T-ADIC EXPONENTIAL SUMS OVER FINITE FIELDS

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Abstract. T-adic exponential sums associated to a Laurent polynomial f are introduced. They interpolate all classical p^m-power order exponential sums associated to f. The Hodge bound for the Newton polygon of L-functions of T-adic exponential sums is established. This bound enables us to determine, for all m, the Newton polygons of L-functions of p^m-power order exponential sums associated to an f which is ordinary for $m = 1$. Deeper properties of L-functions of T-adic exponential sums are also studied. Along the way, new open problems about the T-adic exponential sum itself are discussed.

1. Introduction

1.1. **Classical exponential sums.** We first recall the definition of classical exponential sums over finite fields of characteristic p with values in a p-adic field.

Let p be a fixed prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $q = p^a$ be a power of p, $\mathbb{F}_q$ the finite field of q elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and $\mathbb{Z}_q$ the ring of integers of $\mathbb{Q}_q$.

Fix a positive integer n. Let $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ be a Laurent polynomial in n variables of the form

$$f(x) = \sum_u a_u x^u,\ a_u \in \mu_{q-1},\ x^u = x_1^{u_1} \cdots x_n^{u_n},$$

where μ_k denotes the group of k-th roots of unity in $\overline{\mathbb{Q}}_p$.

Definition 1.1. *Let ψ be a locally constant character of $\mathbb{Z}_p$ of order p^m with values in $\overline{\mathbb{Q}}_p$, and let $\pi_\psi = \psi(1) - 1$. The sum*

$$S_{f,\psi}(k) = \sum_{x \in \mu^n_{q^k-1}} \psi(Tr_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x)))$$

is called a p^m-power order exponential sum on the n-torus $\mathbb{G}^n_m$ over $\mathbb{F}_{q^k}$. The generating function

$$L_{f,\psi}(s) = L_{f,\psi}(s; \mathbb{F}_q) = \exp\Big(\sum_{k=1}^{\infty} S_{f,\psi}(k)\frac{s^k}{k}\Big) \in 1 + s\mathbb{Z}_p[\pi_\psi][[s]]$$

is called the L-function of p^m-power order exponential sums over $\mathbb{F}_q$ associated to $f(x)$.

Note that the above exponential sum for $m \geq 1$ is still an exponential sum over a finite field as we just sum over the subset of roots of unity (corresponding to the elements of a finite field via the Teichmüller lifting), not over the whole finite residue ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$. The exponential sum over the whole finite ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$ and its generating function as m varies is the subject of Igusa's zeta function, see Igusa [17].

In general, the above L-function $L_{f,\psi}(s)$ of exponential sums is rational in s. But, if f is non-degenerate, then $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial, as was shown in [1,2] for ψ of order p, and in [20] for all ψ. By a result of [12], if p is large enough, then f is generically non-degenerate. For non-degenerate f, the location of the zeros of $L_{f,\psi}(s)^{(-1)^{n-1}}$ becomes an important issue. The p-adic theory of such L-functions was developed by Dwork, Bombieri [8], Adolphson-Sperber [1,2], the second author [26,27], and Blache [7] for ψ of order p. More recently initial part of the theory was extended to all ψ by Liu-Wei [20] and Liu [19].

The p-adic theory of the above exponential sum for $n = 1$ and ψ of order p has a long history and has been studied extensively in the literature. For instance, in the simplest case that $f(x) = x^d$, the exponential sum was studied by Gauss, see Berndt-Evans [3] for a comprehensive survey. By the Hasse-Davenport relation for Gauss sums, the L-function is a polynomial whose zeros are given by roots of Gauss sums. Thus, the slopes of the L-function are completely determined by the Stickelberger theorem for Gauss sums. The roots of the L-function have explicit p-adic formulas in terms of p-adic Γ-function via the Gross-Koblitz formula [13]. These ideas can be extended to treat the so-called diagonal f case for general n, see Wan [27]. These elementary cases have been used as building bricks to study the deeper non-diagonal $f(x)$ via various decomposition theorems, which are the main ideas of Wan [26,27]. In the case $n = 1$ and ψ of order p, further progresses about the slopes of the L-function were made in Zhu [32,33], Blache and Ferard [5], and Liu [21].

1.2. **T-adic exponential sums.** We now define the T-adic exponential sum, state our main results, and put forward some new questions.

Definition 1.2. *For a positive integer k, the T-adic exponential sum of f over $\mathbb{F}_{q^k}$ is the sum:*

$$S_f(k,T) = \sum_{x \in \mu_{q^k-1}^n} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_p[[T]].$$

The T-adic L-function of f over $\mathbb{F}_q$ is the generating function

$$L_f(s,T) = L_f(s,T;\mathbb{F}_q) = \exp(\sum_{k=1}^{\infty} S_f(k,T)\frac{s^k}{k}) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

The T-adic exponential sum interpolates classical exponential sums of p^m-order over finite fields for all positive integers m. In fact, we have

$$S_f(k, \pi_\psi) = S_{f,\psi}(k).$$

Similarly, one can recover the classical L-function of the p^m-order exponential sum from the T-adic L-function by the formula

$$L_f(s, \pi_\psi) = L_{f,\psi}(s).$$

We view $L_f(s, T)$ as a power series in the single variable s with coefficients in the complete discrete valuation ring $\mathbb{Q}_p[[T]]$ with uniformizer T.

Definition 1.3. *The T-adic characteristic function of f over $\mathbb{F}_q$, or C-function of f for short, is the generating function*

$$C_f(s, T) = \exp(\sum_{k=1}^{\infty} -(q^k - 1)^{-n} S_f(k, T) \frac{s^k}{k}) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

The C-function $C_f(s, T)$ and the L-function $L_f(s, T)$ determine each other. They are related by

$$L_f(s, T) = \prod_{i=0}^{n} C_f(q^i s, T)^{(-1)^{n-i-1}\binom{n}{i}},$$

and

$$C_f(s, T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s, T)^{\binom{n+j-1}{j}}.$$

In §4, we prove

Theorem 1.4 (analytic continuation)**.** *The C-function $C_f(s, T)$ is T-adic entire in s. As a consequence, the L-function $L_f(s, T)$ is T-adic meromorphic in s.*

The above theorem tells that the C-function behaves T-adically better than the L-function. In fact, in the T-adic setting, the C-function is a more natural object than the L-function. Thus, we shall focus more on the C-function.

