Upper Bounds on the Number of Determining Modes, Nodes, and Volume Elements for the Navier-Stokes Equations

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Dedicated to Professor Ciprian Foias on the Occasion of his 60\(^{th}\) Birthday

Abstract

In this paper we present improved upper bounds on the number of determining Fourier modes, determining nodes and volume elements for the Navier-Stokes equations. The bounds presented here seem to agree with some of the heuristic physical estimates conjectured by Manley and Treve.

1 Introduction

The conventional theory of turbulence asserts that turbulent flows are monitored by a finite number of degrees of freedom. The notions of determining modes, [5], [6] determining nodes [7], [8], [9], and determining volume elements [8],[10] are rigorous attempts to identify those parameters that control turbulent flows (see Sections 3, 4 and 5 for the definitions of determining modes, nodes and volume elements respectively). Since the problem of global existence and uniqueness of strong solutions to the \(3D\) Navier-Stokes Equations (NSE) is still an open question, most of the research on estimating these parameters has been concentrated on the \(2D\) case. Here we will give an improvement of the previously reported estimates on the number of determining modes, nodes and volume elements [5], [9] and [10] respectively.

In this paper we will consider only the case with periodic boundary conditions. The same techniques, however, work for other boundary conditions as well, but they lead to larger upper bounds. It is not clear yet whether this is merely a mathematical technicality or if

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it is due to the effect of the physical boundaries. We will denote by $G$ the Grashof number (see Section 2), which plays an analogous role as the Reynolds number, and which will be our bifurcation parameter. It is known that if the Grashof number is small enough, then the NSE possess a unique, globally stable, steady state solution \( \text{cf. [19]} \). It is expected that as the Grashof number $G$ increases, this steady state goes through a sequence of bifurcations that lead to chaotic dynamics (see however [17]). Therefore, it is natural to use the Grashof number $G$ as a parameter for measuring the complexity of the dynamics.

There have been many studies to estimate the number of degrees of freedom of the solutions for the NSE in terms of the Grashof number. A sharp upper bound for the fractal, as well as the Hausdorff dimension of the global attractor was established in [2] (see also references therein). This upper bound is of the order of $G^{2/3}(1 + \log G)^{1/3}$. Also, it has been reported in [5] that there exists an upper bound for the number of determining Fourier modes for the $2D$ NSE, with periodic boundary conditions, of the order of $G(1 + \log G)^{1/3}$. In [9], we obtain an upper bound for the number of determining nodes of the order $G^2(1 + \log G)$, and for the number of determining volume elements, [10], of the order $G^2$.

In this paper we slightly improve the estimate on the number of determining modes and obtain an upper bound of the order $G$. Similarly, we improve our previous estimates on the number of determining nodal values and of determining volume elements to $G$. These new estimates are in agreement with the heuristic estimates, which are based on physical arguments, that have been conjectured by Manley and Treve ([5], [16], [20]).

The question as to the sharpness of these estimates remains open. However, in practice experimentalists take their measurements at one point (node), or sometimes at few, but small number of points in the physical domain. Then they rely on the \textit{generic} embedding theorem of Takens [18] to unfold the attractor and to understand the characteristics of the dynamics. It is worth mentioning, however, that Takens’ theorem is only a generic theorem and that there are some interesting physical examples in fluid flows where it degenerates. Such examples occur, for instance, in the presence of certain symmetries [4]. It has been recently shown, however, by [13] that for the $1D$ complex Ginzburg-Landau equation, two nodes are determining if they are sufficiently close. The idea of the proof of [13] is valid for many other $1D$ dissipative parabolic PDEs (see for instance [3]). Also, a straight forward extension of the proof given in [13] has been reported for the $2D$ NSE, and the results imply that there is a small closed determining curve, [15]. This result holds under the assumption that the solutions of the NSE are spatially analytic. However, one can show, in a straight forward manner, that these results are also valid if the solutions enjoy the unique continuation property, which is a weaker assumption on the space of solutions. The results of [13] indicate that the minimal number of determining nodes may not have anything to do with the dimension of the global attractor. On the other hand, if one takes enough nodes, one can parameterize the inertial manifold (when it exists) in terms of the nodal values of the solution, and hence, show that the dynamics of the nodal values is equivalent to that of the PDE [8]. Even though we still do not know whether the $2D$ NSE has an inertial manifold, one can show that there is a parameterization of the global attractor for the $2D$ NSE, with periodic boundary conditions, in terms of the nodal values of the solutions on the attractor [21]. The proof of the last statement uses ideas from [8], [11] and [12]. The latter statement gives an affirmative answer to the assertion conjectured by Foias and Temam [7]: \textit{there exists a finite number of nodes such that the solutions on the global attractor, for the $2D$ NSE, are determined in a unique fashion by their values at these nodes.}
We organize the paper as follows. In Section 2 we recall some of the basic properties of the 2D NSE with periodic boundary conditions. In Section 3 we recall the notion of determining modes. We also show how to eliminate the logarithmic term found in the upper bound for the number of determining modes in [5]. In Section 4 we recall the notion of determining nodes and give an upper bound for their number. Though the idea for the proof is essentially the same as in [9], there are some major differences in our estimates. A similar analysis of the number of determining volume elements is given in Section 5. To make the paper self-contained we include an appendix that contains the proof of certain interpolation estimates that are needed here.

