Quiz 4

1. Let

\[ A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}. \]

(a) Find a fundamental matrix for \( \dot{x}(t) = Ax(t) \).

By computing and factoring the characteristic polynomial for \( A \) we find it has eigenvalues \( \lambda_1 = -3 \) and \( \lambda_2 = 2 \), each with multiplicity 1. Because the multiplicities are 1, for each eigenvalue the generalized eigenspace coincides with the usual eigenspace. By computing the null space of \( A - \lambda_1 E \) we find that \( A \) has \( \lambda_1 \) eigenspace spanned by

\[ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \]

and similarly we find that \( A \) has \( \lambda_2 \) eigenspace spanned by

\[ \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \]

Next we compute

\[ e^{tA} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{-3t}e^{t(A+3E)} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{-3t}(E + 0 + \cdots) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix} \]

and similarly

\[ e^{tA} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix}. \]

Since the two eigenvectors are linearly independent and span \( \mathbb{R}^2 \) and since the exponentials just calculated solve the given homogeneous ODE, we have found that

\[ \begin{pmatrix} e^{-3t} & 2e^{2t} \\ -2e^{-3t} & e^{2t} \end{pmatrix} \]

is a fundamental matrix for the ODE.
(b) Solve
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{pmatrix} =
\begin{pmatrix}
1 & 2 \\
2 & -2 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix} +
\begin{pmatrix}
e^{-3t} \\
-2e^{-3t} \\
\end{pmatrix}
\]
with initial data \( \vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).

We will use the method of variation of constants to construct a particular solution \( \vec{\psi}(t) \) to the inhomogeneous solution with initial data \( \vec{\psi}(0) = \vec{0} \) and then add the appropriate solution to the homogeneous problem in order to satisfy the given initial data.

To apply the method of variation of constants we will use the fundamental matrix from part (a). We will need the inverse matrix \( \Phi(t)^{-1} \). This can be obtained quickly by using the fact that the inverse of an invertible matrix is its adjugate matrix divided by its determinant. We find \( \det \Phi(t) = 5e^{-t} \) and
\[
\Phi(t)^{-1} = \frac{1}{5} \begin{pmatrix}
e^{3t} & -2e^{3t} \\
2e^{-2t} & e^{-2t} \\
\end{pmatrix}.
\]

Next we calculate
\[
\Phi(s)^{-1} = \begin{pmatrix}
e^{-3s} \\
-2e^{-3s} \\
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
so
\[
\int_0^t \Phi(s)^{-1} \begin{pmatrix} e^{-3s} \\
-2e^{-3s} \\
\end{pmatrix} ds = \begin{pmatrix} t \\ 0 \end{pmatrix},
\]
and thus we arrive at
\[
\vec{\psi}(t) = \Phi(t) \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} te^{-3t} \\
-2te^{-3t} \\
\end{pmatrix}.
\]

Last we need to add the solution \( \vec{\phi}(t) \) to the homogeneous problem with initial value \( \vec{\phi}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \). This \( \vec{\phi}(t) \) will be some linear combination of the columns of \( \Phi(t) \) and
conveniently we see it is simply the second column itself. Therefore we find

\[ \vec{x}(t) = \begin{pmatrix} te^{-3t} + 2e^{2t} \\ -2te^{-3t} + e^{2t} \end{pmatrix} \]

solves the given initial value problem.