Knowing the analytic continuation of $C_f(s, T)$, we are then interested in the location of its zeros. More precisely, we would like to determine the T-adic Newton polygon of this entire function $C_f(s, T)$. This is expected to be a complicated problem in general. It is open even in the simplest case $n = 1$ and $f(x) = x^d$ is a monomial if $p \not\equiv 1 \pmod{d}$. What we can do is to give an explicit combinatorial lower bound depending only on q and Δ, called the q-Hodge bound $\mathrm{HP}_q(\Delta)$. This polygon will be described in detail in §3.

Let $\mathrm{NP}_T(f)$ denote the T-adic Newton polygon of the C-function $C_f(s, T)$. In §5, we prove

Theorem 1.5 (Hodge bound)**.** *We have*

$$\mathrm{NP}_T(f) \ \geq \ \mathrm{HP}_q(\Delta).$$

This theorem shall give several new results on classical exponential sums, as we shall see in §2. In particular, this extends, in one stroke, all known ordinariness results for ψ of order p to all ψ of any p-power order. It demonstrates the significance of the T-adic L-function. It also gives rise to the following definition.

Definition 1.6. *The Laurent polynomial f is called T-adically ordinary if* $\mathrm{NP}_T(f) \ = \ \mathrm{HP}_q(\Delta)$.

We shall show that the classical notion of ordinariness implies T-adic ordinariness. But it is possible that a non-ordinary f is T-adically ordinary. Thus, it remains of interest to study exactly when f is T-adically ordinary. For this purpose, in §6, we extend the facial decomposition theorem in Wan [26] to the T-adic case. Let Δ be the convex closure in $\mathbb{R}^n$ of the origin and the exponents of the non-zero monomials in the Laurent polynomial $f(x)$. For any closed face σ of Δ, we let f_σ denote the sum of monomials of f whose exponent vectors lie in σ.

Theorem 1.7 (T-adic facial decomposition)**.** *The Laurent polynomial f is T-adically ordinary if and only if for every closed face σ of Δ of codimension* 1 *not containing the origin, the restriction f_σ is T-adically ordinary.*

In §7, we briefly discuss the variation of the C-function $C_f(s,T)$ and its Newton polygon when the reduction of f moves in an algebraic family over a finite field. The main questions are the generic ordinariness, generic Newton polygon, the analogue of the Adolphson-Sperber conjecture [1], Wan's limiting conjecture [27], Dwork's unit root conjecture [10] in the T-adic and π_ψ-adic case. We shall give an overview about what can be proved and what is unknown, including a number of conjectures. Basically, a lot can be proved in the ordinary case, and a lot remain to be proved in the non-ordinary case.

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2. Applications

In this section we give several applications of the T-adic exponential sum to classical exponential sums.

Theorem 2.1 (integrality theorem)**.** *We have*

$$L_f(s,T) \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

and

$$C_f(s,T) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

Proof. Let $|\mathbb{G}_m^n|$ be the set of closed points of $\mathbb{G}_m^n$ over $\mathbb{F}_q$, and $a \mapsto \hat{a}$ the Teichmüller lifting. It is easy to check that the T-adic L-function has the Euler product expansion

$$L_f(s,T) = \prod_{x \in |\mathbb{G}_m^n|} \frac{1}{\left(1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg(x)}}/\mathbb{Q}_p}(f(\hat{x}))} s^{\deg(x)}\right)} \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where $\hat{x} = (\hat{x}_1, \cdots, \hat{x}_n)$. The theorem now follows. □

The above proof shows that the L-function $L_f(s,T)$ is the L-function $L(s,\rho_f)$ of the following continuous (p,T)-adic representation of the arithmetic fundamental group:

$$\rho_f : \pi_1^{\mathrm{arith}}(\mathbb{G}_m^n/\mathbb{F}_q) \longrightarrow \mathrm{GL}_1(\mathbb{Z}_p[[T]]),$$

defined by

$$\rho_f(\mathrm{Frob}_x) = (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg(x)}}/\mathbb{Q}_p}(f(\hat{x}))}.$$

The rank one representation ρ_f is transcendental in nature. Its L-function $L(s,\rho_f)$ seems to be beyond the reach of ℓ-adic cohomology, where ℓ is a prime different from p. However, the specialization of ρ_f at the special point $T = \pi_\psi$ is a character of finite order. Thus, the specialization

$$L(s,\rho_f)|_{T=\pi_\psi} = L_{f,\psi}(s)$$

can indeed be studied using Grothendieck's ℓ-adic trace formula [14]. This gives another proof that the L-function $L_{f,\psi}(s)$ is a rational function in s. But the T-adic L-function $L_f(s,T)$ itself is certainly out of the reach of ℓ-adic cohomology as it is truly transcendental.

Let $\mathrm{NP}_{\pi_\psi}(f)$ denote the π_ψ-adic Newton polygon of the C-function $C_f(s,\pi_\psi)$. The integrality of $C_f(s,T)$ immediately gives the following theorem.

Theorem 2.2 (rigidity bound)**.** *If ψ is non-trivial, then*

$$\mathrm{NP}_{\pi_\psi}(f) \;\geq\; \mathrm{NP}_T(f).$$

Proof. Obvious. □

A natural question is to ask when $\mathrm{NP}_{\pi_\psi}(f)$ coincides with its rigidity bound.

Theorem 2.3 (transfer theorem)**.** *If $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{NP}_T(f)$ holds for one non-trivial ψ, then it holds for all non-trivial ψ.*

Proof. By the integrality of $C_f(s,T)$, the T-adic Newton polygon of $C_f(s,T)$ coincides with the π_ψ-adic Newton polygon of $C_f(s,\pi_\psi)$ if and only if for every vertex (i,e) of the T-adic Newton polygon of $C_f(s,T)$, the coefficients of s^i in $C_f(s,T)$ differs from T^e by a unit in $\mathbb{Z}_p[[T]]^\times$. It follows that if the coincidence happens for one non-trivial ψ, it happens for all non-trivial ψ. The theorem is proved. □

Definition 2.4. *We call f rigid if $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{NP}_T(f)$ for one (and hence for all) non-trivial ψ.*

In [22], cooperating with his students, the first author showed that f is generically rigid if $n=1$ and p is sufficiently large. So the rigid bound is the best possible bound. In contrast, the weaker Hodge bound $\mathrm{HP}_q(\Delta)$ is only best possible if $p \equiv 1 \pmod{d}$, where d is the degree of f.

