Functional Equations of L-Functions for Symmetric Products of the Kloosterman Sheaf *

Lei Fu
Institute of Mathematics, Nankai University, Tianjin, P. R. China
leifu@nankai.edu.cn

Daqing Wan
Department of Mathematics, University of California, Irvine, CA 92697
dwan@math.uci.edu

Abstract

We determine the (arithmetic) local monodromy at 0 and at ∞ of the Kloosterman sheaf using local Fourier transformations and Laumon's stationary phase principle. We then calculate ϵ-factors for symmetric products of the Kloosterman sheaf. Using Laumon's product formula, we get functional equations of L-functions for these symmetric products, and prove a conjecture of Evans on signs of constants of functional equations.

Key words: Kloosterman sheaf, ϵ-factor, ℓ-adic Fourier transformation.

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Introduction

Let $p \neq 2$ be a prime number and let $\mathbb{F}_p$ be the finite field with p elements. Fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Denote the projective line over $\mathbb{F}_p$ by $\mathbb{P}^1$. For any power q of p, let $\mathbb{F}_q$ be the finite subfield of $\mathbb{F}$ with q elements. Let ℓ be a prime number different from p. Fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^*$. For any $x \in \mathbb{F}_q^*$, we define the one variable Kloosterman sum by

$$\mathrm{Kl}_2(\mathbb{F}_q, x) = \sum_{x\in\mathbb{F}_q^*} \psi\left(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(\lambda + \frac{x}{\lambda}\right)\right).$$

In [3], Deligne constructs a lisse $\overline{\mathbb{Q}}_l$-sheaf Kl_2 of rank 2 on $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$, which we call the Kloosterman sheaf, such that for any $x \in \mathbb{G}_m(\mathbb{F}_q) = \mathbb{F}_q^*$, we have

$$\mathrm{Tr}(F_x, \mathrm{Kl}_{2,\bar{x}}) = -\mathrm{Kl}_2(\mathbb{F}_q, x),$$

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where F_x is the geometric Frobenius element at the point x. For a positive integer k, the L-function $L(\mathbb{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ of the k-th symmetric product of Kl_2 was first studied by Robba [15] via Dwork's p-adic methods. Motivated by applications in coding theory, by connections with modular forms, p-adic modular forms and Dwork's unit root zeta functions, there has been a great deal of recent interests to understand $L(\mathbb{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ as much as possible for all k and for all p. This quickly raises a large number of interesting new problems.

Let $j : \mathbb{G}_m \to \mathbb{P}^1$ be the inclusion. We shall be interested in the L-function

$$M_k(p, T) := L(\mathbb{P}^1, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)), T).$$

This is the non-trivial factor of $L(\mathbb{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$. The trivial factor of $L(\mathbb{G}_m, \mathrm{Sym}^k(\mathrm{Kl}_2), T)$ was completely determined in Fu-Wan [6]. By general theory of Grothendieck-Deligne, the non-trivial factor $M_k(p, T)$ is a polynomial in T with integer coefficients, pure of weight $k + 1$. Its degree $\delta_k(p)$ can be easily extracted from Fu-Wan [7] Proposition 2.3, Lemmas 4.1 and 4.2:

$$\delta_k(p) = \begin{cases} \frac{k-1}{2} - \left[\frac{k}{2p} + \frac{1}{2}\right] & \text{if } k \text{ is odd,} \\ 2\left([\frac{k-2}{4}] - [\frac{k}{2p}]\right) & \text{if } k \text{ is even.} \end{cases}$$

For fixed k, the variation of $M_k(p, T)$ as p varies should be explained by an automorphic form, see Choi-Evans [2] and Evans [4] for the precise relations in the cases $k \leq 7$ and Fu-Wan [8] for a motivic interpretation for all k. For $k \leq 4$, the degree $\delta_k(p) \leq 1$ and $M_k(p, T)$ can be determined easily. For $k = 5$, the degree $\delta_5(p) = 2$ for $p > 5$. The quadratic polynomial $M_5(p, T)$ is explained by an explicit modular form [14]. For $k = 6$, the degree $\delta_6(p) = 2$ for $p > 6$. The quadratic polynomial $M_6(p, T)$ is again explained by an explicit modular form [9]. For $k = 7$, the degree $\delta_7(p) = 3$ for $p > 7$. The cubic polynomial $M_7(p, T)$ is conjecturally explained in a more subtle way by an explicit modular form in Evans [4]. We will return to this conjecture later in the introduction.

For fixed p, the variation of $M_k(p, T)$ as k varies p-adically should be related to p-adic automorphic forms and p-adic L-functions. No progress has been made along this direction. The p-adic limit of $M_k(p, T)$ as k varies p-adically links to an important example of Dwork's unit root zeta function, see the introduction in Wan [18]. The polynomial $M_k(p, T)$ can be used to determine the weight distribution of certain codes, see Moisio [12][13], and this has been studied extensively for small p and small k. The p-adic Newton polygon (the p-adic slopes) of $M_k(p, T)$ remains largely mysterious.

By Katz [10] 4.1.11, we have $(\mathrm{Kl}_2)^\vee = \mathrm{Kl}_2 \otimes \overline{\mathbb{Q}}_\ell(1)$. So for any natural number k, we have

$$(\mathrm{Sym}^k(\mathrm{Kl}_2))^\vee = \mathrm{Sym}^k(\mathrm{Kl}_2) \otimes \overline{\mathbb{Q}}_\ell(k).$$

General theory (confer [11] 3.1.1) shows that $M_k(p,T)$ satisfies the functional equation

$$M_k(p,T) = cT^\delta M_k\left(p, \frac{1}{p^{k+1}T}\right),$$

where

$$\begin{aligned} c &= \prod_{i=0}^{2} \det(-F, H^i(\mathbb{P}^1_{\mathbb{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)))^{(-1)^{i+1}}, \\ \delta &= -\chi(\mathbb{P}^1_{\mathbb{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2)) = \delta_k(p), \end{aligned}$$

and F denotes the Frobenius correspondence. Applying the functional equation twice, we get

$$c^2 = p^{(k+1)\delta}.$$

Based on numerical computation, Evans [4] suggests that the sign of c should be $-\left(\frac{p}{105}\right)$ (the Jacobi symbol) for $k = 7$, and $-\left(\frac{p}{1155}\right)$ for $k = 11$. In this paper, we determine c for all k and all $p > 2$. The main result of this paper is the following theorem.

Theorem 0.1. *Let $p > 2$ be an odd prime. If k is even, we have*

$$c = p^{(k+1)([\frac{k-2}{4}]-[\frac{k}{2p}])}.$$

If k is odd, we have

$$c = (-1)^{\frac{k-1}{2}+[\frac{k}{2p}+\frac{1}{2}]} p^{\frac{k+1}{2}(\frac{k-1}{2}-[\frac{k}{2p}+\frac{1}{2}])} \left(\frac{-2}{p}\right)^{[\frac{k}{2p}+\frac{1}{2}]} \prod_{j\in\{0,1,\ldots,[\frac{k}{2}]\},\ p\nmid 2j+1} \left(\frac{(-1)^j(2j+1)}{p}\right).$$

Corollary 0.2. *If k is even and $p > 2$, the sign of c is always 1. If k is odd and $p > k$, the sign of c is*

$$(-1)^{\frac{k-1}{2}} \prod_{j\in\{0,1,\ldots,[\frac{k}{2}]\},\ p\nmid 2j+1} \left(\frac{(-1)^j(2j+1)}{p}\right).$$

In the above corollary, if we take $k = 7$, we see that the sign of c for $p > 7$ is

$$-\left(\frac{1\cdot(-3)\cdot 5\cdot(-7)}{p}\right) = -\left(\frac{105}{p}\right) = -\left(\frac{p}{105}\right);$$

if we take $k = 11$, we see that the sign of c for $p > 11$ is

$$-\left(\frac{1\cdot(-3)\cdot 5\cdot(-7)\cdot 9\cdot(-11)}{p}\right) = -\left(\frac{-1155}{p}\right) = -\left(\frac{p}{1155}\right),$$

consistent with Evans' calculation.

