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Rank one case of Dwork's conjecture.

The two papers of Wan under review prove an important and long-outstanding conjecture of B. Dwork in the $p$-adic analytic theory of $L$-functions of sheaves on schemes over finite fields. Wan proves that if $X$ is a smooth affine scheme of finite type over a finite field, then the pure slope $L$-functions of any overconvergent $\sigma$-module on $X$ are meromorphic functions on the $p$-adic affine line. This completes earlier partial results of the same author [Ann. of Math. (2) 150 (1999), no. 3, 867–927; MR 2001a:11108], and we refer to the review of that paper for another exposition of the ideas surrounding Dwork's conjecture and Wan's approach to proving it.

For the duration of this review, $X$ will denote a smooth affine scheme of finite type over the finite field $F_q$ of order $q$ (where $q = p^d$ for some prime $p$ and some integer $d \geq 1$). If $x$ is a closed point of $X$, let $\kappa(x)$ denote the residue field of $x$, let $\deg(x)$ denote the degree of $\kappa(x)$ over $F_q$, and let $\overline{\kappa}(x)$ be an algebraic closure of $\kappa(x)$. Let $\text{Frob}_x$ denote the geometric Frobenius automorphism of $\overline{\kappa}(x)$ over $\kappa(x)$; this is the inverse of the map which raises any element of $\overline{\kappa}(x)$ to the $(q^\deg(x))$th power. This automorphism is a topological generator of the Galois group $\text{Gal}(\overline{\kappa}(x)/\kappa(x))$.

Before elaborating on the meaning of Wan's result, it will help to first recall the general $l$-adic theory of $L$-functions of sheaves on the etale site of $X$. Thus we let $l$ be a prime, and let $\mathcal{F}$ be an $l$-adic local system (that is, a lisse sheaf of $\mathbb{Z}_l$-modules) on the etale site of $X$. The stalk of $\mathcal{F}$ at a closed point $x$ of $X$ is a free $\mathbb{Z}_l$-module of finite rank equipped with a continuous action of $\text{Gal}(\overline{\kappa}(x)/\kappa(x))$. The local $L$-factor of $\mathcal{F}$ at $x$ is defined to be the reciprocal of the characteristic polynomial of $\text{Frob}_x$ acting on the stalk $\mathcal{F}_x$, computed with respect to the variable $T^{\deg(x)}$. The $L$-function of $\mathcal{F}$ is then the infinite product of all such local $L$-factors: $L(\mathcal{F}, T) = \prod_{x \in |X|} \det(\text{Id} - \text{Frob}_x T^{\deg(x)}, \mathcal{F}_x)^{-1}$.

(A Here $|X|$ denotes the set of closed points of $X$.) Since there are only finitely many closed points of $X$ of any given degree, this product yields a well-defined element of the formal power-series ring $\mathbb{Z}_l[[T]]$.

A fundamental theorem of Grothendieck states that if $l$ is distinct from $p$, then the power series $L(\mathcal{F}, T)$ is in fact a rational function of $T$ [see C. Houzel, in Cohomologie $l$-adique et fonctions $L$, Exp. XIV, Lecture Notes in Math., 589, Springer, Berlin, 1977; see MR 58#10907; see also P. Deligne, Cohomologie etale, Lecture Notes in Math., 569, Springer, Berlin, 1977; MR 57#3132 (Chapter 3)].
One case of note occurs when $\mathcal{F}$ equals the constant sheaf $\mathbb{Z}_l$; in this case the $L$-function is equal to the product $\prod_{x \in X} (1 - T^\deg(x))^{-1}$, which is the so-called zeta function of $X$, denoted $\zeta(X, T)$. It is a generating function for the number of points of $X$ defined over all finite extensions of $\mathbb{F}_q$. Grothendieck’s theorem implies in particular that $\zeta(X, T)$ is a rational function; this is one of the celebrated Weil conjectures.