We now pause to describe the relationship between the Newton polygons of $C_f(s,\pi_\psi)$ and $L_{f,\psi}(s)^{(-1)^{n-1}}$. We need the following definitions.

Definition 2.5. *A convex polygon with initial point* $(0,0)$ *is called algebraic if it is the graph of a* $\mathbb{Q}$*-valued function defined on* $\mathbb{N}$ *or on an interval of* $\mathbb{N}$*, and its slopes are of finite multiplicity and of bounded denominator.*

Definition 2.6. *For an algebraic polygon with slopes* $\{\lambda_i\}$*, we define its slope series to be* $\sum_i t^{\lambda_i}$*.*

It is clear that an algebraic polygon is uniquely determined by its slope series. So the slope series embeds the set of algebraic polygons into the ring $\varinjlim_d \mathbb{Z}[[t^{\frac{1}{d}}]]$. The image is $\varinjlim_d \mathbb{N}[[t^{\frac{1}{d}}]]$. It is closed under addition and multiplication. Therefore one can define an addition and a multiplication on the set of algebraic polygons.

Lemma 2.7. *Suppose that* f *is non-degenerate. Then the* q*-adic Newton polygon of* $C_f(s,\pi_\psi;\mathbb{F}_q)$ *is the product of the* q*-adic Newton polygon of* $L_{f,\psi}(s;\mathbb{F}_q)^{(-1)^{n-1}}$ *and the algebraic polygon* $\frac{1}{(1-t)^n}$*.*

Proof. Note that the C-value $C_f(s,\pi_\psi)$ and the L-function $L_{f,\psi}(s)$ determine each other. They are related by

$$L_{f,\psi}(s) = \prod_{i=0}^{n} C_f(q^i s, \pi_\psi)^{(-1)^{n-i-1}\binom{n}{i}},$$

and

$$C_f(s,\pi_\psi)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_{f,\psi}(q^j s)^{\binom{n+j-1}{j}}.$$

Suppose that

$$L_{f,\psi}(s)^{(-1)^{n-1}} = \prod_{i=1}^{d}(1-\alpha_i s).$$

Then

$$C_f(s,\pi_\psi) = \prod_{j=0}^{\infty}\prod_{i=1}^{d}(1-\alpha_i q^j s)^{\binom{n+j-1}{j}}.$$

Let λ_i be the q-adic order of α_i. Then the q-adic order of $\alpha_i q^j$ is $\lambda_i + j$. So the slope series of the q-adic Newton polygon of $L_{f,\psi}(s)^{(-1)^{n-1}}$ is

$$S(t) = \sum_{i=1}^{d} t^{\lambda_i},$$

and the slope series of the q-adic Newton polygon of $C_f(s, \pi_\psi)$ is

$$\sum_{j=0}^{+\infty}\sum_{i=0}^{d}\binom{n+j-1}{j}t^{\lambda_i+j} = \frac{1}{(1-t)^n}S(t).$$

The lemma now follows. □

We combine the rigidity bound and the Hodge bound to give the following theorem.

Theorem 2.8. *If ψ is non-trivial, then*

$$\mathrm{NP}_{\pi_\psi}(f) \ \geq\ \mathrm{NP}_T(f) \ \geq\ \mathrm{HP}_q(\Delta).$$

Proof. Obvious. □

If we drop the middle term, we arrive at the Hodge bound

$$\mathrm{NP}_{\pi_\psi}(f) \ \geq\ \mathrm{HP}_q(\Delta)$$

of Adolphson-Sperber [2] and Liu-Wei [20].

Theorem 2.9. *If $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ holds for one non-trivial ψ, then f is rigid, T-adically ordinary, and the equality holds for all non-trivial ψ.*

Proof. Suppose that $\mathrm{NP}_{\pi_{\psi_0}}(f) = \mathrm{HP}_q(\Delta)$ for a non-trivial ψ_0. Then, by the last theorem, we have

$$\mathrm{NP}_{\pi_{\psi_0}}(f) \ =\ \mathrm{NP}_T(f) \ =\ \mathrm{HP}_q(\Delta).$$

So f is rigid and T-adically ordinary, and

$$\mathrm{NP}_{\pi_\psi}(f) \ =\ \mathrm{NP}_T(f) \ =\ \mathrm{HP}_q(\Delta)$$

holds for all nontrivial ψ. The theorem is proved. □

Definition 2.10. *We call f ordinary if $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ holds for one (and hence for all) non-trivial ψ.*

The notion of ordinariness now carries much more information than what we had known. From this, we see that the T-adic exponential sum provides a new framework to study all p^m-power order exponential sums simultaneously. Instead of the usual way of extending the methods for ψ of order p to the case of higher order, the T-adic exponential sum has the novel feature that it can sometimes transfer a known result for one non-trivial ψ to all non-trivial ψ. This philosophy is carried out further in the paper [22].

Example 2.1. *Let*

$$f(x) = x_1 + x_2 + \cdots + x_n + \frac{\alpha}{x_1x_2\cdots x_n},\ \alpha \in \mu_{q-1}.$$

Then, by the result of Sperber [25] and our new information on ordinariness, we have

$$\mathrm{NP}_{\pi_\psi}(f) \ =\ \mathrm{HP}_q(\Delta)$$

for all non-trivial ψ.

3. The q-Hodge polygon

In this section, we describe explicitly the q-Hodge polygon mentioned in the introduction. Recall that $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ is a Laurent polynomial in n variables of the form

$$f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u, \ a_u \in \mathbb{Z}_q, \ a_u^q = a_u.$$

We stress that the non-zero coefficients of $f(x)$ are roots of unity in $\mathbb{Z}_q$, thus correspond in a unique way to Teichmüller liftings of elements of the finite field $\mathbb{F}_q$. If the coefficients of $f(x)$ are arbitrary elements in $\mathbb{Z}_q$, much of the theory still holds, but it is more complicated to describe the results. We have made the simplifying assumption that the non-zero coefficients are always roots of unity in this paper.