In the case $k=7$, Evans proposed a precise description of $M_7(p,T)$ in terms of modular forms. For $k=7$ and $p>7$, the polynomial $M_7(p,T)$ has degree 3. Write

$$M_7(p,T)=1+a_pT+d_pT^2+e_pT^3.$$

The functional equation and our sign determination show that one of the reciprocal roots for $M_7(p,T)$ is $\left(\frac{p}{105}\right)p^4$ and $e_p=-\left(\frac{p}{105}\right)p^{12}$. Denote the other two reciprocal roots by λ_p and μ_p which are Weil numbers of weight 8. We deduce that

$$a_p=-\left(\left(\frac{p}{105}\right)p^4+\lambda_p+\mu_p\right),\ \lambda_p\mu_p=p^8,\ |\lambda_p|=|\mu_p|=p^4.$$

To explain the numerical calculation of Evans, Katz suggests that there exists a two dimensional representation

$$\rho:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}(\overline{\mathbb{Q}}_\ell^2)$$

unramified for $p>7$ and a Dirichlet character χ such that

$$\begin{aligned}\alpha_p^2&=\chi(p)\left(\frac{p}{105}\right)\frac{\lambda_p}{p^4},\\ \beta_p^2&=\chi(p)\left(\frac{p}{105}\right)\frac{\mu_p}{p^4},\\ \alpha_p\beta_p&=\chi(p),\end{aligned}$$

where α_p and β_p are the eigenvalues of the geometric Frobenius element at p under ρ. We then have

$$\begin{aligned}1-\left(\frac{p}{105}\right)\frac{a_p}{p^4}&=2+\left(\frac{p}{105}\right)\frac{\lambda_p}{p^4}+\left(\frac{p}{105}\right)\frac{\mu_p}{p^4}\\ &=\overline{\chi}(p)(2\alpha_p\beta_p+\alpha_p^2+\beta_p^2)\\ &=\overline{\chi}(p)(\alpha_p+\beta_p)^2.\end{aligned}$$

Set $b(p)=p(\alpha_p+\beta_p)$. Evans [4] conjectured that $b(p)$ is the p-th Hecke eigenvalue for a weight 3 newform f on $\Gamma_0(525)$. Our a_p equals $-c_pp^2$ in [4].

Our proof of Theorem 0.1 naturally splits into two parts, corresponding to the two ramification points at 0 and ∞. Let t be the coordinate of $\mathbb{A}^1=\mathbb{P}^1-\{\infty\}$. For any closed point x in $\mathbb{P}^1$, let $\mathbb{P}^1_{(x)}$ be the henselization of $\mathbb{P}^1$ at x. By Laumon's product formula [11] 3.2.1.1, we have

$$c=p^{k+1}\prod_{x\in|\mathbb{P}^1|}\epsilon(\mathbb{P}^1_{(x)},j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(x)}},dt|_{\mathbb{P}^1_{(x)}}),$$

where $|\mathbb{P}^1|$ is the set of all closed points of $\mathbb{P}^1$. When $x \neq 0, \infty$, the sheaf $\mathrm{Sym}^k(\mathrm{Kl}_2)|_{\mathbb{P}^1_{(x)}}$ is lisse and the order of dt at x is 0. So by [11] 3.1.5.4 (ii) and (v), we have

$$\epsilon(\mathbb{P}^1_{(x)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(x)}}, dt|_{\mathbb{P}^1_{(x)}}) = 1$$

for $x \neq 0, \infty$. Therefore

$$c = p^{k+1}\epsilon(\mathbb{P}^1_{(0)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}})\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}).$$

In §1, we prove the following.

Proposition 0.3. *We have*

$$\epsilon(\mathbb{P}^1_{(0)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) = (-1)^k p^{\frac{k(k+1)}{2}}.$$

In §2, we prove the following.

Proposition 0.4. $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})$ *equals*

$$p^{-(k+1)(\frac{k+8}{4}+[\frac{k}{2p}])}$$

if $k = 2r$ for an even r,

$$p^{-(k+1)(\frac{k+6}{4}+[\frac{k}{2p}])}$$

if $k = 2r$ for an odd r, and

$$(-1)^{\frac{k+1}{2}+[\frac{k}{p}]-[\frac{k}{2p}]}p^{-\frac{k+1}{2}(\frac{k+5}{2}+[\frac{k}{p}]-[\frac{k}{2p}])}\left(\frac{-2}{p}\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\prod_{j\in\{0,1,\ldots,[\frac{k}{2}]\},\ p\nmid 2j+1}\left(\frac{(-1)^j(2j+1)}{p}\right)$$

if $k = 2r+1$.

We deduce from the above two propositions the constant c as stated in Theorem 0.1 using the following facts:

$$\left[\frac{k-2}{4}\right] = \begin{cases} \frac{k-4}{4} & \text{if } k = 2r \text{ for an even } r, \\ \frac{k-2}{4} & \text{if } k = 2r \text{ for an odd } r, \end{cases}$$

$$\left[\frac{k}{p}\right] - \left[\frac{k}{2p}\right] = \left[\frac{k}{2p} + \frac{1}{2}\right] \text{ if } k \text{ is odd.}$$

To get Proposition 0.4, we first have to determine the local (arithmetic) monodromy of Kl_2 at ∞. This is Theorem 2.1 in §2, which is of interest itself, and is proved by using local Fourier transformations and Laumon's stationary phase principle.

1 Calculation of $\epsilon(\mathbb{P}^1_{(0)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}})$

Let η_0 be the generic point of $\mathbb{P}^1_{(0)}$, let $\bar{\eta}_0$ be a geometric point located at η_0, and let V be an $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{Gal}(\bar{\eta}_0/\eta_0)$. Suppose the inertia subgroup I_0 of $\mathrm{Gal}(\bar{\eta}_0/\eta_0)$ acts unipotently on V. Consider the ℓ-adic part of the cyclotomic character

$$t_\ell : I_0 \to \mathbb{Z}_\ell(1),\ \sigma \mapsto \left(\frac{\sigma(\sqrt[\ell^n]{t})}{\sqrt[\ell^n]{t}}\right).$$

Note that for any σ in the inertia subgroup, the ℓ^n-th root of unity $\frac{\sigma(\sqrt[\ell^n]{t})}{\sqrt[\ell^n]{t}}$ does not depends on the choice of the ℓ^n-th root $\sqrt[\ell^n]{t}$ of t. Since I_0 acts on V unipotently, there exists a nilpotent homomorphism

$$N : V(1) \to V$$

such that the action of $\sigma \in I_0$ on V is given by $\exp(t_\ell(\sigma).N)$. Fix a lifting $F \in \mathrm{Gal}(\bar{\eta}_0/\eta_0)$ of the geometric Frobenius element in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$.

Lemma 1.1. *Notation as above. Let $V = \mathrm{Kl}_{2,\bar{\eta}_0}$. There exists a basis $\{e_0, e_1\}$ of V such that*

$$F(e_0) = e_0,\ F(e_1) = pe_1$$

$$N(e_0) = 0,\ N(e_1) = e_0.$$

Proof. This is the $n = 2$ case of Proposition 1.1 in [6]. □

Lemma 1.2. *Keep the notation in Lemma 1.1. Let $\{f_0, \dots, f_k\}$ be the basis of $\mathrm{Sym}^k(V) = \mathrm{Sym}^k(\mathrm{Kl}_{2,\bar{\eta}_0})$ defined by $f_i = \frac{1}{i!}e_0^{k-i}e_1^i$. We have*

$$F(f_i) = p^i f_i, N(f_i) = f_{i-1},$$

where we regard f_{i-1} as 0 if $i = 0$.

Proof. Use the fact that for any $v_1, \dots, v_k \in V$, we have the following identities in $\mathrm{Sym}^k(V)$:

$$\begin{aligned} F(v_1 \cdots v_k) &= F(v_1) \cdots F(v_k), \\ N(v_1 \cdots v_k) &= \sum_{i=1}^{k} v_1 \cdots v_{i-1} N(v_i) v_{i+1} \cdots v_k. \end{aligned}$$

□

Corollary 1.3. *The sheaf* $\mathrm{Sym}^k(\mathrm{Kl}_2)|_{\eta_0}$ *has a filtration*

$$0=\mathcal{F}_{-1}\subset\mathcal{F}_0\subset\cdots\subset\mathcal{F}_k=\mathrm{Sym}^k(\mathrm{Kl}_2)|_{\eta_0}$$

such that

$$\mathcal{F}_i/\mathcal{F}_{i-1}\cong\overline{\mathbb{Q}}_l(-i)$$

for any $i=0,\ldots,k$.