This paper is dedicated to Professor Ciprian Foias on the occasion of his 60th birthday to thank him for his endless support and encouragement to us and to express our admiration for his scientific work in the various fields of the mathematical sciences.

2 Functional Setting and Preliminary Results

We consider the 2D NSE for a viscous incompressible fluid in $\mathbb{R}^2$ with periodic boundary conditions on the square $\Omega = (0, L) \times (0, L)$. The governing equations are

\[
\begin{aligned}
\frac{du}{dt} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f & \text{in } \mathbb{R}^2 \times (0, \infty) \\
\nabla \cdot u &= 0 & \text{in } \mathbb{R}^2 \times (0, \infty) \\
u (x_1, x_2, t) &= u (x_1, x_2 + L, t) \\
u (x_1, x_2, t) &= u (x_1 + L, x_2, t),
\end{aligned}
\]

where $f = f(x, t)$, the volume force, and $\nu > 0$, the kinematic viscosity, are given. We denote by $u = u(x, t)$ the velocity vector, and $p = p(x, t)$ the pressure which are the unknowns. Further, we assume that the integrals of $u$ and $f$ vanish on $\Omega$ for all time (i.e., $u$ and $f$ have zero mean in $\Omega$).

Following the notation in [1], [14], [19], we set

\[
\mathcal{V} = \{ u : \mathbb{R}^2 \to \mathbb{R}^2, \text{ vector-valued trigonometric polynomials} \}
\]

with period $L$, $\nabla \cdot u = 0$, and $\int_\Omega u dx = 0$.

Further, we set

\[
H = \text{the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2,
\]

\[
V = \text{the closure of } \mathcal{V} \text{ in } (H^1(\Omega))^2,
\]

where $H^l(\Omega)$ ($l = 1, 2, \ldots$) denote the usual $L^2$-Sobolev spaces. $H$ is a Hilbert space with the inner product and norm

\[
(u, v) = \int_\Omega u(x) \cdot v(x) dx, \quad |u| = \left( \int_\Omega |u(x)|^2 dx \right)^{1/2},
\]

respectively, and $u(x) \cdot v(x)$ is the usual Euclidean scalar product. Thanks to the Poincaré inequality, $V$ is also a Hilbert space with inner product and norm

\[
((u, v)) = \sum_{i,j=1}^2 \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \quad \|v\|^2 = \sum_{i,j=1}^2 \int_\Omega \left| \frac{\partial v_i}{\partial x_j} \right|^2 dx,
\]

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respectively. Let $P$ denote the orthogonal projection in $L^2(\Omega) \times L^2(\Omega)$ onto $H$. We denote by $A$ the Stokes operator

$$Au = -P\Delta u,$$

(notice that in the periodic case $Au = -\Delta u$) and the bilinear operator

$$B(u, v) = P((u \cdot \nabla)v)$$

for all $u, v$ in $\mathcal{D}(A) = V \cap (H^3(\Omega) \times H^3(\Omega))$. We recall that the operator $A$ is a self-adjoint positive definite operator with compact inverse. Thus there exists a complete orthonormal set $w_j$ of eigenfunctions of $A$ such that $Aw_j = \lambda_j w_j$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, where $\lambda_1 = \frac{(2\pi)^2}{L^2}$, and $\lambda_j = O(j)$ for $j \to \infty$. The NSE, (2.1), are equivalent to the functional differential equation in $H$

$$\frac{du}{dt} + \nu Au + B(u, u) = f,$$

(2.2)

where from now on $f = Pf$, and it is assumed that $f$ satisfies $f \in L^\infty((0, \infty); H)$. That is, $\sup_{t \geq 0} |f(t)| < \infty$, (for details see for example [1], [14], [19]).

Let

$$F = \limsup_{t \to \infty} \left( \int_{\Omega} |f(t, x)|^2 dx \right)^{1/2}.$$ 

Following [5] we define the generalized Grashof number $Gr$ as

$$Gr = \frac{F}{\lambda_1 \nu^2} = \frac{L^2 F}{4\pi^2 \nu^2}.$$ 

The generalized Grashof number will play an analogous role as the Reynolds number and will be our bifurcation parameter. In what follows all our estimates will be in terms of the generalized Grashof number. Notice that if $f$ is time independent, then $Gr$ is the Grashof number $G = \frac{L^2 |f|}{4\pi^2 \nu^2}$. For questions related to existence, uniqueness, and regularity of solutions the reader is referred for instance to [1], [14], [19] and the references therein.