It is natural to ask what happens in the case of $p = l$. If we consider the constant sheaf $\mathcal{F} = \mathbb{Z}_p$, then the $L$-function of $\mathcal{F}$ is again the zeta function of $X$ (in the case of a constant sheaf, the local $L$-factors are insensitive to the choice of coefficient ring), and, as already noted, is a rational function. In fact, this result was proved by Dwork [Amer. J. Math. 82 (1960), 631–648; MR 25#3914] prior to Grothendieck’s general theorem on the rationality of $L$-functions, by a method quite different from that of Grothendieck. Dwork worked in the context of $l = p$, rather than that of $l \neq p$, and established that $\zeta(X, T)$ is a meromorphic function on the $p$-adic affine $T$-line. On the other hand, a trivial estimate shows that $\zeta(X, T)$ also converges to a holomorphic function in a small disk around the origin of the complex affine $T$-line. A general lemma then implies that since its coefficients are integers, $\zeta(X, T)$ is necessarily a rational function.

The key point in Dwork’s argument is thus to prove that $\zeta(X, T)$ is a meromorphic function of the $p$-adic variable $T$, and it is natural to ask whether this result remains true for the $L$-functions of $p$-adic local systems $\mathcal{F}$ on the étale site of $X$ other than the constant local system $\mathbb{Z}_p$. In fact it is not true, as Wan demonstrated [Ann. of Math. (2) 143 (1996), no. 3, 469–498; MR 97c:14021]. All that one can say for an arbitrary $p$-adic local system $\mathcal{F}$ is that the $L$-function $L(\mathcal{F}, T)$ is meromorphic on the $p$-adic closed unit $T$-disc. (Note that a priori, the $L$-function is an element of $\mathbb{Z}_p[[T]]$, and so describes a holomorphic function on the open unit $T$-disc.)

The question naturally arises: is there a more restricted class of $p$-adic local systems, which includes the constant local system, for which the $L$-function is a $p$-adic meromorphic function on the affine $T$-line? Before describing an (in fact, more than one) answer to this question, we have to recall the notion of a $\sigma$-module on $X$, the special cases of unit-root and overconvergent $\sigma$-modules, and the definition of their $L$-functions. This we now do.

Fix a complete discrete valuation ring $R$ whose fraction field is of characteristic zero, and whose residue field is equal to $\mathbb{F}_q$. We may lift $X$ to a formally smooth formal $R$-scheme $\mathcal{X}$. If $F_X$ denotes the endomorphism of $\mathcal{X}$ which raises sections of the structure sheaf $\mathcal{O}_X$ to
the qth power (the dth power of the absolute Frobenius endomorphism of X), then we may lift \( F_X \) to an endomorphism \( \sigma \) of the R-scheme \( X \). Now suppose that \( \mathcal{F} \) is a local system of finite-rank free \( R \)-modules on the etale site of \( X \). We may deform \( \mathcal{F} \) in a unique fashion to a local system on the etale site of \( X \) (which we continue to denote by \( \mathcal{F} \)), and then form the tensor product \( \mathcal{M} = \mathcal{O}_X \otimes_R \mathcal{F} \), which is a locally free sheaf on \( X \), equipped with the \( \sigma \)-linear operator \( \varphi = \sigma \otimes \text{Id}_\mathcal{F} \).

A pair \((\mathcal{M}, \varphi)\) consisting of a finite-rank locally free sheaf \( \mathcal{M} \) on \( X \) together with a \( \sigma \)-linear endomorphism \( \varphi \) of \( \mathcal{M} \) is called a \( \sigma \)-module on \( X \). (The choice of lifting \( X \) is suppressed in the terminology.) The \( \sigma \)-linear endomorphism \( \varphi \) induces a morphism of locally free sheaves \( \sigma^* \mathcal{M} \to \mathcal{M} \); if this map is furthermore an isomorphism, the pair \((\mathcal{M}, \varphi)\) is called a unit-root \( \sigma \)-module. The construction of the preceding paragraph yields an equivalence of categories between the local systems of finite-rank free \( R \)-modules on the etale site of \( X \) and the unit-root \( \sigma \)-modules on \( X \) thus extends the category of finite-rank free \( R \)-modules on the etale site of \( X \).

Among all \( \sigma \)-modules, those that are overconvergent play a special role. To define these, we begin by writing the affine formal scheme \( X \) as the formal spectrum of a \( p \)-adically complete formally smooth \( R \)-algebra \( A \) of topologically finite type. A construction of Monsky and Washnitzer (as extended by van der Put) yields an \( R \)-subalgebra of \( A \) which we denote by \( A^\dagger \), whose definition we briefly recall. (We also remark that in the papers under review, the ring we are denoting by \( A \) is denoted by \( A_0 \), while the ring we are denoting by \( A^\dagger \) is denoted by \( A \).)