Proof. This follows from Lemma 1.2 by taking $\mathcal{F}_i$ to be the sheaf on η_0 corresponding to the galois representation $\mathrm{Span}(f_0,\ldots,f_i)$ of $\mathrm{Gal}(\bar{\eta}_0/\eta_0)$. □

The following is Proposition 0.3 in the introduction.

Proposition 1.4. *We have*

$$\epsilon(\mathbb{P}^1_{(0)},j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}},dt|_{\mathbb{P}^1_{(0)}})=(-1)^kp^{\frac{k(k+1)}{2}}.$$

Proof. Let $u:\eta_0\to\mathbb{P}^1_{(0)}$ and $v:\{0\}\to\mathbb{P}^1_{(0)}$ be the immersions. By [11] 3.1.5.4 (iii) and (v), we have

$$\begin{aligned}\epsilon(\mathbb{P}^1_{(0)},u_*\overline{\mathbb{Q}}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}}) &= 1,\\ \epsilon(\mathbb{P}^1_{(0)},v_*\overline{\mathbb{Q}}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}}) &= \det(-F_0,\overline{\mathbb{Q}}_\ell(-i))^{-1}=-\frac{1}{p^i}.\end{aligned}$$

We have an exact sequence

$$0\to u_!\overline{\mathbb{Q}}_\ell(-i)\to u_*\overline{\mathbb{Q}}_\ell(-i)\to v_*\overline{\mathbb{Q}}_\ell(-i)\to 0.$$

It follows from [11] 3.1.5.4 (ii) that we have

$$\begin{aligned}\epsilon(\mathbb{P}^1_{(0)},u_!\overline{\mathbb{Q}}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}}) &= \frac{\epsilon(\mathbb{P}^1_{(0)},u_*\overline{\mathbb{Q}}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}})}{\epsilon(\mathbb{P}^1_{(0)},v_*\overline{\mathbb{Q}}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}})}\\ &= -p^i.\end{aligned}$$

By Corollary 1.3 and [11] 3.1.5.4 (ii), we have

$$\begin{aligned}\epsilon(\mathbb{P}^1_{(0)},j_!(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}},dt|_{\mathbb{P}^1_{(0)}}) &= \prod_{i=0}^k\epsilon(\mathbb{P}^1_{(0)},u_!(\mathcal{F}_i/\mathcal{F}_{i+1}),dt|_{\mathbb{P}^1_{(0)}})\\ &= \prod_{i=0}^k\epsilon(\mathbb{P}^1_{(0)},u_!\mathbb{Q}_\ell(-i),dt|_{\mathbb{P}^1_{(0)}})\\ &= \prod_{i=0}^k(-p^i).\end{aligned}$$

Moreover, by Lemma 1.2, we have

$$v^*(j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}) \cong \overline{\mathbb{Q}}_\ell,$$

and hence

$$\epsilon(\mathbb{P}^1_{(0)}, v_*v^*(j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}), dt|_{\mathbb{P}^1_{(0)}}) = -1.$$

So we have

$$\begin{aligned}
& \epsilon(\mathbb{P}^1_{(0)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) \\
= \; & \epsilon(\mathbb{P}^1_{(0)}, j_!(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}})\epsilon(\mathbb{P}^1_{(0)}, v_*v^*(j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(0)}}), dt|_{\mathbb{P}^1_{(0)}}) \\
= \; & \prod_{i=1}^k (-p^i) \\
= \; & (-1)^k p^{\frac{k(k+1)}{2}}.
\end{aligned}$$

□

2 Calculation of $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})$

We first introduce some notations. Fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^*$ and define Kl_2 as in the introduction. Fix a separable closure $\overline{\mathbb{F}_p(t)}$ of $\mathbb{F}_p(t)$. Let x be an element in $\overline{\mathbb{F}_p(t)}$ satisfying $x^p - x = t$. Then $\mathbb{F}_p(t,x)$ is galois over $\mathbb{F}_p(t)$. We have a canonical isomorphism

$$\mathbb{F}_p \stackrel{\cong}{\to} \mathrm{Gal}(\mathbb{F}_p(t,x)/\mathbb{F}_p(t))$$

which sends each $a \in \mathbb{F}_p$ to the element in $\mathrm{Gal}(\mathbb{F}_p(t,x)/\mathbb{F}_p(t))$ defined by $x \mapsto x + a$. Let $\mathcal{L}_\psi$ be the galois representation defined by

$$\mathrm{Gal}(\overline{\mathbb{F}_p(t)}/\mathbb{F}_p(t)) \to \mathrm{Gal}(\mathbb{F}_p(t,x)/\mathbb{F}_p(t)) \stackrel{\cong}{\to} \mathbb{F}_p \stackrel{\psi^{-1}}{\to} \overline{\mathbb{Q}}_\ell^*.$$

It is unramfied outside ∞ and totally wild at ∞ with Swan conductor 1. This galois representation defines a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1$ which we still denote by $\mathcal{L}_\psi$. Let X be an $\mathbb{F}_p$-scheme. Any section f in $\mathcal{O}_X(X)$ defines an $\mathbb{F}_p$-algebra homomorphism

$$\mathbb{F}_p[t] \to \mathcal{O}_X(X), \; t \mapsto f,$$

and hence an $\mathbb{F}_p$-morphism of schemes

$$f : X \to \mathbb{A}^1.$$

We denote the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $f^*\mathcal{L}_\psi$ on X by $\mathcal{L}_\psi(f)$. For any $f_1, f_2 \in \mathcal{O}_X(X)$, we have

$$\mathcal{L}_\psi(f_1) \otimes \mathcal{L}_\psi(f_2) \cong \mathcal{L}_\psi(f_1 + f_2).$$

Recall that $p \neq 2$. Let y be an element in $\overline{\mathbb{F}_p(t)}$ satisfying $y^2 = t$. Then $\mathbb{F}_p(t, y)$ is galois over $\mathbb{F}_p(t)$. We have a canonical isomorphism

$$\{\pm 1\} \xrightarrow{\cong} \mathrm{Gal}(\mathbb{F}_p(t, y)/\mathbb{F}_p(t))$$

which sends -1 to the element in $\mathrm{Gal}(\mathbb{F}_p(t, y)/\mathbb{F}_p(t))$ defined by $y \mapsto -y$. Let

$$\chi : \{\pm 1\} \to \overline{\mathbb{Q}}_\ell^*$$

be the (unique) nontrivial character. Define $\mathcal{L}_\chi$ to be the galois representation defined by

$$\mathrm{Gal}(\overline{\mathbb{F}_p(t)}/\mathbb{F}_p(t)) \to \mathrm{Gal}(\mathbb{F}_p(t, y)/\mathbb{F}_p(t)) \xrightarrow{\cong} \{\pm 1\} \xrightarrow{\chi^{-1}} \overline{\mathbb{Q}}_\ell^*.$$

It is unramified outside 0 and ∞, and tamely ramified at 0 and ∞. This galois representation defines a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_m$ which we still denote by $\mathcal{L}_\chi$.

Let $\theta : \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^*$ be a character of the galois group of the finite field. Denote by $\mathcal{L}_\theta$ the galois representation

$$\mathrm{Gal}(\overline{\mathbb{F}_p(t)}/\mathbb{F}_p(t)) \to \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \xrightarrow{\theta} \overline{\mathbb{Q}}_\ell^*.$$

It is unramified everywhere, and hence defines a lisse $\overline{\mathbb{Q}}_l$-sheaf on $\mathbb{P}^1$ which we still denote by $\mathcal{L}_\theta$.

