This is a condensed write-up of some of the material covered on 11/13 and 11/16, excluding examples and pictures.

We have been studying ODE systems of the form

\[ \dot{x}(t) = \vec{f}(x(t), t) \]

with the usual assumptions that the vector field \( \vec{f} : D \to \mathbb{R}^n \) is continuous on an open set \( D \subseteq \mathbb{R}^{n+1} \) and that the partial derivative \( \frac{\partial f_i}{\partial x_j} \), for each \( 1 \leq i, j \leq n \), exists and is continuous on \( D \).

Given an open interval \( I \subseteq \mathbb{R} \) and a function \( \vec{\phi} : I \to \mathbb{R}^n \) such that \( (t, \vec{\phi}(t)) \in D \) for all \( t \in I \) (but not necessarily a solution of (1)), we define the linearization of the system (1) at \( \vec{\phi} \) to be the linear system

\[ \dot{\vec{\psi}}(t) = df|_{(\vec{\phi}(t), t)} \vec{\psi}(t), \]

where

\[ df|_{(\vec{\phi}(t), t)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{\phi}(t), t) & \frac{\partial f_1}{\partial x_2}(\vec{\phi}(t), t) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{\phi}(t), t) \\ \frac{\partial f_2}{\partial x_1}(\vec{\phi}(t), t) & \frac{\partial f_2}{\partial x_2}(\vec{\phi}(t), t) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{\phi}(t), t) \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{\phi}(t), t) & \frac{\partial f_n}{\partial x_2}(\vec{\phi}(t), t) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{\phi}(t), t) \end{pmatrix}. \]

Observe that if \( \vec{\phi} \) solves (1), then \( \vec{\phi} + \vec{\psi} \) also solves (1) iff \( \vec{\psi} \) solves

\[ \dot{\vec{\psi}}(t) = df|_{(\vec{\phi}(t), t)} \vec{\psi}(t) + Q_{\vec{\phi}(t)}[\vec{\psi}(t)], \]

where

\[ Q_{\vec{\phi}(t)}[\vec{\psi}(t)] = \vec{f}(\vec{\phi}(t) + \vec{\psi}(t), t) - \vec{f}(\vec{\phi}(t), t) - df|_{(\vec{\phi}(t), t)} \vec{\psi}(t). \]
Note that (4) differs from (3) by the “error” term $Q_{\vec{\phi}(t)}[\vec{\psi}(t)]$, so the linearized equation (3) can be thought of as an approximation to the exact equation (4) when $\vec{\psi}(t)$ is small so that the error is even smaller.

**Theorem 6.** Make the above (usual) assumptions on $D$ and $\vec{f}$. Suppose additionally that the solution to (1) with initial data $\vec{x}(t_0) = \vec{x}_0$ exists on an open interval $I$, and let $[a, b] \subset I$ be a closed subinterval including $t_0$. Then there exists $\delta > 0$ such that $\forall \vec{v} \in \mathbb{R}^n$ with $|\vec{v}| < \delta$, the solution to (1) with initial data $\vec{x}(t_0) = \vec{x}_0 + \vec{v}$ exists on $(a, b)$ and is differentiable in $\vec{v}$.

**Proof.** We’ll give just a sketch, assuming $n = 1$; the general case does not require any different ideas. By assumption we have a solution $\phi : I \to \mathbb{R}$ solving (1) with initial data $\phi(t_0) = x_0$. We want to show that for any small $v$ (how small will be specified by $\delta$, which we will need to pick) we can find a solution $\tilde{\phi} : (a, b) \to \mathbb{R}$ satisfying $\tilde{\phi}(t_0) = x_0 + v$ and that this solution depends differentiably on $v$.

The existence portion of Picard’s theorem guarantees that a solution with this initial data exists but cannot tell us that the solution exists on all $(a, b)$. Instead we will prove the existence of $\tilde{\phi}$ *iteratively* as follows. For $v$ small the solution we want should be a small perturbation of $\phi$. Our first approximation to $\tilde{\phi}(t)$ will be $\phi(t) + v$. This function will satisfy the initial data but probably will not solve the equation.