Let Δ be the convex polyhedron in $\mathbb{R}^n$ associated to f, which is generated by the origin and the exponent vectors of the non-zero monomials of f. Let $C(\Delta)$ be the cone in $\mathbb{R}^n$ generated by Δ. Define the degree function $u \mapsto \deg(u)$ on $C(\Delta)$ such that $\deg(u) = 1$ when u lies on a codimensional 1 face of Δ that does not contain the origin, and such that

$$\deg(ru) = r \deg(u), \ r \in \mathbb{R}_{\geq 0}, \ u \in C(\Delta).$$

We call it the degree function associated to Δ. We have $\deg(u + v) \leq \deg(u) + \deg(v)$ if $u, v \in C(\Delta)$, and the equality holds if and only if u and v are co-facial. In other words, the number

$$c(u, v) := \deg(u) + \deg(v) - \deg(u + v)$$

is 0 if $u, v \in C(\Delta)$ are co-facial, and is positive otherwise. We call that number $c(u, v)$ the co-facial defect of u and v. Let

$$M(\Delta) := C(\Delta) \cap \mathbb{Z}^n$$

be the set of lattice points in the cone $C(\Delta)$. Let D be the denominator of the degree function, which is the smallest positive integer such that

$$\deg M(\Delta) \subset \frac{1}{D}\mathbb{Z}.$$

For every natural number k, we define

$$W(k) := W_\Delta(k) = \#\{u \in M(\Delta) | \deg(u) = k/D\}$$

to be the number of lattice points of degree $\frac{k}{D}$ in $M(\Delta)$. For prime power $q = p^a$, the q-Hodge polygon of f is the polygon with vertices $(0, 0)$ and

$$(\sum_{j=0}^{i} W(j), a(p-1) \sum_{j=0}^{i} \frac{j}{D} W(j)), \ i = 0, 1, \cdots.$$

It is also called the q-Hodge polygon of Δ and denoted by $\mathrm{HP}_q(\Delta)$. It depends only on q and Δ. It has a side of slope $a(p-1)\frac{j}{D}$ with horizontal length $W(j)$ for each non-negative integer j.

4. Analytic continuation

In this section, we prove the T-adic analytic continuation of the C-function $C_f(s,T)$. The idea is to employ Dwork's trace formula in the T-adic case.

Note that the Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is cyclic of order $a = \log_p q$. There is an element in the Galois group whose restriction to μ_{q-1} is the p-power morphism. It is of order a, and is called the Frobenius element. We denote that element by σ.

We define a new variable π by the relation $E(\pi) = 1 + T$, where

$$E(\pi) = \exp(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i}) \in 1 + \pi\mathbb{Z}_p[[\pi]]$$

is the Artin-Hasse exponential series. Thus, π and T are two different uniformizers of the T-adic local ring $\mathbb{Q}_p[[T]]$. It is clear that for $\alpha \in \mathbb{Z}_q$, we have

$$E(\pi\alpha) \in 1 + \pi\mathbb{Z}_q[[\pi]],$$

and for $\beta \in \mathbb{Z}_p$, we have

$$E(\pi)^{\beta} \in 1 + \pi\mathbb{Z}_p[[\pi]].$$

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $\mathbb{Z}_q[[\pi]]$ but keeping π fixed. The Artin-Hasse exponential series has a kind of commutativity expressed as the following lemma.

Lemma 4.1 (Commutativity)**.** *We have the following commutative diagram*

$$\begin{array}{ccc} \mu_{q-1} & \stackrel{E(\pi\cdot)}{\rightarrow} & \mathbb{Z}_q[[\pi]] \\ \mathrm{Tr}\downarrow & & \downarrow \mathrm{Norm} \\ \mu_{p-1} & \stackrel{E(\pi)^{\cdot}}{\rightarrow} & \mathbb{Z}_p[[\pi]]. \end{array}$$

That is, if $x \in \mu_{q-1}$, then

$$E(\pi)^{x+x^p+\cdots+x^{p^{a-1}}} = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}).$$

Proof. Since for $x \in \mu_{q-1}$,

$$\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+i}},$$

we have

$$E(\pi)^{x+x^p+\cdots+x^{p^{a-1}}} = \exp(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \sum_{j=0}^{a-1} x^{p^{j+i}}) = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}).$$

The lemma is proved. □

Definition 4.2. *Let $\pi^{1/D}$ be a fixed D-th root of π. Define*

$$L(\Delta) = \{ \sum_{u \in M(\Delta)} b_u \pi^{\deg(u)} x^u : \; b_u \in \mathbb{Z}_q[[\pi^{1/D}]]\},$$

and

$$B=\{\sum_{u\in M(\Delta)} b_u\pi^{\deg(u)}x^u\in L(\Delta),\ \mathrm{ord}_T(b_u)\to+\infty \text{ if } \deg(u)\to+\infty\}.$$

The spaces $L(\Delta)$ and B are T-adic Banach algebras over the ring $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. The monomials $\pi^{\deg(u)}x^u$ $(u\in M(\Delta))$ form an orthonormal basis (resp., a formal basis) of B (resp., $L(\Delta)$) . The algebra B is contained in the larger Banach algebra $L(\Delta)$. If $u\in\Delta$, it is clear that $E(\pi x^u)\in L(\Delta)$. Write

$$E_f(x):=\prod_{a_u\neq 0}E(\pi a_u x^u),\ \text{if } f(x)=\sum_{u\in\mathbb{Z}^n}a_u x^u.$$

This is an element of $L(\Delta)$ since $L(\Delta)$ is a ring.

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $L(\Delta)$ but keeping $\pi^{1/D}$ as well as the variables x_i's fixed. From the commutativity of the Artin-Hasse exponential series, one can infer the following lemma.

Lemma 4.3 (Dwork's splitting lemma)**.** *If $x\in\mu_{q^k-1}$, then*

$$E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))}=\prod_{i=0}^{ak-1}E_f^{\sigma^i}(x^{p^i}),$$

where a is the order of $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$.

Proof. We have

$$E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))}=\prod_{a_u\neq 0}E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(a_u x^u)}$$

$$=\prod_{a_u\neq 0}\prod_{i=0}^{ak-1}E(\pi(a_u x^u)^{p^i})=\prod_{i=0}^{ak-1}E_f^{\sigma^i}(x^{p^i}).$$

The lemma is proved. □

Definition 4.4. *We define a map*

$$\psi_p: L(\Delta)\to L(\Delta),\ \sum_{u\in M(\Delta)}b_u x^u\mapsto\sum_{u\in M(\Delta)}b_{pu}x^u.$$

It is clear that the composition map $\psi_p\circ E_f$ sends B to B.

Lemma 4.5. *Write*

$$E_f(x)=\sum_{u\in M(\Delta)}\alpha_u(f)\pi^{\deg(u)}x^u.$$

Then, $\psi_p\circ E_f(\pi^{\deg(u)}x^u)$

$$=\sum_{w\in M(\Delta)}\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\pi^{(p-1)\deg(w)}\pi^{\deg(w)}x^w,\ u\in M(\Delta),$$

where $c(pw-u,u)$ is the co-facial defect of $pw-u$ and u.