Theorem 2.1. *Notation as above. Let η_∞ be the generic point of $\mathbb{P}^1_{(\infty)}$. Then $\mathrm{Kl}_2|_{\eta_\infty}$ is isomorphic to the restriction to η_∞ of the sheaf*

$$[2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_{\theta_0},$$

where $[2] : \mathbb{G}_m \to \mathbb{G}_m$ is the morphism defined by $x \mapsto x^2$, and

$$\theta_0 : \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^*$$

is the character sending the geometric Frobenius element F in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ to the Gauss sum

$$\theta_0(F) = g(\chi, \psi) = -\sum_{x \in \mathbb{F}_p^*} \left(\frac{x}{p}\right) \psi(x).$$

Proof. By [8] Proposition 1.1, we have

$$\mathrm{Kl}_2 = \mathcal{F}\left(j_!\mathcal{L}_\psi\left(\frac{1}{t}\right)\right)|_{\mathbb{G}_m}, \tag{1}$$

where $\mathcal{F}$ is the ℓ-adic Fourier transformation and $j : \mathbb{G}_m \to \mathbb{A}^1$ is the inclusion. Let

$$\pi_1, \pi_2 : \mathbb{G}_m \times_{\mathbb{F}_p} \mathbb{G}_m \to \mathbb{G}_m$$

be the projections. Using the proper base change theorem and the projection formula ([1] XVII 5.2.6 and 5.2.9), one can verify

$$[2]^*\left(\mathcal{F}\left(j_!\mathcal{L}_\psi\left(\frac{1}{t}\right)\right)|_{\mathbb{G}_m}\right) \cong R\pi_{2!}\left(\mathcal{L}_\psi\left(\frac{1}{t}+tt'^2\right)\right)[1], \tag{2}$$

where

$$\frac{1}{t}+tt'^2 : \mathbb{G}_m \times_{\mathbb{F}_p} \mathbb{G}_m \to \mathbb{A}^1$$

is the morphism corresponding to the $\mathbb{F}_p$-algebra homomorphism

$$\mathbb{F}_p[t] \to \mathbb{F}_p[t, 1/t, t', 1/t'],\ t \mapsto \frac{1}{t}+tt'^2.$$

Consider the isomorphism

$$\tau : \mathbb{G}_m \times_{\mathbb{F}_p} \mathbb{G}_m \to \mathbb{G}_m \times_{\mathbb{F}_p} \mathbb{G}_m,\ (t, t') \mapsto \left(\frac{t}{t'}, t'\right).$$

We have $\pi_2\tau = \pi_2$. So

$$R\pi_{2!}\left(\mathcal{L}_\psi\left(\frac{1}{t}+tt'^2\right)\right) \cong R(\pi_2\tau)_!\tau^*\left(\mathcal{L}_\psi\left(\frac{1}{t}+tt'^2\right)\right) \cong R\pi_{2!}\mathcal{L}_\psi\left(\left(\frac{1}{t}+t\right)t'\right). \tag{3}$$

Consider the morphism

$$g : \mathbb{G}_m \to \mathbb{A}^1,\ t \mapsto \frac{1}{t}+t.$$

Again using the proper base change theorem and the projection formula, one can verify

$$\mathcal{F}(Rg_!\overline{\mathbb{Q}}_\ell) \cong R\pi_{2!}\mathcal{L}_\psi\left(\left(\frac{1}{t}+t\right)t'\right)[1]. \tag{4}$$

From the isomorphisms (1)-(4), we get

$$[2]^*\mathrm{Kl}_2 \cong \mathcal{F}(Rg_!\overline{\mathbb{Q}}_\ell)|_{\mathbb{G}_m}.$$

By Lemma 2.2 below, the stationary phase principle of Laumon [11] 2.3.3.1 (iii), and [11] 2.5.3.1, as representations of $\mathrm{Gal}(\bar{\eta}_{\infty'}/\eta_{\infty'})$, we have

$$\begin{aligned}\mathcal{H}^0(\mathcal{F}(Rg_!\overline{\mathbb{Q}}_\ell))_{\bar{\eta}_{\infty'}} &\cong \mathcal{F}^{(2,\infty')}(\mathcal{L}_\chi)\bigoplus\mathcal{F}^{(-2,\infty')}(\mathcal{L}_\chi)\\ &\cong (\mathcal{L}_\psi(2t')\otimes\mathcal{F}^{(0,\infty')}(\mathcal{L}_\chi))\bigoplus(\mathcal{L}_\psi(-2t')\otimes\mathcal{F}^{(0,\infty')}(\mathcal{L}_\chi))\\ &\cong (\mathcal{L}_\psi(2t')\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0})\bigoplus(\mathcal{L}_\psi(-2t')\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0}).\end{aligned}$$

Hence

$$([2]^*\mathrm{Kl}_2)|_{\eta_\infty}\cong(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0})|_{\eta_\infty}\bigoplus(\mathcal{L}_\psi(-2t)\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0})|_{\eta_\infty}.$$

Note that this decomposition of $([2]^*\mathrm{Kl}_2)|_{\eta_\infty}$ is non-isotypical. By [16] Proposition 24 on p. 61, and the fact that $\mathrm{Kl}_2|_{\eta_\infty}$ is irreducible (since its Swan conductor is 1), we have

$$\mathrm{Kl}_2|_{\eta_\infty}\cong[2]_*(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0})|_{\eta_\infty}.$$

We have

$$\begin{aligned}[2]_*(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi\otimes\mathcal{L}_{\theta_0}) &\cong [2]_*\left(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi\otimes[2]^*\mathcal{L}_{\theta_0}\right)\\ &\cong [2]_*\left(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi\right)\otimes\mathcal{L}_{\theta_0}.\end{aligned}$$

Here we use the fact that $[2]^*\mathcal{L}_{\theta_0}\cong\mathcal{L}_{\theta_0}$. Hence

$$\mathrm{Kl}_2|_{\eta_\infty}\cong\left([2]_*(\mathcal{L}_\psi(2t)\otimes\mathcal{L}_\chi)\otimes\mathcal{L}_{\theta_0}\right)|_{\eta_\infty}.$$

□

Lemma 2.2. *For the morphism*

$$g:\mathbb{G}_m\to\mathbb{A}^1,\ t\mapsto\frac{1}{t}+t,$$

the following holds:

(i) $Rg_!\overline{\mathbb{Q}}_\ell$ is a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1$ which is lisse outside the rational points 2 and -2.

(ii) $Rg_!\overline{\mathbb{Q}}_\ell$ is unramified at ∞.

(iii) Let P be one of the rational points 2 or -2, and let $\tilde{\mathbb{A}}^1_{(P)}$ be the henselization of $\mathbb{A}^1$ at P. We have

$$(Rg_!\overline{\mathbb{Q}}_\ell)|_{\tilde{\mathbb{A}}^1_{(P)}}\cong\overline{\mathbb{Q}}_\ell\oplus\mathcal{L}_{\chi,!},$$

where $\mathcal{L}_{\chi,!}$ denotes the extension by 0 of the Kummer sheaf $\mathcal{L}_\chi$ on the generic point of $\tilde{\mathbb{A}}^1_{(P)}$ to $\tilde{\mathbb{A}}^1_{(P)}$.

Proof. We have

$$\frac{\partial g}{\partial t} = -\frac{1}{t^2} + 1.$$

So $\frac{\partial g}{\partial t}$ vanishes at the points $t = \pm 1$. We have

$$g(\pm 1) = \pm 2,$$
$$\frac{\partial^2 g}{\partial t^2}(\pm 1) = \pm 2 \neq 0.$$

It follows that g is tamely ramified above ± 2 with ramification index 2, and g is étale elsewhere. Consider the morphism

$$\bar{g} : \mathbb{P}^1 \to \mathbb{P}^1,\ [t_0 : t_1] \mapsto [t_0 t_1 : t_0^2 + t_1^2].$$

We have $\bar{g}^{-1}(\infty) = \{0, \infty\}$. Hence

$$\bar{g}^{-1}(\mathbb{A}^1) = \mathbb{G}_m.$$

It is clear that

$$\bar{g}|_{\mathbb{G}_m} = g.$$

So $g : \mathbb{G}_m \to \mathbb{A}^1$ is a finite morphism of degree 2. Near 0, the morphism $\bar{g}$ can be expressed as

$$t \mapsto \frac{t}{1+t^2}.$$

Hence $\bar{g}$ is unramified at 0. Similarly $\bar{g}$ is also unramified at ∞. Our lemma follows from these facts. $\square$

Remark 2.3. The first attempt to determine the monodromy at ∞ of the $(n-1)$-variable Kloosterman sheaf $\mathrm{Kl}_n|_{\eta_\infty}$ is done in Fu-Wan [7] Theorem 1.1, where we deduce from Katz [10] that

$$\mathrm{Kl}_n|_{\eta_\infty} \cong \left([n]_* (\mathcal{L}_\psi(nt) \otimes \mathcal{L}_{\chi^{n-1}}) \otimes \mathcal{L}_\theta \otimes \overline{\mathbb{Q}}_\ell \left(\frac{1-n}{2} \right) \right) |_{\eta_\infty}$$

for some character $\theta : \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^*$, and an explicit description of θ^2 is given. Using induction on n, [8] Proposition 1.1, and adapting the argument in [5] to non-algebraically closed ground field, we can get an explicit description of θ. See [5] where the monodromy of the more general hypergeometric sheaf is treated (over algebraically closed field).