We let \( R(X_1, \ldots, X_n) \) denote the \( p \)-adic completion of the polynomial ring \( R[X_1, \ldots, X_n] \). Thus \( R(X_1, \ldots, X_n) \) is the subring of \( R[[X_1, \ldots, X_n]] \) consisting of those power series \( \sum_I a_I X_1^{i_1} \cdots X_n^{i_n} \) (the sum ranges over all multi-indices \( I = (i_1, \ldots, i_n) \)) for which the \( a_I \) tend to zero as the degree \( |I| = i_1 + \cdots + i_n \) of the multi-index \( I \) tends to infinity. In analytic terms, \( R(X_1, \ldots, X_n) \) consists of those power series in \( R[[X_1, \ldots, X_n]] \) which converge on the \( p \)-adic \( n \)-dimensional closed unit ball.

We let \( R(X_1, \ldots, X_n)^\dagger \) denote the subring of \( R(X_1, \ldots, X_n) \) consisting of power series whose coefficients \( a_I \) satisfy the condition

\[
\liminf_{|I|} v(a_I)/|I| > 0.
\]

(Here \( v \) denotes the discrete valuation on \( R \).) In analytic terms, \( R(X_1, \ldots, X_n)^\dagger \) consists of those power series in \( R[[X_1, \ldots, X_n]] \) which converge on some \( p \)-adic \( n \)-dimensional ball whose radius is strictly greater than one. (One says that such a power series overconverges on the closed unit ball.)
We can find an isomorphism between the ring $A$ (of which $X$ is the formal spectrum) and a quotient $R\langle X_1, \ldots, X_n \rangle/(f_1, \ldots, f_r)$, for some elements $f_i$ of $R[X_1, \ldots, X_n]$. We then define $A^\dagger$ to be the quotient $R\langle X_1, \ldots, X_n \rangle^\dagger/(f_1, \ldots, f_r)$, for some elements $f_i$ of $R[X_1, \ldots, X_n]$. Geometrically, if we use the preceding description of $A$ as a quotient to regard $X$ (or more properly, the rigid analytic generic fibre of $X$) as being a closed analytic subspace of the $n$-dimensional unit ball, then we may regard the ring $A^\dagger$ as corresponding to the “germ of all rigid analytic extensions of $X$ to a ball of radius strictly greater than one”; we denote this germ by $X^\dagger$.

We may choose our lift $\sigma$ of $F_X$ so that it extends to an endomorphism of $X^\dagger$. (More precisely, we choose $\sigma$ so that the corresponding endomorphism of $A$ restricts to an endomorphism of $A^\dagger$.) Thus we may define the notion of a $\sigma$-module on $X^\dagger$. (Precisely, this will be a locally free $A^\dagger$-module equipped with a $\sigma$-linear endomorphism.) We may restrict such an object to $X$, obtaining a $\sigma$-module on $X$. (Precisely, we tensor the given $A^\dagger$-module with $A$, or alternatively, take its $p$-adic completion.) We say that a $\sigma$-module on $X$ is overconvergent if it arises as the restriction of a $\sigma$-module on $X^\dagger$ in this way.

Algebraic geometry provides a rich source of overconvergent $\sigma$-modules on $X$. For example, if $f: Y \to X$ is a formally smooth proper morphism of formally smooth $R$-schemes, lifting the smooth proper morphism $Y \to X$ of $\mathbb{F}_q$-schemes, then the relative rigid cohomology sheaves of $Y$ over $X$ are overconvergent $\sigma$-modules on $X$.

If $(M, \varphi)$ is any $\sigma$-module on $X$ (overconvergent or not) we can define its local $L$-factor at a closed point $x$ of $X$. Let $\tilde{x}$ denote the Teichmuller lift of $x$, which is a point of $\mathcal{X}$ with values in a finite extension of $R$. The $\deg(x)$th power of $\varphi$ then induces a linear (as opposed to $\sigma$-linear) endomorphism of the fibre $M_{\tilde{x}}$, which we denote $\varphi^\deg(x)_{\tilde{x}}$, and the local $L$-factor is defined to be the inverse of the corresponding characteristic polynomial, computed with respect to the variable $T^\deg(x)$. The $L$-function of $(M, \varphi)$ is then defined to be the product of all such local $L$-factors: $L(\varphi, T) = \prod_{x \in |X|} (\det(\Id - \varphi^\deg(x)_{\tilde{x}}T^\deg(x), M_{\tilde{x}}))^{-1}$. If $(M, \varphi)$ is a unit-root $\sigma$-module, then this $L$-function coincides with that of the corresponding etale local system of $R$-modules. Extending the result for etale local systems, one can show that the $L$-function of a $\sigma$-module on $X$ is a meromorphic function on the $p$-adic closed unit $T$-disk. However, as already noted, even in the unit-root case, one cannot say anything more in general.