We will next look for a function $\psi(t)$ to add to our first try to correct it to the exact solution: $\tilde{\phi}(t) = \phi(t) + v + \psi(t)$. We’ll impose initial data $\psi(t_0) = 0$ so that $\tilde{\phi}(t_0) = x_0 + v$. We also need $\tilde{\phi}$ to solve (1); equivalently, replacing $\psi$ in (4) above by $v + \psi$ here, we need

\[
\dot{\psi}(t) = \frac{\partial f}{\partial x}(\phi(t), t) \psi(t) + \frac{\partial f}{\partial x}(\phi(t), t) v + Q_{\phi(t)}[v + \psi(t)].
\]

Now for each integer $k \geq 0$ we define $\psi_k : (a, b) \to \mathbb{R}$ inductively by taking $\psi_0$ to be identically 0 and $\psi_{k+1}$ to be the solution on $(a, b)$ to

\[
\dot{\psi}_{k+1}(t) = \frac{\partial f}{\partial x}(\phi(t), t) \psi_{k+1}(t) + \frac{\partial f}{\partial x}(\phi(t), t) v + Q_{\phi(t)}[v + \psi_k(t)].
\]
To see that this equation really does have a solution defined on all \((a, b)\), you need to recognize that the equation is **linear** in the unknown function \(\psi_{k+1}\) and identify the \((1 \times 1)\) matrix of coefficients, which we usually denote by \(A(t)\), along with the inhomogeneity, which we usually denote by \(g(t)\).

The matrix \(A(t)\) is defined and continuous on \((a, b)\) (why?). The inhomogeneity depends on \(v\) and \(\psi_k\). By taking \(\delta\) small enough we can ensure that \(g(t)\) is also defined and continuous on \((a, b)\). The starting point for the proof of this last claim is an argument, using the facts that \([a, b]\) is closed and \(D\) is open, that there’s a sufficiently small \(\delta > 0\) so that \(\phi(t) + v \in D\) for all \(t \in [a, b]\) and for all \(v \in \mathbb{R}\) with \(|v| < \delta\); then one proceeds by induction on \(k\) using our results on linear equations and making an estimate for the error \(Q\).

Using the same tools one can also show that for each \(k\) the partial derivative \(\partial \psi_k / \partial v\) exists, that for each \(t \in (a, b)\) the sequence \(\{\psi_k(t)\}_{k=0}^{\infty}\) converges, and that the limit function

\[
(9) \quad \psi(t) := \lim_{k \to \infty} \psi_k(t)
\]

solves (7) and is also differentiable in \(v\), proving the theorem. Some details will be filled in by future homework exercises. \(\square\)

As a corollary we get continuous dependence of the solution on the initial position for fixed initial time. Keep in mind that we need \(\vec{f}(\vec{x}, t)\) to be differentiable in all spatial coordinates \(x_1, \ldots, x_n\) for the linearization to exist even. This is obvious from its definition. Somewhat less obvious, until you examine the details of the proof, is that we need all these partial derivatives to be continuous in order for the proof to go through. Because our book always makes this assumption anyway, I decided to take full advantage and give you the stronger result (which is crucial in many applications). It is possible though to get continuity with weaker assumptions (as is necessary for other applications).

We also have differentiability in the initial time.

**Theorem 10.** Make the above (usual) assumptions on \(D\) and \(\vec{f}\). Suppose additionally that the solution to (1) with initial data \(\vec{x}(t_0) = \vec{x}_0\) exists on an open interval \(I\), and let \([a, b] \subset I\)
be a closed subinterval including $t_0$. Then there exists $\delta > 0$ such that $\forall \tau \in \mathbb{R}$ with $|\tau| < \delta$, the solution to (1) with initial data $\bar{x}(t_0 + \tau) = \bar{x}_0$ exists on $(a, b)$ and is differentiable in $\tau$.

This can be proven in a similar fashion to the first theorem. One can start with $\phi(t + t_0 - \tau)$ as a first approximation, which will satisfy the new initial condition but not (necessarily) the ODE. One then finds a function $\psi$ to add as before.