Lemma 2.4. *Keep the notation in Theorem 2.1. Let*

$$\theta_1 : \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^*$$

be the character defined by

$$\theta_1(\sigma) = \chi\left(\frac{\sigma(\sqrt{-1})}{\sqrt{-1}}\right)$$

for any $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. *Note that the above expression is independent of the choice of the square root* $\sqrt{-1} \in \mathbb{F}$ *of* -1.

(i) If $k = 2r$ *is even,* $\mathrm{Sym}^k(\mathrm{Kl}_2)|_{\eta_\infty}$ *is isomorphic to the restriction to* η_∞ *of the sheaf*

$$\left(\mathcal{L}_{\chi^r} \otimes \mathcal{L}_{\theta_0^{2r}\theta_1^r}\right) \oplus \left(\bigoplus_{i=0}^{r-1}[2]_*\mathcal{L}_\psi((4i-4r)t) \otimes \mathcal{L}_{\theta_0^{2r}\theta_1^i}\right).$$

(ii) If $k = 2r+1$ *is odd,* $\mathrm{Sym}^k(\mathrm{Kl}_2)|_{\eta_\infty}$ *is isomorphic to the restriction to* η_∞ *of the sheaf*

$$\bigoplus_{i=0}^{r}[2]_*\left(\mathcal{L}_\psi((4i-4r-2)t) \otimes \mathcal{L}_\chi\right) \otimes \mathcal{L}_{\theta_0^{2r+1}\theta_1^{i+1}}.$$

Proof. By Theorem 2.1, it suffices to calculate the restriction to η_∞ of $\mathrm{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi))$. Let y, z, w be elements in $\overline{\mathbb{F}_p(t)}$ satisfying

$$y^2 = t,\ z^p - z = y,\ w^2 = y.$$

Fix a square root $\sqrt{-1}$ of -1 in $\mathbb{F}$. Then $\mathbb{F}_p(z, w, \sqrt{-1})$ and $\mathbb{F}_p(y)$ are galois extensions of $\mathbb{F}_p(t)$. Let $G = \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(t))$ and $H = \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(y))$. Then H is normal in G, and we have canonical isomorphisms

$$G/H \xrightarrow{\cong} \mathrm{Gal}(\mathbb{F}_p(y)/\mathbb{F}_p(t)) \xrightarrow{\cong} \{\pm 1\}.$$

Consider the case where $\sqrt{-1}$ does not lie in $\mathbb{F}_p$. We have an isomorphism

$$\mathbb{F}_p \times \{\pm 1\} \times \{\pm 1\} \xrightarrow{\cong} H = \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(y))$$

which maps $(a, \mu', \mu'') \in \mathbb{F}_p \times \{\pm 1\} \times \{\pm 1\}$ to the element $g_{(a,\mu',\mu'')} \in \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(y))$ defined by

$$g_{(a,\mu',\mu'')}(z) = z + a,\ g_{(a,\mu',\mu'')}(w) = \mu' w,\ g_{(a,\mu',\mu'')}(\sqrt{-1}) = \mu''\sqrt{-1}.$$

(In the case where $\sqrt{-1}$ lies in $\mathbb{F}_p$, we have $\mathbb{F}_p(z, w, \sqrt{-1}) = \mathbb{F}_p(z, w)$, and we have an isomorphism

$$\mathbb{F}_p \times \{\pm 1\} \xrightarrow{\cong} H = \mathrm{Gal}(\mathbb{F}_p(z, w)/\mathbb{F}_p(y))$$

which maps $(a, \mu) \in \mathbb{F}_p \times \{\pm 1\}$ to the element $g_{(a,\mu)} \in \mathrm{Gal}(\mathbb{F}_p(z, w)/\mathbb{F}_p(y))$ defined by

$$g_{(a,\mu)}(z) = z + a,\ g_{(a,\mu)}(w) = \mu w.$$

The following argument works for this case with slight modification. We leave to the reader to treat this case.) Let V be a one dimensional $\overline{\mathbb{Q}}_\ell$-vector space with a basis e_0. Define an action of H on V by

$$g_{(a,\mu',\mu'')}(e_0) = \psi(-2a)\chi(\mu'^{-1})e_0.$$

Then $[2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)$ is just the composition of $\mathrm{Ind}_H^G(V)$ with the canonical homomorphism

$$\mathrm{Gal}(\overline{\mathbb{F}_p(t)}/\mathbb{F}_p(t)) \to \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(t)) = G.$$

Let g be the element in $G = \mathrm{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(t))$ defined by

$$g(z) = -z,\ g(w) = \sqrt{-1}w,\ g(\sqrt{-1}) = \sqrt{-1}.$$

Then the image of g in G/H is a generator of the cyclic group G/H. So G is generated by $g_{(a,\mu',\mu'')} \in H$ $((a, \mu', \mu'') \in \mathbb{F}_p \times \{\pm 1\} \times \{\pm 1\})$ and g. The space $\mathrm{Ind}_H^G(V)$ has a basis $\{e_0, e_1\}$ with

$$\begin{aligned}
g(e_0) &= e_1, \\
g_{(a,\mu',\mu'')}(e_0) &= \psi(-2a)\chi(\mu'^{-1})e_0, \\
g_{(a,\mu',\mu'')}(e_1) &= \psi(2a)\chi(\mu'^{-1}\mu''^{-1})e_1, \\
g(e_1) &= g^2(e_0) = g_{(0,-1,1)}(e_0) = -e_0.
\end{aligned}$$

Suppose $k = 2r$ is even. $\mathrm{Sym}^k(\mathrm{Ind}_H^G(V))$ has a basis

$$\{e_1^k, g(e_1^k), e_0e_1^{k-1}, g(e_0e_1^{k-1}), \ldots, e_0^{r-1}e_1^{r+1}, g(e_0^{r-1}e_1^{r+1}), e_0^r e_1^r\},$$

and for each $i = 0, 1, \ldots, r$, we have

$$\begin{aligned}
g_{(a,\mu',\mu'')}(e_0^i e_1^{k-i}) &= \psi(-2ia)\chi(\mu'^{-i})\psi(2(k-i)a)\chi(\mu'^{-(k-i)}\mu''^{-(k-i)})e_0^i e_1^{k-i} \\
&= \psi(2(k-2i)a)\chi(\mu'^{-k})\chi(\mu''^{-(k-i)})e_0^i e_1^{k-i}.
\end{aligned}$$

Using the fact that k is even and $\chi^2 = 1$, we get

$$g_{(a,\mu',\mu'')}(e_0^i e_1^{k-i}) = \psi(2(k-2i)a)\chi(\mu''^i)e_0^i e_1^{k-i}.$$

In particular, we have

$$g_{(a,\mu',\mu'')}(e_0^r e_1^r) = \chi(\mu''^r)e_0^r e_1^r.$$

Moreover, we have

$$g(e_0^r e_1^r) = e_1^r(g(e_1^r)) = (-1)^r e_0^r e_1^r.$$

It follows that

$$\mathrm{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)) \cong (\mathcal{L}_{\chi^r} \otimes \mathcal{L}_{\theta_1^r}) \oplus \left(\bigoplus_{i=0}^{r-1} [2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^i}) \right).$$

We have

$$[2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^i}) \cong [2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes [2]^*\mathcal{L}_{\theta_1^i}) \cong [2]_*\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^i}.$$

So we have

$$\mathrm{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)) \cong (\mathcal{L}_{\chi^r} \otimes \mathcal{L}_{\theta_1^r}) \oplus \left(\bigoplus_{i=0}^{r-1} [2]_*\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^i} \right).$$

Suppose $n = 2r+1$ is odd. $\mathrm{Sym}^k(\mathrm{Ind}_H^G(V))$ has a basis

$$\{e_1^k, g(e_1^k), e_0 e_1^{k-1}, g(e_0 e_1^{k-1}), \ldots, e_0^r e_1^{r+1}, g(e_0^r e_1^{r+1})\}.$$

Using the same calculation as above, we get

$$\mathrm{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)) \cong \bigoplus_{i=0}^{r} [2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_{\theta_1^{i+1}}.$$

Lemma 2.4 follows by twisting the above expressions of $\mathrm{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi))$ by $\mathcal{L}_{\theta_0^k}$. □

Lemma 2.5. *Assume $a \in \mathbb{F}_p$ is nonzero. We have the following identities.*

(i) $\epsilon(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}) = \frac{1}{p^2}$.