In contrast with the case of a general $\sigma$-module on $X$, a fundamental result states that the $L$-function of an overconvergent $\sigma$-module on $X$ is a meromorphic function on the $p$-adic affine $T$-line. This includes as a special case Dwork’s result that the zeta function of $X$ is
meromorphic. (Indeed, the $\sigma$-module corresponding to the constant sheaf $R$ on $X$ is just the rank-one free sheaf $\mathcal{O}_X$ on $X$, with $\varphi$ taken to be $\sigma$, and this $\sigma$-module is manifestly overconvergent (it is the base-change to $A$ of the ring $A^\dagger$ equipped with its endomorphism $\sigma$).

A proof of this result can be found in Section 3 and Section 10 of the first paper under review, based on results of Monsky, which were in turn inspired by Dwork’s original argument proving the meromorphicity of the zeta function. The underlying fact on which the proof depends is that the restriction of analytic functions from a ball of given radius to a ball of strictly smaller radius (and hence the restriction of functions from $X^\dagger$ to $X$) is a compact operator, and hence has a good spectral theory. Building (in a rather elaborate fashion) on this observation, one is able to write the $L$-function of an overconvergent $\sigma$-module as a ratio of products of finitely many Fredholm determinants of compact operators. Since the Fredholm determinant of a compact operator is an entire function, we find that the $L$-function is meromorphic.

This result explains the important role of overconvergent $\sigma$-modules in the $p$-adic theory of $L$-functions. In particular, we see that the overconvergent $\sigma$-modules thus do provide a natural class of sheaves, containing the constant sheaf, for which the $L$-function is meromorphic, yielding an answer to the question raised above. However, this is not the end of the story with regard to $p$-adic analytic properties of the $L$-functions of $\sigma$-modules. In the course of his detailed study of zeta functions of varieties moving in families, Dwork found that in some situations, an overconvergent $\sigma$-module could break up into pure slope parts (in a sense to be described below). Although these pure slope $\sigma$-modules need not be overconvergent, he was nevertheless led to conjecture that their $L$-functions are meromorphic on the $p$-adic affine $T$-line. Wan proves this conjecture in the papers under review. Thus we obtain a second (and richer) answer to the question raised above: the collection of all pure slope parts of overconvergent $\sigma$-modules yields a class of $\sigma$-modules whose $L$-functions are meromorphic.

Before defining the notion of the pure slope parts of a $\sigma$-module (when they exist), it is simpler (and more general) to first define its pure slope $L$-functions. Let $(M, \varphi)$ be a $\sigma$-module on $X$. Each local factor of the $L$-function of $(M, \varphi)$ is equal to the inverse of the characteristic polynomial of $\varphi^\deg(x)$ acting on $M_{\xi}$ (for some closed point $x$ of $X$), computed with respect to the variable $T^\deg(x)$. Let $P_x(T)$ denote this characteristic polynomial (it is an element of the polynomial ring $R[T]$). Then in general the zeroes of $P_x(T)$ (which lie
in some finite extension of \( R \) will not all be of the same valuation. We may factor \( P_x(T) \) into a product of elements of \( R[T] \), each factor having the property that its zeroes are all of the same valuation. Thus we may write \( P_x(T) = \prod_z P_{x,z}(T) \), where all the zeroes of \( P_{x,z}(T) \) have valuation equal to \(-s\). (Here \( s \) runs over all elements of \( \mathbb{Q} \), but the factor \( P_{x,z}(T) \) is equal to 1 for all but finitely many values of \( s \); \( s \) appears with a minus sign in the definition of \( P_{x,z}(T) \) so as to ensure that it is the reciprocal zeroes of \( P_{x,z}(T) \) that have valuation equal to \( s \).) We then define, for any rational number \( s \), the \( L \)-function of \((M, \varphi)\) of pure slope \( s \) to be the product \( L_s(\varphi, T) = \prod_{x \in |X|} P_{x,s}(T)^{-1} \). Note that \( L(\varphi, T) = \prod_{s \in \mathbb{Q}} L_s(\varphi, T) \). (The slope of an element of \( R \) (or an extension of \( R \)) simply means its valuation; this explains the use of the word slope to describe \( L_s(\varphi, T) \).) The Grothendieck specialization theorem for the Newton polygon of \((M, \varphi)\) implies that in fact only finitely many slopes appear among the reciprocal zeroes of the \( P_x(T) \) as \( x \) ranges over all the closed points of \( X \), and so the decomposition of \( L(\varphi, T) \) as a product of pure slope \( L \)-functions involves only finitely many nontrivial terms.

