Extra Questions for Exam Review - Answers

1. \[ \begin{array}{|c|c|c|c|c|c|} \hline P & Q & R & P \land \lnot Q & R \lor \lnot P & (P \land \lnot Q) \implies (R \lor \lnot P) \\ \hline T & T & T & F & T & T \\ T & T & F & F & F & T \\ T & F & T & T & T & T \\ T & F & F & T & F & F \\ F & T & T & F & T & T \\ F & T & F & F & T & T \\ F & F & T & F & T & T \\ F & F & F & F & T & T \\ \hline \end{array} \]

2. No. \( A = \{2a + 1 : a \in \mathbb{Z}\} \) is the set of all odd integers. \( B = \{a : 2a + 1 \in \mathbb{Z}\} = \{\frac{1}{2}(m - 1) : m \in \mathbb{Z}\} \) is the set of integers and half-integers.

3. (a) \( \mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset, \{4\}, \{10\}, \{4,10\}\} \).
   
   (b) Yes.
   
   (c) It is true. Here is a proof.
   \[
   X \in \mathcal{P}(A \cap B) \iff X \subseteq A \cap B \iff X \subseteq A \text{ and } X \subseteq B \\
   \iff X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\
   \iff X \in \mathcal{P}(A) \cap \mathcal{P}(B).
   \]

4. \[
\left((A^c \cup B) \cap (A^c \cup C^c)\right)^c = (A^c \cup B)^c \cup (A^c \cup C^c)^c \\
= ((A^c)^c \cap B^c) \cup ((A^c)^c \cap (C^c)^c) \\
= (A \cap B^c) \cup (A \cap C) \\
= A \cap (B^c \cup C) \\
= A \cap (B \cup C)^c \\
\]

   (de Morgan)
   
   (de Morgan ×2)
   
   (distributive law)
   
   (de Morgan)

5. The statement is true.
   \[
   x \in (A \cup B) \setminus (A \cap B) \iff x \in A \cup B \text{ and } x \in (A \cap B)^c = A^c \cup B^c \\
   \iff (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B).
   \]

   We therefore need one of the two statements in each set of parantheses to be true. But we can’t have \( x \in A \) and \( x \notin A \), so we conclude
   \[
   x \in (A \cup B) \setminus (A \cap B) \iff (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B) \\
   \iff x \in (A \setminus B) \cup (B \setminus A).
   \]

6. (a) For all integers \( x \), there exist integers \( y \) and \( r \) such that \( x = y + r \) and \( r \) is between 1 and 4.
   
   (b) Given \( x \in \mathbb{Z} \), let \( r = 1 \) and \( y = x - 1 \). Then \( r, y \) are integers satisfying the claim.
   
   (c) \( \exists x \in \mathbb{Z}, \text{ such that } \forall y, r \in \mathbb{Z} \text{ we have } (x \neq y + r) \lor (r \leq 0) \lor (r \geq 5). \)
7. (a) Let \( x, y \in W \). Then \( \exists m, n \in \mathbb{Z} \) such that \( x = m^2, y = n^2 \). But then
\[
xy = m^2n^2 = (mn)^2 \in W.
\]

(b) \( W \) is closed under addition if \( \forall x, y \in W \) we have \( x + y \in W \). This is false. For example \( 1 = 1^2 \) and \( 4 = 2^2 \) are both in \( W \), yet \( 1 + 4 = 5 \not\in W \) as it is not a perfect square.

8. (a) The positive divisors of 6 are 1, 2, 3 and 6: we have \( 6 = 1 + 2 + 3 \).

The positive divisors of 28 are 1, 2, 4, 7, 14 and 28: we have \( 28 = 1 + 2 + 4 + 7 + 14 \).

(b) Let \( 2^n - 1 = p \) be prime. Clearly \( p = 3 \), from \( n = 2 \), is the smallest such prime. It follows that \( p \) is odd. The positive divisors of \( 2^{n-1} (2^n - 1) = 2^{n-1} p \) are then
\[
1, 2, 2^2, 2^3, \ldots, 2^{n-2}, 2^{n-1}, p, 2p, 2^2 p, \ldots, 2^{n-2} p, 2^{n-1} p.
\]
The sum of all of these except the last is then
\[
\sum_{k=0}^{n-1} 2^k + \sum_{k=0}^{n-2} 2^k p = \frac{1 - 2^n}{1 - 2} + \frac{(1 - 2^{n-1}) p}{1 - 2} = 2^n - 1 + \frac{1 - 2^{n-1}}{1 - 2} p
\]
\[
= 2^n - 1 + (2^{n-1} - 1)(2^n - 1) = 2^n - 1,
\]
as required.

9. (a) Base case \( (n = 4) \): \( 4! = 24 > 16 = 2^4 \) is true.

Induction step: Assume that, for some \( n \in \mathbb{N}_{\geq 4} \), we have \( n! > 2^n \). Then
\[
(n + 1)! = (n + 1)n! > (n + 1) \cdot 2^n > 5 \cdot 2^n > 2 \cdot 2^n = 2^{n+1}.
\]
By induction, we have \( n! > 2^n, \forall n \in \mathbb{N}_{\geq 4} \).