(ii) $\epsilon(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell, dt^2|_{\mathbb{P}^1_{(\infty)}}) = \frac{1}{p^3}$.

(iii) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = -\frac{g(\chi,\psi)}{p^2}$.

(iv) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}) = -\frac{g(\chi,\psi)}{p^3}\left(\frac{-2}{p}\right)$.

(v) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathcal{L}_\psi(at) \otimes \mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}) = \frac{1}{p^2}\left(\frac{2a}{p}\right)$.

(vi) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\psi(at)|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}) = \frac{1}{p^2}$.

(vii) $\epsilon(\mathbb{P}^1_{(\infty)}, [2]_*\overline{\mathbb{Q}}_\ell|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = -\frac{g(\chi,\psi)}{p^4}$.

(viii) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*(\mathcal{L}_\psi(at) \otimes \mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = -\frac{g(\chi,\psi)}{p^3}\left(\frac{2a}{p}\right)$.

(ix) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = \frac{g(\chi,\psi)^2}{p^4}\left(\frac{-2}{p}\right)$.

(x) $\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\psi(at)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = -\frac{g(\chi,\psi)}{p^3}$.

Proof. Let K_∞ be the completion of the field $k(\eta_\infty)$, let $\mathcal{O}_\infty$ be the ring of integers in K_∞, and let $s=\frac{1}{t}$. Then s is a uniformizer of K_∞. Denote the inclusion $\eta_\infty \to \mathbb{P}^1_{(\infty)}$ also by j. Let V be a $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1 on η_∞, and let $\phi : K_\infty^* \to \overline{\mathbb{Q}}_\ell^*$ be the character corresponding to V via the reciprocity law. The Artin conductor $a(\phi)$ of ϕ is defined to be the smallest integer m such that $\phi|_{1+s^m\mathcal{O}_\infty}=1$. For any nonzero meromorphic differential 1-form $\omega = f ds$ on $\mathbb{P}^1_{(\infty)}$, define the order $v_\infty(\omega)$ of ω to be the valuation $v_\infty(f)$ of f. By [11] 3.1.5.4 (v), we have

$$\epsilon(\mathbb{P}^1_{(\infty)}, j_*V, \omega) = \begin{cases} \phi(s^{v_\infty(\omega)})p^{v_\infty(\omega)} & \text{if } \phi|_{\mathcal{O}_\infty^*}=1, \\ \int_{s^{-(a(\phi)+v_\infty(\omega))}\mathcal{O}_\infty^*} \phi^{-1}(z)\psi(\mathrm{Res}_\infty(z\omega))dz & \text{if } \phi|_{\mathcal{O}_\infty^*}\neq 1, \end{cases}$$

where Res_∞ denotes the residue of a meromorphic 1-form at ∞, and the integral is taken with respect to the Haar measure dz on K_∞ normalized by $\int_{\mathcal{O}_\infty} dz = 1$.

Note that $dt = -\frac{ds}{s^2}$ has order -2 at ∞ and $dt^2 = -\frac{2ds}{s^3}$ has order -3. Applying the first case of the above formula for the ϵ-factor, we get (i) and (ii).

(iii) Taking $a=t=\frac{1}{s}$ and $b=z$ in the explicit reciprocity law in [17] XIV §3 Proposition 8, we see the character

$$\chi' : K_\infty^* \to \overline{\mathbb{Q}}_l^*$$

corresponding to $\mathcal{L}_\chi$ is given by

$$\chi'(z) = \chi^{-1}\left(\left(\frac{\bar{c}}{p}\right)\right),$$

where

$$c = (-1)^{-v_\infty(z)}\frac{z^{-1}}{s^{-v_\infty(z)}}$$

which is a unit in $\mathcal{O}_\infty$, $\bar{c}$ is the residue class of c in $\mathcal{O}_\infty/s\mathcal{O}_\infty \cong \mathbb{F}_p$, and $\left(\frac{\bar{c}}{p}\right)$ is the Legendre symbol of $\bar{c}$. Note that our formula for c is the reciprocal of the formula in [17] because the reciprocity map in [17] maps uniformizers in K to arithmetic Frobenius elements in $\mathrm{Gal}(\overline{K}_\infty/K_\infty)^{\mathrm{ab}}$, whereas the reciprocity map in [11] maps uniformizers in K to geometric Frobenius elements. One can verify $a(\chi')=1$. For any $z\in s\mathcal{O}_\infty^*$, write

$$z = s(r_0 + r_1 s + \cdots)$$

with $r_i \in \mathbb{F}_p$ and $r_0 \neq 0$. We then have

$$\begin{aligned} \bar{c} &= -r_0^{-1}, \\ \mathrm{Res}_\infty(zdt) &= -r_0. \end{aligned}$$

So we have

$$\begin{aligned}
\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) &= \int_{s\mathcal{O}_\infty^*} \chi'^{-1}(z)\psi(\mathrm{Res}_\infty(zdt))dz \\
&= \int_{s\mathcal{O}_\infty^*} \chi\left(\left(\frac{-r_0^{-1}}{p}\right)\right)\psi(-r_0)dz \\
&= \int_{s\mathcal{O}_\infty^*} \left(\frac{-r_0}{p}\right)\psi(-r_0)dz \\
&= \sum_{r_0\in\mathbb{F}_p^*}\int_{r_0 s(1+s\mathcal{O}_\infty)} \left(\frac{-r_0}{p}\right)\psi(-r_0)dz \\
&= \sum_{r_0\in\mathbb{F}_p^*} \left(\frac{-r_0}{p}\right)\psi(-r_0)\int_{r_0 s(1+s\mathcal{O}_\infty)} dz \\
&= \frac{1}{p^2}\sum_{r_0\in\mathbb{F}_p^*} \left(\frac{-r_0}{p}\right)\psi(-r_0) \\
&= -\frac{g(\chi,\psi)}{p^2}.
\end{aligned}$$

(iv) We can use the same method as in (iii), or use the formula [11] 3.1.5.5 to get

$$\begin{aligned}
\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}) &= \epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, 2tdt|_{\mathbb{P}^1_{(\infty)}}) \\
&= \chi'\left(\frac{2}{s}\right)p^{v_\infty(\frac{2}{s})}\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\
&= \left(\frac{-2}{p}\right)\cdot\frac{1}{p}\cdot\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\
&= -\frac{g(\chi,\psi)}{p^3}\left(\frac{-2}{p}\right)
\end{aligned}$$

(v) Taking a to be $at = \frac{a}{s}$ and $b = z$ in the explicit reciprocity law in [17] XIV §5 Proposition 15, we see the character

$$K_\infty^* \to \overline{\mathbb{Q}}_l^*$$

corresponding to $\mathcal{L}_\chi(at)$ is

$$z \mapsto \psi^{-1}\left(-\mathrm{Res}_\infty\left(\frac{a}{s}\cdot\frac{dz}{z}\right)\right).$$

(We add the negative sign to the formula in [17] since the reciprocity map in [17] is different from the one used in [11].) So the character

$$\phi : K_\infty^* \to \overline{\mathbb{Q}}_l^*$$

corresponding to $\mathcal{L}_\chi(at)\otimes\mathcal{L}_\chi$ is given by

$$\phi(z) = \psi^{-1}\left(-\mathrm{Res}_\infty\left(\frac{a}{s}\cdot\frac{dz}{z}\right)\right)\chi^{-1}\left(\left(\frac{\bar{c}}{p}\right)\right),$$

where $c = (-1)^{-v_\infty(z)} \frac{z^{-1}}{s^{-v_\infty(z)}}$. One can verify $a(\phi) = 2$. For any $z \in s\mathcal{O}_\infty^*$, write

$$z = s(r_0 + r_1 s + \cdots)$$

with $r_i \in \mathbb{F}_p$ and $r_0 \neq 0$. We then have

$$\begin{aligned} \mathrm{Res}_\infty\left(\frac{a}{s} \cdot \frac{dz}{z}\right) &= \frac{ar_1}{r_0}, \\ \bar{c} &= -r_0^{-1}, \\ \mathrm{Res}_\infty(zdt^2) &= -2r_1. \end{aligned}$$