Proof. This follows directly from the definition of ψ_p and $E_f(x)$. □

Definition 4.6. *Define*

$$\psi := \sigma^{-1} \circ \psi_p \circ E_f : B \longrightarrow B,$$

and its a-th iterate

$$\psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}).$$

Note that ψ is linear over $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$, but semi-linear over $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. On the other hand, ψ^a is linear over $\mathbb{Z}_q[[\pi^{1/D}]]$. By the last lemma, ψ^a is completely continuous in the sense of Serre [24].

Theorem 4.7 (Dwork's trace formula)**.** *For every positive integer k,*

$$(q^k - 1)^{-n} S_f(k, T) = \mathrm{Tr}_{B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]}(\psi^{ak}).$$

Proof. Let $g(x) \in B$. We have

$$\psi^{ak}(g) = \psi_p^{ak}(g \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i})).$$

Write

$$\prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_u x^u.$$

One computes that

$$\psi^{ak}(\pi^{\deg(v)} x^v) = \sum_{u \in M(\Delta)} \beta_{q^k u - v} \pi^{\deg(v)} x^u.$$

Thus,

$$\mathrm{Tr}(\psi^{ak} | B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}.$$

But, by Dwork's splitting lemma, we have

$$(q^k - 1)^{-n} S_f(k, T) = (q^k - 1)^{-n} \sum_{x \in \mu_{q^k-1}^n} \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}.$$

The theorem now follows. □

Theorem 4.8 (Analytic trace formula)**.** *We have*

$$C_f(s, T) = \det(1 - \psi^a s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]).$$

In particular, the T-adic C-function $C_f(s, T)$ is T-adic analytic in s.

Proof. It follows from the last theorem and the well known identity

$$\det(1-\psi^a s)=\exp(-\sum_{k=1}^{\infty}\mathrm{Tr}(\psi^{ak})\frac{s^k}{k}).$$

□

This theorem gives another proof that the coefficients of $C_f(s,T)$ and $L_f(s,T)$ as power series in s are T-adically integral.

Corollary 4.9. *For each non-trivial ψ, the C-value $C_f(s,\pi_\psi)$ is p-adic entire in s and the L-function $L_{f,\psi}(s)$ is rational in s.*

Proof. Obvious. □

5. The Hodge bound

The analytic trace formula in the previous section reduces the study of $C_f(s,T)$ to the study of the operator ψ^a. We consider ψ first. Note that ψ operates on B and is linear over $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$.

Theorem 5.1. *The T-adic Newton polygon of* $\det(1-\psi s\mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ *lies above the polygon with vertices* $(0,0)$ *and*

$$(a\sum_{k=0}^{i}W(k),a(p-1)\sum_{k=0}^{i}\frac{k}{D}W(k)),\ i=0,1,\cdots.$$

Proof. Let $\xi_1,\xi_2,\cdots,\xi_a$ be a normal basis of $\mathbb{Q}_q$ over $\mathbb{Q}_p$. Write

$$(\xi_j\alpha_{pw-u}(f))^{\sigma^{-1}}=\sum_{i=0}^{a-1}\alpha_{(i,w),(j,u)}(f)\xi_i,\ \alpha_{(i,w),(j,u)}(f)\in\mathbb{Z}_p[[\pi^{1/D}]].$$

Then $\psi(\xi_j\pi^{\deg(u)}x^u)$

$$=\sum_{i=0}^{a-1}\sum_{w\in M(\Delta)}\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)}\pi^{(p-1)\deg(w)}\xi_i\pi^{\deg(w)}x^w.$$

That is, the matrix of ψ over $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$ with respect to the orthonormal basis $\{\xi_j\pi^{\deg(u)}x^u\}_{0\le j<a,u\in M(\Delta)}$ is

$$A=(\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)}\pi^{(p-1)\deg(w)})_{(i,w),(j,u)}.$$

So, the T-adic Newton polygon of $\det(1-\psi s\mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ lies above the polygon with vertices $(0,0)$ and

$$(a\sum_{k=0}^{i}W(k),a(p-1)\sum_{k=0}^{i}\frac{k}{D}W(k))\ (i=0,1,\cdots).$$

Theorem 5.1 is proved. □

We are now ready to prove the Hodge bound for the Newton polygon.

Theorem 5.2. *We have*

$$\mathrm{NP}_T(f) \ \geq \ \mathrm{HP}_q(\Delta).$$

Proof. By the above theorem, it suffices to prove that the T-adic Newton polygon of $\det(1 - \psi^a s^a \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$ coincides with that of $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$. Note that

$$\det(1 - \psi^a s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \mathrm{Norm}(\det(1 - \psi^a s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])),$$

where the norm map is the norm from $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ to $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$. The theorem now follows from the equality

$$\prod_{\zeta^a=1} \det(1 - \psi\zeta s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \det(1 - \psi^a s^a \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]).$$

□

6. Facial decomposition

In this section, we extend the facial decomposition theorem in [26]. Recall that the operator $\psi = \sigma^{-1} \circ (\psi_p \circ E_f)$ is only semi-linear over $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. But its second factor $\psi_p \circ E_f$ is clearly linear and so $\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$ is well defined. We begin with the following theorem.

Theorem 6.1. *The T-adic Newton polygon of $C_f(s,T)$ coincides with $\mathrm{HP}_q(\Delta)$ if and only if the T-adic Newton polygon of det$(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$ coincides with the polygon with vertices $(0,0)$ and*

$$(\sum_{k=0}^{i} W(k), (p-1)\sum_{k=0}^{i} \frac{k}{D} W(k)), \ i = 0, 1, \cdots .$$

Proof. In the proof of Theorem 5.2, we showed that the T-adic Newton polygon of $C_f(s^a, T)$ coincides with that of $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$. Note that

$$\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \mathrm{Norm}(\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])),$$

where the norm map is the norm from $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ to $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$. The theorem is equivalent to the statement that the T-adic Newton polygon of $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ coincides with the polygon with vertices $(0,0)$ and

$$(\sum_{k=0}^{i} aW(k), a(p-1)\sum_{k=0}^{i} \frac{k}{D} W(k)), \ i = 0, 1, \cdots$$

if and only if the T-adic Newton polygon of $\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ does. Therefore it suffices to show that the determinant of the matrix

$$(\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)})_{0 \leq i,j < a, \deg(w), \deg(u) \leq \frac{k}{D}}$$

is not divisible by T in $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$ if and only if the determinant of the matrix

$$(\alpha_{pw-u}(f)\pi^{c(pw-u,u)})_{\deg(w),\deg(u)\leq\frac{k}{D}}$$

is not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. The theorem now follows from the fact that the former determinant is the norm of the latter from $\mathbb{Q}_q[[\pi^{\frac{1}{D}}]]$ to $\mathbb{Q}_p[[\pi^{\frac{1}{D}}]]$ up to a sign. □

We now define the open facial decomposition $F(\Delta)$. It is the decomposition of $C(\Delta)$ into a disjoint union of relatively open cones generated by the relatively open faces of Δ whose closure does not contain the origin. Note that every relatively open cone generated by co-facial vectors in $C(\Delta)$ is contained in a unique element of $F(\Delta)$.