In certain situations \( M \) may admit a decomposition series by \( \varphi \)-invariant submodules, \( 0 \subset M_{s_1} \subset \cdots \subset M_{s_k} \), indexed by a strictly increasing sequence of rational numbers \( s_i \), such that each subquotient \((M_{s_i}/M_{s_{i-1}}, \varphi_i)\) is a \( \sigma \)-module whose slopes at any closed point of \( X \) are all equal to \( s_i \). (Here \( \varphi_i \) denotes the \( \sigma \)-linear endomorphism of the subquotient induced by \( \varphi \).) If this is the case, then \( L_{s_i}(\varphi, T) = L(\varphi, T) \), and we refer to the subquotients \((M_{s_i}/M_{s_{i-1}}, \varphi_i)\) as the pure slope parts of \((M, \varphi)\). An arbitrary \( \sigma \)-module need not break up into pure slope parts in this way; nevertheless, its \( L \)-function does factor as a product of pure slope \( L \)-functions, as described in the preceding paragraph.

As we stated at the very beginning, the main theorem of the two papers under review is the following: if \((M, \varphi)\) is an overconvergent \( \sigma \)-module on the smooth affine finite type \( \mathbb{F}_q \)-scheme \( X \), then for any rational number \( s \), the pure slope \( L \)-function \( L_s(\varphi, T) \) is a meromorphic function on the \( p \)-adic affine \( T \)-line. In particular, if \((M, \varphi)\) does break up into pure slope parts, then each of these pure slope parts will have a meromorphic \( L \)-function, establishing the conjecture of Dwork.

As already mentioned, the novelty (and difficulty) of this result stems from the fact that these pure slope parts may not be overconvergent, even though \((M, \varphi)\) is (not to mention that in general, there is no pure slope decomposition of \((M, \varphi)\) at all).

We now sketch the proof of Wan’s result. The first of the two papers under review explains the reduction of the theorem to a certain key
lemma, which encapsulates the nub of the problem. The first step of this reduction is to show that by twisting, and applying Katz’s isogeny theorem and the Hodge-Newton decomposition (both applications being somewhat subtle in the context under consideration, since the property of being overconvergent need not be preserved by isogeny, and the subquotients of an overconvergent \( \sigma \)-module which occur in the Hodge-Newton decomposition need not be overconvergent in general), one may assume that \((\mathcal{M}, \varphi)\) admits a partial pure slope decomposition of the following type: we may assume that \((\mathcal{M}, \varphi)\) contains a nontrivial \( \varphi \)-invariant unit-root \( \sigma \)-submodule \( N \), for which the quotient \( \mathcal{M}/N \) is a \( \sigma \)-module whose slopes at all closed points of \( X \) are strictly positive. We let \( \rho \) denote the restriction of \( \varphi \) to \( N \).

The second step of the reduction is based on the following idea: assuming that \((\mathcal{M}, \varphi)\) contains a pure slope zero submodule \((N, \rho)\) (as we can, by invoking the reduction step of the preceding paragraph), we find that \( L(\varphi, T) = L_0(\varphi, T) \times \prod_{s>0} L_s(\varphi, T) = L(\rho, T) \times \prod_{s>0} L_s(\varphi, T) \). If \( s > 0 \) then the coefficients of \( T \) appearing in the local factors of the \( L_s(\varphi, T) \) are of greater absolute value than the coefficients of \( T \) appearing in the local factors of \( L(\rho, T) \). Thus one feels that (in some sense) \( L(\rho, T) \) stands less chance of being meromorphic than do the factors \( L_s(\varphi, T) \), for \( s > 0 \) and its total \( L \)-function is known to be meromorphic, since \((\mathcal{M}, \varphi)\) is overconvergent. Suppose that \((\mathcal{M}', \varphi')\) is another overconvergent \( \sigma \)-module; then \( L(\rho \otimes \varphi', T) \) is a meromorphic function on the \( p \)-adic affine \( T \)-line.