So we have

$$\begin{aligned} & \epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}) \\ = & \int_{s\mathcal{O}_\infty^*} \phi^{-1}(z)\psi(\mathrm{Res}_\infty(zdt^2))dz \\ = & \int_{s\mathcal{O}_\infty^*} \psi\left(-\frac{ar_1}{r_0}\right)\chi\left(\left(\frac{-r_0^{-1}}{p}\right)\right)\psi(-2r_1)dz \\ = & \int_{s\mathcal{O}_\infty^*} \left(\frac{-r_0}{p}\right)\psi\left(-r_1\left(\frac{a}{r_0}+2\right)\right)dz \\ = & \sum_{r_0, r_1 \in \mathbb{F}_p, r_0 \neq 0} \int_{s(r_0+r_1 s)(1+s^2\mathcal{O}_\infty)} \left(\frac{-r_0}{p}\right)\psi\left(-r_1\left(\frac{a}{r_0}+2\right)\right)dz \\ = & \sum_{r_0, r_1 \in \mathbb{F}_p, r_0 \neq 0} \left(\frac{-r_0}{p}\right)\psi\left(-r_1\left(\frac{a}{r_0}+2\right)\right)\int_{s(r_0+r_1 s)(1+s^2\mathcal{O}_\infty)} dz \\ = & \frac{1}{p^3} \sum_{r_0, r_1 \in \mathbb{F}_p, r_0 \neq 0} \left(\frac{-r_0}{p}\right)\psi\left(-r_1\left(\frac{a}{r_0}+2\right)\right) \\ = & \frac{1}{p^3} \sum_{r_0 \in \mathbb{F}_p^*} \left(\frac{-r_0}{p}\right) \sum_{r_1 \in \mathbb{F}_p} \psi\left(-r_1\left(\frac{a}{r_0}+2\right)\right) \\ = & \frac{1}{p^3} \cdot \left(\frac{-\frac{-a}{2}}{p}\right) \cdot p \\ = & \frac{1}{p^2}\left(\frac{2a}{p}\right). \end{aligned}$$

We omit the proof of (vi), which is similar to the proof of (v).

(vii) We have $[2]_*\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell \oplus j_*\mathcal{L}_\chi$. So

$$\epsilon(\mathbb{P}^1_{(\infty)}, [2]_*\overline{\mathbb{Q}}_\ell|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = \epsilon(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}).$$

We then use (i) and (iii).

(viii) We can define ϵ-factors for virtual sheaves on $\mathbb{P}^1_{(\infty)}$. By [11] 3.1.5.4 (iv), we have

$$\epsilon(\mathbb{P}^1_{(\infty)}, [2]_*([j_*(\mathcal{L}_\psi(at)\otimes\mathcal{L}_\chi)]-[\overline{\mathbb{Q}}_\ell])|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) = \epsilon(\mathbb{P}^1_{(\infty)}, ([j_*(\mathcal{L}_\psi(at)\otimes\mathcal{L}_\chi)]-[\overline{\mathbb{Q}}_\ell])|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}}).$$

Hence

$$\begin{aligned} & \epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*(\mathcal{L}_\psi(at)\otimes\mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\ = \ & \frac{\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathcal{L}_\psi(at)\otimes\mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt^2|_{\mathbb{P}^1_{(\infty)}})}{\epsilon(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell, dt^2|_{\mathbb{P}^1_{(\infty)}})}\epsilon(\mathbb{P}^1_{(\infty)}, [2]_*\overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}). \end{aligned}$$

We then apply the formulas (ii), (v), and (vii).

We omit the proof of (ix) and (x), which is similar to the proof of (viii). □

Lemma 2.6. *We have*

$$\begin{aligned} & \epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathcal{L}_{\chi^r}\otimes\mathcal{L}_{\theta_0^{2r}\theta_1^r})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\ = & \begin{cases} \frac{g(\chi,\psi)^{-4r}}{p^2} & \text{if } r \text{ is even,} \\ -\frac{g(\chi,\psi)^{-2r+1}}{p^2}\left(\frac{-1}{p}\right) & \text{if } r \text{ is odd,} \end{cases} \\ & \epsilon(\mathbb{P}^1_{(\infty)}, j_*([2]_*\mathcal{L}_\psi((4i-4r)t)\otimes\mathcal{L}_{\theta_0^{2r}\theta_1^i})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\ = & \begin{cases} -\frac{g(\chi,\psi)^{-6r+1}}{p^4}\left(\frac{-1}{p}\right)^i & \text{if } p|i-r, \\ -\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i & \text{if } p\not|i-r, \end{cases} \\ & \epsilon(\mathbb{P}^1_{(\infty)}, j_*([2]_*\mathcal{L}_\psi((4i-4r-2)t)\otimes\mathcal{L}_\chi)\otimes\mathcal{L}_{\theta_0^{2r+1}\theta_1^{i+1}})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\ = & \begin{cases} \frac{g(\chi,\psi)^{-4r}}{p^4}\left(\frac{-2}{p}\right) & \text{if } p|2i-2r-1, \\ -\frac{g(\chi,\psi)^{-2r}}{p^3}\left(\frac{(-1)^{i+1}(2i-2r-1)}{p}\right) & \text{if } p\not|2i-2r-1. \end{cases} \end{aligned}$$

Proof. Let F_∞ be the geometric Frobenius element at ∞. We have

$$\theta_0(F_\infty) = g(\chi,\psi),\ \theta_1(F_\infty) = \left(\frac{-1}{p}\right).$$

Using the notation [11] 3.1.5.1, we have

$$\begin{aligned} a(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}) &= -2, \\ a(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) &= -1, \\ a(\mathbb{P}^1_{(\infty)}, [2]_*\overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}) &= -3, \\ a(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\psi(at)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) &= -1,\ (a\in\mathbb{F}_p^*) \\ a(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) &= -2, \\ a(\mathbb{P}^1_{(\infty)}, j_*[2]_*(\mathcal{L}_\psi(at)\otimes\mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) &= -1,\ (a\in\mathbb{F}_p^*). \end{aligned}$$

So by [11] 3.1.5.6, we have

$$
\begin{aligned}
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathcal{L}_{\chi^r}\otimes\mathcal{L}_{\theta_0^{2r}\theta_1^r})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})\\
=&\left\{\begin{array}{ll}((\theta_0^{2r}\theta_1^r)(F_\infty))^{-2}\epsilon(\mathbb{P}^1_{(\infty)}, \overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } r \text{ is even,}\\ ((\theta_0^{2r}\theta_1^r)(F_\infty))^{-1}\epsilon(\mathbb{P}^1_{(\infty)}, j_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } r \text{ is odd,}\end{array}\right.\\
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*([2]_*\mathcal{L}_\psi((4i-4r)t)\otimes\mathcal{L}_{\theta_0^{2r}\theta_1^i})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})\\
=&\left\{\begin{array}{ll}((\theta_0^{2r}\theta_1^i)(F_\infty))^{-3}\epsilon(\mathbb{P}^1_{(\infty)}, [2]_*\overline{\mathbb{Q}}_\ell, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } p|i-r,\\ ((\theta_0^{2r}\theta_1^i)(F_\infty))^{-1}\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\psi((4i-4r)t)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } p\not|i-r,\end{array}\right.\\
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*([2]_*(\mathcal{L}_\psi((4i-4r-2)t)\otimes\mathcal{L}_\chi)\otimes\mathcal{L}_{\theta_0^{2r+1}\theta_1^{i+1}})|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})\\
=&\left\{\begin{array}{ll}((\theta_0^{2r+1}\theta_1^{i+1})(F_\infty))^{-2}\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*\mathcal{L}_\chi|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } p|2i-2r-1,\\ ((\theta_0^{2r+1}\theta_1^{i+1})(F_\infty))^{-1}\epsilon(\mathbb{P}^1_{(\infty)}, j_*[2]_*(\mathcal{L}_\psi((4i-4r-2)t)\otimes\mathcal{L}_\chi)|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) & \text{if } p\not|2i-2r-1.\end{array}\right.
\end{aligned}
$$

We then apply the formulas in Lemma 2.5. □

The following is Proposition 0.4 in the introduction.

Proposition 2.7. $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})$ *equals*

$$p^{-(k+1)(\frac{k+8}{4}+[\frac{k}{2p}])}$$

if $k=2r$ *for an even* r,

$$p^{-(k+1)(\frac{k+6}{4}+[\frac{k}{2p}])}$$

if $k=2r$ *for an odd* r, *and*

$$(-1)^{\frac{k+1}{2}+[\frac{k}{p}]-[\frac{k}{2p}]}p^{-\frac{k+1}{2}(\frac{k+5}{2}+[\frac{k}{p}]-[\frac{k}{2p}])}\left(\frac{-2}{p}\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\prod_{j\in\{0,1,\ldots,[\frac{k}{2}]\},\ p\not|2j+1}\left(\frac{(-1)^j(2j+1)}{p}\right)$$

if $k=2r+1$.