Lemma 6.2. *Let $\sigma \in F(\Delta)$, and $u \in \sigma$. Then $\alpha_u(f_{\bar{\sigma}}) \equiv \alpha_u(f)(\mod \pi^{1/D})$, where $f_{\bar{\sigma}}$ is the sum of monomials of f whose exponent vectors lie in the closure $\bar{\sigma}$ of σ.*

Proof. Let $v_1, \cdots, v_j$ be exponent vectors of monomials of f such that $a_1v_1 + \cdots + a_jv_j = u$ with $a_1 > 0, \cdots, a_j > 0$. It suffices to show that either $v_1, \cdots v_j$ lie in the closure of σ, or their contribution to $\alpha_u(f)$ is $\equiv 0(\mod \pi^{1/D})$. Suppose that their contribution to $\alpha_u(f)$ is $\not\equiv 0(\mod \pi^{1/D})$. Then $v_1, \cdots, v_j$ must be co-facial. So the interior of the cone generated by those vectors is contained in a unique element of $F(\Delta)$. As that interior has a common point u with σ, it must be σ. It follows that $v_1, \cdots v_j$ lie in the closure of σ. The lemma is proved. □

Lemma 6.3. *Let $\sigma, \tau \in F(\Delta)$ be distinct. Let $w \in \sigma$, and $u \in \tau$. Suppose that the dimension of σ is no greater than that of τ. Then $pw - u$ and u are not co-facial, i.e., $c(pw - u, u) > 0$.*

Proof. Suppose that $pw - u$ and u are co-facial. Then the interior of the cone generated by $pw - u$ and u is contained in a unique element of $F(\Delta)$. As that interior has a common point w with σ, it must be σ. It follows that u lies in the closure of σ. As σ and τ are distinct, u lies in the boundary of σ. This implies that the dimension of τ is less than that of σ, which is a contradiction. Therefore $pw - u$ and u are not co-facial. The lemma is proved. □

For $\sigma \in F(\Delta)$, we define

$$M(\sigma) = M(\Delta) \cap \sigma = \mathbb{Z}^n \cap \sigma$$

be the set of lattice points in the cone σ.

Theorem 6.4 (Open facial decomposition)**.** *The T-adic Newton polygon of $C_f(s, T)$ coincides with $\mathrm{HP}_q(\Delta)$ if and only if for every $\sigma \in F(\Delta)$, the determinants of the matrices*

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\deg(w),\deg(u)\leq\frac{k}{D}},\ k = 0, 1, \cdots$$

are not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$, where $\bar{\sigma}$ is the closure of σ.

Proof. By Theorem 6.1, the T-adic Newton polygon of $C_f(s,T)$ coincides with the q-Hodge polygon of f if and only if the determinants of the matrices

$$A^{(k)} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w,u\in M(\Delta),\deg(w),\deg(u)\leq \frac{k}{D}},\ k=0,1,\cdots$$

are not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. Write

$$A^{(k)}_{\sigma,\tau} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w\in M(\sigma),u\in M(\tau),\deg(w),\deg(u)\leq \frac{k}{D}}.$$

The facial decomposition shows that $A^{(k)}$ has the block form $(A^{(k)}_{\sigma,\tau})_{\sigma,\tau\in F(\Delta)}$. The last lemma shows that the block form modulo $\pi^{\frac{1}{D}}$ is triangular if we order the cones in $F(\Delta)$ in dimension-increasing order. It follows that $\det A^{(k)}$ is not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ if and only if for all $\sigma\in F(\Delta)$, $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. By Lemma 6.2, modulo $\pi^{\frac{1}{D}}$, $A^{(k)}_{\sigma,\sigma}$ is congruent to the matrix

$$\{\alpha_{pw-u}(f_{\bar\sigma})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\deg(w),\deg(u)\leq \frac{k}{D}}.$$

So $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ if and only if the determinant of the matrix

$$\{\alpha_{pw-u}(f_{\bar\sigma})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\deg(w),\deg(u)\leq \frac{k}{D}}$$

is not divisible by T in $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$. The theorem is proved. □

The closed facial decomposition Theorem 1.7 follows from the open decomposition theorem and the fact that

$$F(\Delta) = \bigcup_{\sigma\in F(\Delta),\dim\sigma=\dim\Delta} F(\bar\sigma).$$

A similar π_ψ-adic facial decomposition theorem for $C_f(s,\pi_\psi)$ can be proved in a similar way. Alternatively, it follows from the transfer theorem together with the π_ψ-adic facial decomposition in [26] for ψ of order p.

7. Variation of C-functions in a family

Fix an n-dimensional integral convex polytope $\triangle$ in $\mathbb{R}^n$ containing the origin. For each prime p, let $P(\Delta,\mathbb{F}_p)$ denote the parameter space of all Laurent polynomials $f(x)$ over $\bar{\mathbb{F}}_p$ such that $\Delta(f)=\Delta$. This is a connected rational variety defined over $\mathbb{F}_p$. For each $f\in P(\Delta,\mathbb{F}_p)(\mathbb{F}_q)$, the Teichmüller lifting gives a Laurent polynomial $\tilde f$ whose non-zero coefficients are roots of unity in $\mathbb{Z}_q$. The C-function $C_{\tilde f}(s,T)$ is then defined and T-adically entire. For simplicity of notation, we shall just write $C_f(s,T)$ for $C_{\tilde f}(s,T)$, similarly, $L_f(s,T)$ for $L_{\tilde f}(s,T)$. Thus, our C-function and L-function are now defined for Laurent polynomials over finite fields, via the Teichmüller lifting. We would like to study how $C_f(s,T)$ varies when f varies in the algebraic variety $P(\Delta,\mathbb{F}_p)$.