The proof of the key lemma is the subject of the second paper under review. As Wan points out, the appearance of the auxiliary overconvergent \( \sigma \)-module \((\mathcal{M}', \varphi')\) in the key lemma is harmless, since overconvergent \( \sigma \)-modules are known to give rise to meromorphic \( L \)-functions. Thus the main point is to understand why \( L(\rho, T) \) is meromorphic, when \((N, \rho)\) arises as the pure slope zero part of the overconvergent \( \sigma \)-module \((\mathcal{M}, \varphi)\) in the manner described above, under the additional assumption that \( N \) is free of rank one over \( R \).

If we twist the endomorphism \( \varphi \) of \( \mathcal{M} \) by a unit element of \( R \), then we can suppose that (with respect to some fixed basis of \( N \)) the endomorphism \( \rho \) acts as multiplication by an element \( r \in R \) which is congruent to one modulo the maximal ideal of \( R \). Recalling that, for such an element \( r \), \( \lim_{m \to \infty} r^{p^m+1} = r \), while for any element \( r' \) of \( R \)
of strictly positive slope, \( \lim_{m \to \infty} \rho^{(p^m+1)} = 0 \), one finds that

\[
L(\rho, T) = \lim_{m \to \infty} \prod_{i \geq 1} L(\text{Sym}^{p^m+1-i} \varphi \otimes \Lambda^i \varphi, T)^{(-1)^{i-1}}.
\]

Now each product appearing in this limit is finite (since \( \Lambda^i \varphi = 0 \) if \( i \) is greater than the rank of \( \mathcal{M} \)), and each factor of the product is a meromorphic function, since \( \varphi \) is overconvergent. Thus \( L(\rho, T) \) can be written as a limit of meromorphic functions. The main point in the argument is to control this limit in such a way as to conclude that \( L(\rho, T) \) is itself meromorphic. To this end, Wan extends the theory of overconvergent \( \sigma \)-modules to include certain infinite-rank topological \( R \)-modules, which he calls nuclear overconvergent \( \sigma \)-modules. He then shows that these \( \sigma \)-modules again have \( L \)-functions which are meromorphic, by extending the argument used in the finite-rank case. Finally, he shows that for each fixed value of \( i \), there is an overconvergent nuclear \( \sigma \)-module \( (\mathcal{M}_{\infty,i}, \varphi_{\infty,i}) \), which is in some sense the limit of the finite-rank overconvergent \( \sigma \)-modules \( \text{Sym}^{p^m-i} \varphi \), with the property that \( \lim_{m \to \infty} L(\text{Sym}^{p^m+1-i} \varphi \otimes \Lambda^i \varphi, T) = L(\varphi_{\infty,i} \otimes \Lambda^i \varphi, T) \).

Thus we conclude that \( L(\rho, T) = \prod_i L(\varphi_{\infty,i} \otimes \Lambda^i \varphi, T)^{(-1)^{i-1}} \) is a ratio of products of \( L \)-functions of a collection of overconvergent nuclear \( \sigma \)-modules, and so is meromorphic, as claimed. This completes the proof of the key lemma, and so also of the main theorem.

In fact, the key lemma is proved in a stronger form than we have stated it here. Firstly, the overconvergent \( \sigma \)-module \( (\mathcal{M}, \varphi) \) giving rise to \( \rho \) is allowed to be nuclear, not just finite-dimensional, as is the auxiliary overconvergent \( \sigma \)-module \( (\mathcal{M}', \varphi') \). Secondly, \( \rho \) is replaced by \( \rho^k \) for an arbitrary integer \( k \), and it is proved that the \( L \)-functions \( L(\rho^k \otimes \varphi', T) \) are uniformly (with respect to \( k \)) meromorphic in a certain sense. We refer to the papers themselves for the details of these additional statements, and a discussion of some of their consequences.

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