Proof. By Lemmas 2.4 and 2.6, $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})$ equals

$$\frac{g(\chi,\psi)^{-4r}}{p^2}\prod_{i\in\{0,\ldots,r-1\},\ p|i-r}\left(-\frac{g(\chi,\psi)^{-6r+1}}{p^4}\left(\frac{-1}{p}\right)^i\right)\prod_{i\in\{0,\ldots,r-1\},\ p\not|i-r}\left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right)$$

if $k=2r$ for an even r,

$$-\frac{g(\chi,\psi)^{-2r+1}}{p^2}\left(\frac{-1}{p}\right)\prod_{i\in\{0,\ldots,r-1\},\ p|i-r}\left(-\frac{g(\chi,\psi)^{-6r+1}}{p^4}\left(\frac{-1}{p}\right)^i\right)\prod_{i\in\{0,\ldots,r-1\},\ p\not|i-r}\left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right)$$

if $k=2r$ for an odd r, and

$$\prod_{i\in\{0,\ldots,r\},\ p|2i-2r-1}\left(\frac{g(\chi,\psi)^{-4r}}{p^4}\left(\frac{-2}{p}\right)\right)\prod_{i\in\{0,\ldots,r\},\ p\not|2i-2r-1}\left(-\frac{g(\chi,\psi)^{-2r}}{p^3}\left(\frac{(-1)^{i+1}(2i-2r-1)}{p}\right)\right)$$

if $k=2r+1$. Let's simplify the above expressions. Recall that $g(\chi,\psi)^2 = p\left(\frac{-1}{p}\right)$. If $k=2r$ with r even, we have

$$\begin{aligned}
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\
= &\frac{g(\chi,\psi)^{-4r}}{p^2} \prod_{i\in\{0,\ldots,r-1\},\ p|i-r} \left(-\frac{g(\chi,\psi)^{-6r+1}}{p^4}\left(\frac{-1}{p}\right)^i\right) \prod_{i\in\{0,\ldots,r-1\},\ p\nmid i-r} \left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right) \\
= &\frac{g(\chi,\psi)^{-4r}}{p^2} \prod_{i\in\{0,\ldots,r-1\},\ p|i-r} \frac{g(\chi,\psi)^{-4r}}{p} \prod_{i\in\{0,\ldots,r-1\}} \left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right) \\
= &\frac{g(\chi,\psi)^{-4r}}{p^2} \left(\frac{g(\chi,\psi)^{-4r}}{p}\right)^{[\frac{r}{p}]} \left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\right)^r \left(\frac{-1}{p}\right)^{\frac{r(r-1)}{2}} \\
= &\frac{p^{-2r}}{p^2}\left(\frac{p^{-2r}}{p}\right)^{[\frac{r}{p}]} \frac{\left(p\left(\frac{-1}{p}\right)\right)^{\frac{r(-2r+1)}{2}}}{p^{3r}} \left(\frac{-1}{p}\right)^{\frac{r(r-1)}{2}} \\
= &p^{-r^2-\frac{9}{2}r-2-(2r+1)[\frac{r}{p}]}\left(\frac{-1}{p}\right)^{-\frac{r^2}{2}} \\
= &p^{-(k+1)(\frac{k+8}{4}+[\frac{k}{2p}])}.
\end{aligned}$$

If $k=2r$ with r odd, we have

$$\begin{aligned}
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}}) \\
= &-\frac{g(\chi,\psi)^{-2r+1}}{p^2}\left(\frac{-1}{p}\right) \prod_{i\in\{0,\ldots,r-1\},\ p|i-r} \left(-\frac{g(\chi,\psi)^{-6r+1}}{p^4}\left(\frac{-1}{p}\right)^i\right) \\
&\quad \times \prod_{i\in\{0,\ldots,r-1\},\ p\nmid i-r} \left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right) \\
= &-\frac{g(\chi,\psi)^{-2r+1}}{p^2}\left(\frac{-1}{p}\right) \prod_{i\in\{0,\ldots,r-1\},\ p|i-r} \left(\frac{g(\chi,\psi)^{-4r}}{p}\right) \prod_{i\in\{0,\ldots,r-1\}} \left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\left(\frac{-1}{p}\right)^i\right) \\
= &-\frac{g(\chi,\psi)^{-2r+1}}{p^2}\left(\frac{-1}{p}\right)\left(\frac{g(\chi,\psi)^{-4r}}{p}\right)^{[\frac{r}{p}]}\left(-\frac{g(\chi,\psi)^{-2r+1}}{p^3}\right)^r\left(\frac{-1}{p}\right)^{\frac{r(r-1)}{2}} \\
= &\frac{g(\chi,\psi)^{(-2r+1)(r+1)}}{p^{3r+2}}\left(\frac{-1}{p}\right)^{1+\frac{r(r-1)}{2}}\left(\frac{g(\chi,\psi)^{-4r}}{p}\right)^{[\frac{r}{p}]} \\
= &\frac{\left(p\left(\frac{-1}{p}\right)\right)^{\frac{(-2r+1)(r+1)}{2}}}{p^{3r+2}}\left(\frac{-1}{p}\right)^{1+\frac{r(r-1)}{2}}\left(\frac{p^{-2r}}{p}\right)^{[\frac{r}{p}]} \\
= &p^{-r^2-\frac{7}{2}r-\frac{3}{2}-(2r+1)[\frac{r}{p}]}\left(\frac{-1}{p}\right)^{-\frac{(r-1)(r+3)}{2}} \\
= &p^{-(k+1)(\frac{k+6}{4}+[\frac{k}{2p}])}.
\end{aligned}$$

If $k=2r+1$ is odd, we have

$$\begin{aligned}
&\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_2))|_{\mathbb{P}^1_{(\infty)}}, dt|_{\mathbb{P}^1_{(\infty)}})\\
=\;& \prod_{i\in\{0,\ldots,r\},\ p|2i-2r-1}\left(\frac{g(\chi,\psi)^{-4r}}{p^4}\left(\frac{-2}{p}\right)\right)\prod_{i\in\{0,\ldots,r\},\ p\nmid 2i-2r-1}\left(-\frac{g(\chi,\psi)^{-2r}}{p^3}\left(\frac{(-1)^{i+1}(2i-2r-1)}{p}\right)\right)\\
=\;& \left(-\frac{g(\chi,\psi)^{-2r}}{p^3}\right)^{r+1}\left(-\frac{g(\chi,\psi)^{-2r}}{p}\left(\frac{-2}{p}\right)\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\prod_{i\in\{0,\ldots,r\},\ p\nmid 2i-2r-1}\left(\frac{(-1)^{i+1}(2i-2r-1)}{p}\right)\\
=\;& \left(-p^{-3-r}\left(\frac{-1}{p}\right)^{-r}\right)^{r+1}\left(-p^{-1-r}\left(\frac{-1}{p}\right)^{-r}\left(\frac{-2}{p}\right)\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\prod_{j\in\{0,\ldots,r\},\ p\nmid 2j+1}\left(\frac{(-1)^{r-j}(2j+1)}{p}\right)\\
=\;& (-1)^{r+1+[\frac{k}{p}]-[\frac{k}{2p}]}p^{-(r+1)(r+3+[\frac{k}{p}]-[\frac{k}{2p}])}\left(\frac{-2}{p}\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\left(\frac{-1}{p}\right)^{-r(r+1)-r([\frac{k}{p}]-[\frac{k}{2p}])}\\
&\quad\times\prod_{j\in\{0,\ldots,r\},p\nmid 2j+1}\left(\frac{(-1)^{r-j}(2j+1)}{p}\right)\\
=\;& (-1)^{\frac{k+1}{2}+[\frac{k}{p}]-[\frac{k}{2p}]}p^{-\frac{k+1}{2}(\frac{k+5}{2}+[\frac{k}{p}]-[\frac{k}{2p}])}\left(\frac{-2}{p}\right)^{[\frac{k}{p}]-[\frac{k}{2p}]}\prod_{j\in\{0,\ldots,[\frac{k}{2}]\},\ p\nmid 2j+1}\left(\frac{(-1)^j(2j+1)}{p}\right).
\end{aligned}$$

□

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