r}(f))$, where s and r run through the lattice points in the cone $C(U)$. Let

$$A_a(U) = \prod_{i=0}^{a-1} A_1(U)^{\tau^i} = A_1(U)A_1^{\tau}(U) \cdots A_1^{\tau^{a-1}}(U). \tag{8.3}$$

The Dwork trace formula [2] asserts that

$$\mathrm{L}(f,t)^{(-1)^{n-1}} = \prod_{l=0}^{n} \prod_{|U|=l} \det{}^{\tau^{-1}}(I - tq^{n-l}A_a(U))^{(-1)^{n-l}}. \tag{8.4}$$

Let

$$W_U(k) = \mathrm{card}\{r \in C(U) \cap \mathbf{Z}^n \mid w(r) = \frac{k}{D(\Delta)}\}. \tag{8.5}$$

Then the Newton polygons of $\det(I - tA_a(U))$ computed with respect to q and the Newton polygon of $\det(I - tA_1(U))$ computed with respect to p lie above the polygon with vertices $(0,0)$ and

$$\Big(\sum_{k=0}^{m} W_U(k), \frac{1}{D(\Delta)}\sum_{k=0}^{m} kW_U(k)\Big), \quad m = 0, 1, 2, \cdots. \tag{8.6}$$

Furthermore (Theorem 2.4), one of the above Newton polygons coincides with (8.6) if and only if the other Newton polygon coincides with (8.6). The degree polygon of $\det(I - tA_1(U))$ with respect to a set T of lattice points V_i is defined in §4. Then all of the results in §5 − 7 carry over to

$\det(I-tA_1(U))$. For applications to $\mathrm{L}(f,t)$, one can then use (8.4). In particular, Theorem 3 is true for $\mathrm{L}(f,t)$. That is, there is a positive integer D^* depending only on Δ such that if $p \equiv 1 \pmod{D^*}$ and p is large, then the generic Newton polygon of $\mathrm{L}(f,t)^{(-1)^{n-1}}$ coincides with the lower bound of Adolphson and Sperber (provided, of course, $\mathrm{L}(f,t)^{(-1)^{n-1}}$ is a polynomial for such general f. Adolphson-Sperber [2] showed that this is true if f is non-degenerate and commode with respect to S_2).

We now give several applications to zeta functions. First, let us consider affine hypersurfaces. Let d, $d_1, \cdots, d_n$ be n positive integers. Let $g(x_1, \cdots, x_n)$ be a general polynomial of totoal degree d or a general polynomial of degree d_i with respect to x_i. It is well-known [2] that the zeta function of the affine hypersurface defined by $g(x)=0$ is given by

$$\mathrm{Z}(g(x), qt) = \mathrm{L}(\mathbf{A}^{n+1}, x_{n+1}g(x); t) = \frac{\mathrm{L}(\mathbf{A}^n \times \mathbf{G}_m, x_{n+1}g(x); t)}{1-q^n t}.$$

One checks that for all large prime number p, $g(x)$ is non-degenerate and commode with respect to $x_1, \cdots, x_n$ (in the terminology of [2]). Thus, the theorem of Adolphson and Sperber implies that for all large p, $\mathrm{L}(\mathbf{A}^n \times \mathbf{G}_m, x_{n+1}g(x); t)^{(-1)^n}$ is a polynomial of a certain degree whose Newton polygon lies above the lower bound of Adolphson and Sperber. Now, Theorem 7.3 and Theorem 7.5 with $D=1$ show that the Newton polygon of $L(\mathbf{A}^n \times \mathbf{G}_m, x_{n+1}g(x); t)^{(-1)^n}$ coincides with its lower bound for all large p. This proves Theorem 6 and Theorem 7.