Recall that for a closed face $\sigma \in \Delta$, f_σ denotes the restriction of f to σ. That is, f_σ is the sum of those non-zero monomials in f whose exponents are in σ.

Definition 7.1. *A Laurent polynomial $f \in P(\Delta, \mathbb{F}_p)$ is called non-degenerate if for every closed face σ of Δ of arbitrary dimension which does not contain the origin, the system*

$$\frac{\partial f_\sigma}{\partial x_1} = \cdots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

has no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_p$.

The non-degenerate condition is a geometric condition which insures that the associated Dwork cohomology can be calculated. In particular, it implies that, if ψ is of order p^m, then the L-function $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial in s whose degree is precisely $n!\mathrm{Vol}(\Delta)p^{n(m-1)}$, see [20]. As a consequence, we deduce

Theorem 7.2. *Let $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$. Write*

$$L_f(s, T)^{(-1)^{n-1}} = \sum_{k=0}^{\infty} L_{f,k}(T)s^k, \ L_{f,k}(T) \in \mathbb{Z}_p[[T]].$$

Assume that f is non-degenerate. Then for every positive integer m and all positive integer $k > n!\mathrm{Vol}(\Delta)p^{n(m-1)}$, we have the following congruence in $\mathbb{Z}_p[[T]]$:

$$L_{f,k}(T) \equiv 0 \left(\mod \frac{(1+T)^{p^m} - 1}{T} \right).$$

Proof. Write

$$\frac{(1+T)^{p^m} - 1}{T} = \prod (T - \xi).$$

The non-degenerate assumption implies that

$$L_f(s, \xi)^{(-1)^{n-1}} = \sum_{j=0}^{\infty} L_{f,j}(\xi)s^j,$$

is a polynomials in s of degree $\leq n!\mathrm{Vol}(\Delta)p^{n(m-1)} < k$. It follows that $L_{f,k}(\xi) = 0$ for all ξ. That is, $L_{f,k}(T)$ is divisible by $(T - \xi)$ for ξ. The theorem now follows. $\square$

Definition 7.3. *Let $N(\Delta, \mathbb{F}_p)$ denote the subset of all non-degenerate Laurent polynomials $f \in P(\Delta, \mathbb{F}_p)$.*

The subset $N(\Delta, \mathbb{F}_p)$ is Zariski open in $P(\Delta, \mathbb{F}_p)$. It can be empty for some pair $(\Delta, \mathbb{F}_p)$. But, for a given Δ, $N(\Delta, \mathbb{F}_p)$ is Zariski open dense in $P(\Delta, \mathbb{F}_p)$ for all primes p except for possibly finitely many primes depending on Δ. It is an interesting and independent question to classify the primes p for which $N(\Delta, \mathbb{F}_p)$ is non-empty. This is related to the GKZ discriminant [12]. For simplicity, we shall only consider non-degenerate f in the following.

7.1. **Generic ordinariness.** The first question is how often f is T-adically ordinary when f varies in the non-degenerate locus $N(\Delta, \mathbb{F}_p)$. Let $U_p(\Delta, T)$ be the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is T-adically ordinary, and $U_p(\Delta)$ the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is ordinary. One can prove

Lemma 7.4. *The set $U_p(\Delta)$ is Zariski open in $N(\Delta, \mathbb{F}_p)$.*

One can ask if $U_p(\Delta, T)$ is also Zariski open in $N(\Delta, \mathbb{F}_p)$. We do not know the answer.

Our question is for which p, $U_p(\Delta)$ and $U_p(\Delta, T)$ are Zariski dense in $N(\Delta, \mathbb{F}_p)$. The rigidity bound as well as the Hodge bound imply that

$$U_p(\Delta) \subseteq U_p(\Delta, T).$$

It follows that if $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$, then $U_p(\Delta, T)$ is also Zariski dense in $N(\Delta, \mathbb{F}_p)$.

The Adolphson-Sperber conjecture [1] says that if $p \equiv 1 \pmod D$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. This conjecture was proved to be true in [26] [27] if $n \leq 3$. In particular, this implies

Theorem 7.5. *If $p \equiv 1 \pmod D$ and $n \leq 3$, then $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$.*

For $n \geq 4$, it was shown in [26] [27] that there is an effectively computable positive integer $D^*(\Delta)$ depending only on Δ such that if $p \equiv 1 \pmod{D^*(\Delta)}$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. Thus, we obtain

Theorem 7.6. *For each Δ, there is an effectively computable positive integer $D^*(\Delta)$ such that if $p \equiv 1 \pmod{D^*(\Delta)}$, then $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$.*

The smallest possible $D^*(\Delta)$ is rather subtle to compute in general, and it can be much larger than D. We now state a conjecture giving reasonably precise information on $D^*(\Delta)$.

Definition 7.7. *Let $S(\triangle)$ be the monoid generated by the degree 1 lattice points in $M(\Delta)$, i.e., those lattice points on the codimension 1 faces of Δ not containing the origin. Define the exponent of $\triangle$ by*

$$I(\triangle) = \inf\{d \in \mathbb{Z}_{>0} | dM(\Delta) \subseteq S(\triangle)\}.$$

If $u \in M(\Delta)$, then the degree of Du will be integral but Du may not be a non-negative integral combination of degree 1 elements in $M(\Delta)$ and thus $DM(\Delta)$ may not be a subset of $S(\Delta)$. It is not hard to show that $I(\Delta) \geq D$. In general they are different but they are equal if $n \leq 3$. This explains why the Adolphson-Sperber conjecture is true if $n \leq 3$ and it can be false if $n \geq 4$. The following conjecture is a modified form, and it is a consequence of Conjecture 9.1 in [26].

Conjecture 7.8. *If $p \equiv 1 \bmod I(\triangle)$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. In particular, $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ for such p.*

By the facial decomposition theorem, in proving the above conjecture, it is sufficient to assume that Δ has only one codimension 1 face not containing the origin.

7.2. **Generic Newton polygon.** In the case that $U_p(\Delta, T)$ is empty, we expect the existence of a generic T-adic Newton polygon. For this purpose, we need to re-scale the uniformizer. For $f \in N(\Delta, \mathbb{F}_p)(\mathbb{F}_{p^a})$, the $T^{a(p-1)}$-adic Newton polygon of $C_f(s, T; \mathbb{F}_{p^a})$ is independent of the choice of a for which f is defined over $\mathbb{F}_{p^a}$. We call them the absolute T-adic Newton polygon of f.

