Math 118
Fall 2015
Midterm

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- Each lettered part ((a), (b), etc) is worth 5 points.
- Present your work as clearly as possible. Partial credit will be awarded.
- If you find yourself stuck somewhere, move on and come back to the problem later.
- You are not permitted the use of a calculator, phone, or other electronic aid.
- Academic dishonesty in any form will result in a score of zero.
1. Let $A = \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}$.

(a) Calculate the norm $|A|$ of $A$.

$|A| = 11$

(b) Find a fundamental matrix $\Phi(t)$ for $\dot{\vec{\phi}}(t) = A\vec{\phi}(t)$.

One answer:

$$\Phi(t) = \begin{pmatrix} e^{3t} & 5te^{3t} \\ 0 & e^{3t} \end{pmatrix}$$
(c) Find the solution $\vec{\phi}(t)$ to $\frac{d}{dt} \vec{\phi}(t) = A \vec{\phi}(t)$ satisfying $\vec{\phi}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 2e^{3t} \\ 0 \end{pmatrix}$$

(d) Compute the inverse of the fundamental matrix you found in (b).

$$\Phi^{-1}(t) = \begin{pmatrix} e^{-3t} & -5te^{-3t} \\ 0 & e^{-3t} \end{pmatrix}$$
(e) Find the solution \( \vec{\psi}(t) \) to \( \dot{\vec{\psi}}(t) = A\vec{\psi}(t) + \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix} \) satisfying \( \vec{\psi}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

\[
\begin{pmatrix} \frac{5}{2} t^2 e^{3t} \\ te^{3t} \end{pmatrix}
\]

(f) Find the solution \( \vec{\chi}(t) \) to \( \dot{\vec{\chi}}(t) = A\vec{\chi}(t) + \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix} \) satisfying \( \vec{\chi}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \).

\[
\begin{pmatrix} 2e^{3t} + \frac{5}{2} t^2 e^{3t} \\ te^{3t} \end{pmatrix}
\]
2. Consider the second-order scalar equation \( \ddot{u}(t) + \frac{\dot{u}(t)}{t+3} + |t - 1| u(t) = 0 \).
(a) Rewrite the equation as a first-order system.

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -|t - 1| & -\frac{1}{t+3} \end{pmatrix} \]

with \( x_1(t) = u(t) \) and \( x_2(t) = \dot{u}(t) \)

(b) Write down a solution matrix for the system in (a), for \( t \) in the interval \((10, \infty)\). (Hint: I don’t care which solution matrix. This part is unrelated to part (c).)

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

(c) Does the initial-value problem for the scalar equation with initial data \( u(0) = \dot{u}(0) = 1 \) have a solution? If so, is it unique and what is the maximal time interval on which it is defined? Briefly justify your answers. (This part is completely independent of part (b).)

We can see from the answer to (a) that the coefficient matrix \( A(t) \) for the equivalent linear system is continuous on \( \mathbb{R} \setminus \{-3\} \). By Picard’s theorem we know a unique solution exists. We also know that solutions to homogeneous linear systems exist for as long as the coefficient matrix is defined and continuous, so in fact the solution exists on \((-3, \infty)\).
3. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous nonnegative functions on a closed interval $[a, b]$. Suppose also that $f$ is differentiable on $[a, b]$ and $f'(t)$ is continuous on $[a, b]$ and that $f'(t) \leq f(t)g(t)$ for all $t \in [a, b]$.

(a) Integrate both sides of the above inequality from $a$ to $t$ and apply the fundamental theorem of calculus to derive an integral inequality for $f(t)$.

$$f(t) = f(a) + \int_a^t f'(s) \, ds \leq f(a) + \int_a^t f(s)g(s) \, ds$$

(b) Apply Grönwall’s inequality to your result from (a) to obtain an upper bound for $f(t)$ in terms of $f(a)$ and $t$.

$$f(t) \leq f(a) e^{\int_a^t g(s) \, ds}$$
4. True or false. Justify your answer with a very short proof, possibly by giving an example or counterexample where appropriate.
   (a) For any subset $A$ of $\mathbb{R}^2$, the closure of $A$ is a subset of $A$. (Include the definition of closure in your answer.)

   False. The closure of a subset $A \subset \mathbb{R}^2$ is the set of points in $\mathbb{R}^2$ which are the limits of sequences all of whose terms lie in $A$. As an example showing the claim is false take $A = (0, \infty) \times \{0\}$. Then the point $(0,0)$ does not belong to $A$ but it does belong to the closure, since it is the limit of the sequence $\{(1/k, 0)\}_{k=1}^{\infty} \subset A$.

   (b) Let $\Phi(t)$ be a fundamental matrix for a homogeneous linear first-order system in $\mathbb{R}^n$. Then for every invertible $n \times n$ matrix $C$ with constant real entries, the product $\Phi C$ is another fundamental matrix for the same system.

   True. This is a theorem in Chapter 2 with the word \textit{nonsingular} in place of \textit{invertible}.

   (c) Any two functions $f, g : \mathbb{R} \to \mathbb{R}$ are linearly dependent in the real vector space of functions from $\mathbb{R}$ to $\mathbb{R}$.

   False. Take $f(t) = t$ and $g(x) = 1$, and suppose there are $a, b \in \mathbb{R}$ such that $af(t) + bg(t) = 0$ for all $t \in \mathbb{R}$. Then $at + b = 0$ for all $t \in \mathbb{R}$, but then $b = -at$, so $b$ cannot be constant unless $a = b = 0$, proving linear independence of $\{f, g\}$ in the real vector space of real-valued functions on the real line. Thus we have provided a counterexample to the claim.