Next, we consider projective hypersurfaces. Let $g(x_1, \cdots, x_n)$ be a homogeneous polynomial of degree d and n variables. Let X be the hypersurface in $\mathbf{P}^{n-1}$ defined by $g(x)=0$. Assume that X is smooth. Then, it is well-known that the zeta function for X has the form

$$\mathrm{Z}(X,t) = \frac{R(t)^{(-1)^{n-1}}}{(1-t)(1-qt)\cdots(1-q^{n-2}t)},$$

where $R(t)$ is a polynomial. The relation between $R(t)$ and L-function [2] is known to be

$$\mathrm{L}(\mathbf{A}^n \times \mathbf{G}_m, x_{n+1}g(x); t)^{(-1)^n} = \frac{R(qt)}{R(q^2t)}. \tag{8.7}$$

We now derive an expression of $R(t)$ in terms of Fredholm determinants. Note that $\Delta(x_{n+1}g)$ in $\mathbf{R}^{n+1}$ is n-dimensional, being contained in the hyperplane $dx_{n+1} = x_1 + \cdots + x_n$. In the notation of (8.4) with $f = x_{n+1}g(x)$ (now, $S_2 = \{1, 2, \cdots, n\}$ and $S_1 = \{n+1\}$), we have

$$\mathrm{L}(\mathbf{A}^n \times G_m, x_{n+1}g(x); t)^{(-1)^n} = \prod_{l=0}^{n+1} \prod_{|U|=l} \det{}^{\tau^{-1}}(I - tq^{n+1-l}A_a(U))^{(-1)^{n+1-l}}, \tag{8.8}$$

where U runs over subsets of $\{1,2,\cdots,n,(n+1)\}$. Let

$$S(t)=\prod_{l=0}^{n}\prod_{|U_1|=l}\det{}^{\tau^{-1}}(I-tq^{n-l}A_a(U_1))^{(-1)^{n-l}}, \tag{8.9}$$

where U_1 runs only over subsets of $\{1,2,\cdots,n\}$ (corresponding to the variables x_i with $1\leq i\leq n$). If $n+1\in U$, then $U\cap S_2=\{U-\{n+1\}\}\cap S_2$. Thus, $C(U)=C(U-\{n+1\})$ and $A_1(U)=A_1(U-\{n+1\})$. Then, (8.8) takes the form

$$\mathrm{L}(\mathbf{A}^n\times G_m,x_{n+1}g(x);t)^{(-1)^n}=\frac{S(t)}{S(qt)}. \tag{8.10}$$

Let $G(T)=S(t)/R(qt)$. Comparing (8.10) with (8.7), we deduce that $G(t)=G(qt)$. Since $G(t)$ is a non-zero p-adic meromorphic function with constant term 1. This forces $G(t)=1$. That is, $S(t)=R(qt)$ is a polynomial of expected degree. Theorem 7.6 with $D=1$ and equation (8.9) then imply that the generic Newton polygon of $S(t)$ and $R(qt)$ coincides with its lower bound. This gives a p-adic proof of Theorem 5.

9. Newton Polygons and Frobenius Cycles on Lattice Points

In this section, we give a conjecture which describes all the large prime numbers p for which the Newton polygon coincides generically with its lower bound. This conjecture involves Frobenius cycles defined below. We will work on the chain level.

We assume that the origin 0 is a vertex of Δ and that Δ has only one face Δ_1 of codimension 1 which does not contain the origin. Let $V_1,\cdots,V_J$ be J fixed lattice points on the face Δ_1 which include all vertices of Δ_1. Then Δ is generated by 0 and the V_j. Let $E(V)$ be the set of all non-negative integral linear combinations of the V_i. One sees that $E(V)$ is the monoid generated by the V_i. It is not a free monoid unless Δ is a simplex and the V_i are vertices. Let $E(\Delta)$ be the lattice points in the cone $C(\Delta)$. Then, $E(V)$ is a submonoid of full rank in $E(\Delta)$. Let $E_{\leq k}(\Delta)$ be the subset of $E(\Delta)$ consisting of all lattice points with weight at most k/D.

Let $(u_1,\cdots,u_k)$ be an ordered set of k distinct lattice points in $E(\Delta)$. If

$$pu_1-u_2\in E(V),\ pu_2-u_3\in E(V),\cdots,\ pu_k-u_1\in E(V),$$

we call $(u_1,\cdots,u_k)$ a Frobenius k-cycle. In particular, a Frobenius 1-cycle is a lattice point u_1 such that $(p-1)u_1\in E(V)$. A set of lattice points is said to have a disjoint Frobenius covering if the set can be written as a disjoint union of Frobenius cycles.

Let f be the general Laurent polynomial $f = \sum_j a_j x^{V_j}$. Clearly, the Newton polyhedron of f is Δ. Recall that $A_1(f)$ is the semi-linear nuclear Frobenius matrix associated with f.