Conjecture 7.9. *There is a Zariski open dense subset $G_p(\Delta, T)$ of $N(\Delta, \mathbb{F}_p)$ such that the absolute T-adic Newton polygon of f is constant for all $f \in G_p(\Delta, T)$. Denote this common polygon by $\mathrm{GNP}_T(\Delta, p)$, and call it the generic Newton polygon of $(\triangle, T)$.*

More generally, one expects that much of classical theory for finite rank F-crystals extends to a certain nuclear infinite rank setting. This includes the classical Dieudonne-Manin isogeny theorem, the Grothendieck specialization theorem, the Katz isogeny theorem [18]. All these are essentially understood in the ordinary infinite rank case, but open in the non-ordinary infinite rank case.

Similarly, for each non-trivial ψ, there is a Zariski open dense subset $G_p(\Delta, \psi)$ of $N(\Delta, \mathbb{F}_p)$ such that the $\pi_\psi^{a(p-1)}$-adic Newton polygon of the C-value $C_f(s, \pi_\psi; \mathbb{F}_{p^a})$ is constant for all $f \in G_p(\Delta, \psi)$. Denote this common polygon by $\mathrm{GNP}_p(\Delta, \psi)$, and call it the generic Newton polygon of $(\triangle, \psi)$. The existence of $G_p(\Delta, \psi)$ can be proved, since the non-degenerate assumption implies that the C-function $C_f(s, \pi_\psi)$ is determined by a single finite rank F-crystal via a Dwork type cohomological formula for $L_{f,\psi}(s)$. In the T-adic case, we are not aware of any such finite rank reduction.

Clearly, we have the relation

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p).$$

Conjecture 7.10. *If p is sufficiently large, then*

$$\mathrm{GNP}_p(\Delta, \psi) = \mathrm{GNP}_T(\Delta, p).$$

This conjecture is proved in the case $n = 1$ in [22].

Let $\mathrm{HP}(\Delta)$ denote the absolute Hodge polygon with vertices $(0, 0)$ and

$$\left(\sum_{k=0}^{i} W(k), \sum_{k=0}^{i} \frac{k}{D} W(k)\right), \ i = 0, 1, \cdots.$$

Note that $\mathrm{HP}(\Delta)$ depends only on Δ, not on q any more. It is re-scaled from the q-Hodge polygon $\mathrm{HP}_q(\Delta)$. Clearly, we have the relation

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p) \geq \mathrm{HP}(\Delta).$$

Conjecture 7.8 says that if $p \equiv 1 \pmod{I(\triangle)}$, then both $\mathrm{GNP}_p(\Delta, \psi)$ and $\mathrm{GNP}_T(\Delta, p)$ are equal to $\mathrm{HP}(\Delta)$. In general, the generic Newton polygon lies above $\mathrm{HP}(\Delta)$ but for many Δ it should be getting closer and closer to $\mathrm{HP}(\Delta)$ as p goes to infinity. We now make this more precise. Let $E(\Delta)$ be the monoid generated by the lattice points in Δ. This is a subset of $M(\Delta)$. Generalizing the limiting Conjecture 1.11 in [27] for ψ of order p, we have

Conjecture 7.11. *If the difference $M(\Delta) - E(\Delta)$ is a finite set, then for each non-trivial ψ, we have*

$$\lim_{p\to\infty} \mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta).$$

In particular,

$$\lim_{p\to\infty} \mathrm{GNP}_T(\Delta, p) = \mathrm{HP}(\Delta).$$

This conjecture is equivalent to the existence of the limit. This is because for all primes $p \equiv 1 \pmod{D^*(\Delta)}$, we already have the equality $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$ by Theorem 7.6. A stronger version of this conjecture (namely, Conjecture 1.12 in [27]) has been proved by Zhu [32] [33] [34] in the case $m = 1$ and $n = 1$, see also Blache and Férard [5] [6] and Liu [21] for related further work in the case $m = 1$ and $n = 1$, Hong [15] [16] and Yang [31] for more specialized one variable results. For $n \geq 2$, the conjecture is clearly true for any Δ for which both $D \leq 2$ and the Adolphson-Sperber conjecture holds, because then $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$ for every $p > 2$. There are many such higher dimensional examples [27]. Using free products of polytopes and the above known examples, one can construct further examples [7].

7.3. T-adic Dwork Conjecture.

In this final subsection, we describe the T-adic version of Dwork's conjecture [10] on pure slope zeta functions.

Let Λ be a quasi-projective subvariety of $N(\Delta, \mathbb{F}_p)$ defined over $\mathbb{F}_p$. Let f_λ be a family of Laurent polynomials parameterized by $\lambda \in \Lambda$. For each closed point $\lambda \in \Lambda$, the Laurent polynomial f_λ is defined over the finite field $\mathbb{F}_{p^{\deg(\lambda)}}$. The T-adic entire function $C_{f_\lambda}(s, T)$ has the pure slope factorization

$$C_{f_\lambda}(s, T) = \prod_{\alpha \in \mathbb{Q}_{\geq 0}} P_\alpha(f_\lambda, s),$$

where each $P_\alpha(f_\lambda, s) \in 1 + s\mathbb{Z}_p[[T]][s]$ is a polynomial in s whose reciprocal roots all have $T^{\deg(\lambda)(p-1)}$-slope equal to α.

Definition 7.12. *For $\alpha \in \mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function of the family f_Λ is defined to be the infinite Euler product*

$$L_\alpha(f_\Lambda, s) = \prod_{\lambda \in |\Lambda|} \frac{1}{P_\alpha(f_\lambda, s^{\deg(\lambda)})} \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where $|\Lambda|$ denotes the set of closed points of Λ over $\mathbb{F}_p$.

The T-adic version of Dwork's conjecture is then the following

Conjecture 7.13. *For $\alpha \in \mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function $L_\alpha(f_\Lambda, s)$ is T-adic meromorphic in s.*

In the ordinary case, this conjecture can be proved using the methods in [28] [29] [30]. It would be interesting to prove this conjecture in the general case. The π_ψ-adic version of this conjecture is essentially Dwork's original conjecture, which can be proved as it reduces to finite rank F-crystals. The difficulty of the T-adic version is that we have to work with infinite rank objects, where much less is known in the non-ordinary case.

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