(b) \( \forall n \in \mathbb{N} \),
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.
\]
Base case \( (n = 1) \): \( \frac{1}{1} = \frac{1}{3+1} \) is true.

Induction step: Assume that, for some \( n \in \mathbb{N} \), we have \( \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \).

Then
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} = \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}
\]
\[
= \frac{n(2n+3) + 1}{(2n+1)(2n+3)} = \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}.
\]
By induction, we have the result \( \forall n \in \mathbb{N} \).

10. (a) If \( \frac{p}{q} \) satisfies the equation, then
\[
\frac{p^2}{q^2} + \frac{ap}{q} + b = 0 \implies p^2 + apq + bq^2 = 0.
\]
The second and third terms are divisible by \( q \), and so \( p^2 \) is divisible by \( q \). But \( \text{gcd}(p, q) = 1 \), so \( q \) must be 1.
(b) Suppose that $x^2 - 2 = 0$ has a rational solution. By part (a), this solution must be an integer. But $x^2 = 2$ is clearly false for $x = 0, \pm 1$ ($x^2$ too small), and $|x| \geq 2$ ($x^2$ too large), so there are no integer solutions. It follows that the only solutions to $x^2 - 2 = 0$ (namely $\pm \sqrt{2}$) are irrational.

(c) If $x^2 - n = 0$ has a rational solution, then it is an integer. But then $n = x^2$ is the square of an integer. Thus $\sqrt{n}$ is irrational unless $n$ is a perfect square.

(d) Suppose that $x^2 - x - 1 = 0$ has a rational root $n$. Then $n$ is an integer satisfying
\[ 1 = n^2 - n = n(n - 1). \]

We certainly cannot have $n = 0$, but then we must have $\frac{1}{n} = n - 1$. Thus $\frac{1}{n}$ is an integer, which can only happen if $n \pm 1$. In both cases $\frac{1}{n} \neq n - 1$. Thus all roots of $x^2 - x - 1 = 0$ are irrational.

11. (a) Just compute: for example $(f_1 \circ f_2)(x) = 1 - \frac{1}{1-x} = 1 - (1-x) = x = f_0(x)$.\
\[ \begin{array}{c|ccc} \circ & f_0 & f_1 & f_2 \\
\hline
f_0 & f_0 & f_1 & f_2 \\
f_1 & f_0 & f_1 & f_2 \\
f_2 & f_0 & f_1 & f_2 \\
\end{array} \]

(b) For $\mathbb{Z}_3 = \{0,1,2\}$ with $+$ mod 3, we have
\[ + \begin{array}{ccc} 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array} \]

(c) It is clear that $\mu : \mathbb{Z}_3 \to \{f_0, f_1, f_2\} : n \mapsto f_n$ is a bijective function. Under this identification, the tables in (a) and (b) are identical: but these tables completely describe the operations $\circ$ and $+$, whence $\mu(a+b) = \mu(a) \circ \mu(b)$ for all $a, b \in \mathbb{Z}_3$. The two ‘sets with operation’ therefore behave identically after the relabeling afforded by $\mu$.

12. (a) Suppose that $f(x) = f(y)$. Then $x$ and $y$ are have the same binary expansion and are thus equal. Hence $f$ is injective.\footnote{We only need the choice of terminating expansion to be sure that $f$ is well-defined. Note that $f$ is not surjective: choosing the terminating representation of $x = \frac{1}{2}$ means that $f(\frac{1}{2}) = \{1\}$. However $\frac{1}{2} = \sum_{n=2}^{\infty} \frac{1}{3^n}$ which means that the set $X = \{2,3,4,5,\ldots\}$ is not in the image of $f$.}

(b) $C$ is precisely the set of numbers in $[0,1]$ possessing a ternary expansion consisting only of 0’s and 2’s. Thus $g(X) \in C$ for all $X \subseteq \mathbb{N}$. The uniqueness of the ternary representation of an element of $C$ means that $g$ is injective. Moreover, $g$ is surjective since
\[ x = g(\{ n \in \mathbb{N} : b_n = 2 \text{ in the ternary expansion of } x \}). \]

(c) Since $C \subseteq [0,1]$, we can combine parts (a) and (b) to conclude that
\[ |\mathcal{P}(\mathbb{N})| = |C| \leq |[0,1]| = \epsilon \leq |\mathcal{P}(\mathbb{N})| . \]

CSB allows us conclude that these cardinalities are equal: $|\mathcal{P}(\mathbb{N})| = |C| = \epsilon$.\footnote{We only need the choice of terminating expansion to be sure that $f$ is well-defined. Note that $f$ is not surjective: choosing the terminating representation of $x = \frac{1}{2}$ means that $f(\frac{1}{2}) = \{1\}$. However $\frac{1}{2} = \sum_{n=2}^{\infty} \frac{1}{3^n}$ which means that the set $X = \{2,3,4,5,\ldots\}$ is not in the image of $f$.}