Conjecture 9.1. *Let m be a positive integer and p be a large prime number depending only on Δ. Then the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound at the m^{th} vertex if and only if the set $E_{\leq m}(\Delta)$ has a disjoint covering by Frobenius cycles.*

It can be proved that the conjecture is true if the V_i are vertices of a simplex. For the general case, we have the following partial result which asserts that the condition is necessary. The proof is based on the trivial observation that in order for a determinant to be non-zero, at least one term of the determinant expansion must be non-zero.

Theorem 9.2. *If the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound at the m^{th} vertex, then the set $E_{\leq m}(\Delta)$ has a disjoint covering by Frobenius cycles.*

Proof. Recall that

$$A_1(f) = (a_{s,r}(f)) = (F_{ps-r}(f)\pi^{(w(r)-w(s))}),$$

where $F_u(f)$ is defined by (2.9). The Fredholm determinant of a nuclear matrix can be expanded in the usual way. Write

$$\det(I - tA_1(f)) = \sum_{k=0}^{\infty} c(k)t^k.$$

Then

$$c(k) = (-1)^k \sum \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{k} a_{s_i,\sigma(s_i)}(f),$$

where the outer sum is over all sets of k distinct lattice points in $E(\Delta)$, the inner sum is over all permutations on k letters, and $\epsilon(\sigma)$ is the sign of the permutation.

The m^{th} vertex of the lower bound for the Newton polygon is

$$(N(m), M(m)) = \Big(\sum_{k \leq m/D} W(k), \frac{1}{D} \sum_{k \leq m/D} kW(k) \Big).$$

The block form of $A_1(f)$ shows that we can write

$$c(N(m)) = (-1)^{N(m)} \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{N(m)} a_{s_i,\sigma(s_i)}(f) + o(p^{M(m)}), \tag{9.1}$$

where $s_1, \cdots, s_{N(m)}$ are the elements of the set $E_{\leq m}(\Delta)$ and the ord_p of the second term is greater than $M(m)$. By assumption, the generic Newton polygon of $\det(I - tA_1(f))$ coincides with its lower bound at the m^{th} vertex. This means that $\mathrm{ord}_p c(N(m)) = M(m)$. Thus, at least one of the products in (9.1) is non-zero. Fix such a term corresponding to some permutation σ. Since σ decomposes into disjoint cycles, the non-zero term decomposes into a product of the form

$$a_{u_1,u_2}(f)\cdots a_{u_k,u_1}(f), \quad \sigma(u_1)=u_2,\cdots,\sigma(u_k)=u_1.$$

Since these products are non-zero,

$$F_{pu_1-u_2}(f)\neq 0,\cdots,F_{pu_k-u_1}(f)\neq 0.$$

By the definition of $F_u(f)$, this means that

$$pu_1-u_2\in E(V),\ pu_2-u_3\in E(V),\cdots,\ pu_k-u_1\in E(V).$$

Thus, $E_{\leq m}(V)$ has a disjoint Frobenius covering. The theorem is proved.

Example 9.3. The quotient monoid $E(\Delta)/E(V)$ is finite. Hence, it is a finite abelian group. Denote its annihilator by $D^*(\Delta)$. If now $p\equiv 1 \pmod{D^*(\Delta)}$, then $(p-1)u\in E(V)$ for all $u\in E(\Delta)$. Thus, $E_{\leq m}(\Delta)$ has a disjoint covering by Frobenius 1-cycles. Conjecture 9.1 would then imply that for all $p\equiv 1 \pmod{D^*(\Delta)}$, the generic Newton polygon of $\det(I-tA_1(f))$ coincides with its lower bound.

Example 9.4. Let $f(x)$ be the general Laurent polynomial $f(x)=\sum_{j=1}^{J}a_jx^{V_j}$. For positive real number x, let $\pi_V(x)$ be the number of prime numbers $p\leq x$ such that the generic Newton polygon of $\det(I-tA_1(f))$ coincides with its lower bound. Then Conjecture 9.1 would imply that there is a positive rational number r such that

$$\pi_V(x)=r\frac{x}{\log x}+O(\frac{x}{\log^2 x}).$$

This means for general $f(x)$, the set of ordinary primes has positive rational density.

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