We recall the following inequalities which are satisfied by $B(u, v)$ (cf. [1], [14], [19]).

$$|(B(u, v), w)| \leq c_1 |u|^{1/2} |v|^{1/2} |w||u|^{1/2} |w|^{1/2} |w|^{1/2} \quad \forall u, v, w \in V.$$  

(2.3)

Alternatively, we may use Agmon’s inequality

$$\|u\|_{L^\infty(\Omega)} \leq c_2 |u|^{\frac{1}{\nu}} |Au|^{\frac{1}{\nu}} \quad \forall u \in \mathcal{D}(A),$$

to obtain

$$|(B(u, v), w)| \leq c_3 |u|^{1/2} |Au|^{1/2} |v||w| \quad \forall u \in \mathcal{D}(A), v, w \in V.$$ 

(2.4)

In addition, the operator $B$ enjoys the property $(B(u, v), w) = -(B(u, w), v)$, and in the 2D case with periodic boundary conditions it satisfies:

$$(B(w, w), Aw) = 0 \quad \forall w \in \mathcal{D}(A)$$

(2.5)

(cf. [1], [19]). Differentiating this last expression with respect to $w$ in the direction of $u$, we obtain the useful identity

$$(B(u, w), Aw) + (B(w, u), Aw) + (B(w, w), Au) = 0$$

(2.6)

for all $u, w \in \mathcal{D}(A)$ (see [2]).
3 Determining Modes

The first mathematically rigorous indication that the large time behavior of the solutions to the NSE has a finite number of degrees of freedom was given in [6]. More specifically, the NSE was shown to have a finite number of determining modes.

We denote by $P_m$ the orthogonal projection onto the linear space spanned by $\{w_1, w_2, \ldots, w_m\}$, the first $m$ eigenfunctions of the Stokes operator $A$, and $Q_m = I - P_m$. Further, $c_j$ ($j = 1, 2, \ldots$) will denote adequate dimensionless positive constants. Let $u, v$ solve respectively the Navier-Stokes equations

$$\frac{du}{dt} + \nu Au + B(u, u) = f(t)$$

$$u(0) = u_0,$$

$$\frac{dv}{dt} + \nu Av + B(v, v) = g(t)$$

$$v(0) = v_0,$$

where $f, g$ are given forces in $L^\infty(0, \infty; H)$.

A set of modes $\{w_j\}_{j=1}^m$ is called determining if we have

$$\lim_{t \to \infty} |u(t) - v(t)| = 0,$$

whenever

$$\lim_{t \to \infty} |f(t) - g(t)| = 0,$$

and

$$\lim_{t \to \infty} |P_m u(t) - P_m v(t)| = 0.$$

In [5] an upper bound on the number of determining modes for the 2D NSE was obtained in the form

$$\frac{\lambda_{m+1}}{\lambda_1} \geq c_4 Gr (1 + \log(Gr))^{1/2}. \quad (3.3)$$

However, in that paper (see also [16, 20]) it is argued, heuristically, that the number of determining modes should be of the order $m$, where $m$ satisfies

$$\frac{\lambda_{m+1}}{\lambda_1} \geq c_5 Gr.$$

We will suppose throughout that $u(t), v(t)$ solve Equations (3.1), (3.2) respectively and that $|f(t) - g(t)| \to 0$ as $t \to \infty$. Before we show how to eliminate the logarithmic term in (3.3) we recall a version of the Gronwall lemma that will play a central role. The original version was first used in [5] to prove (3.3). The more general version stated below is proven in [10].

Lemma 3.1 Let $\alpha$ be a locally integrable real valued function on $(0, \infty)$, satisfying for some $0 < T < \infty$ the following conditions:

$$\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau = \gamma > 0$$
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha^{-} (\tau) d\tau = \Gamma < \infty,
\]
where \(\alpha^{-} = \max \{-\alpha, 0\}\). Further, let \(\beta\) be a real valued locally integrable function defined on \((0, \infty)\) such that
\[
\lim_{t \to -\infty} \frac{1}{T} \int_t^{t+T} \beta^+ (\tau) d\tau = 0,
\]
where \(\beta^+ = \max \{\beta, 0\}\). Suppose that \(\xi\) is an absolutely continuous non-negative function on \((0, \infty)\) such that
\[
\frac{d}{dt} \xi + \alpha \xi \leq \beta, \text{ a.e. on } (0, \infty).
\]
Then \(\xi(t) \to 0\) as \(t \to \infty\).

**Theorem 3.2** Suppose that \(m\) satisfies
\[
\frac{\lambda_{m+1}}{\lambda_1} \geq \sqrt{3c_3 Gr}.
\]
Then the number of determining modes is not larger than \(m\). That is, if
\[
\lim_{t \to -\infty} |P_m u(t) - P_m v(t)| = 0,
\]
then \(\lim_{t \to -\infty} \|u(t) - v(t)\| = 0\).

Proof. The idea of the proof (as in [5]) is to apply Lemma 3.1. Let \(w(t) = u(t) - v(t)\), \(\delta(t) = P_m w(t)\) and \(\Delta(t) = Q_m w(t)\). Then by assumption \(|\delta| \to 0\) as \(t \to \infty\). Subtracting Equation (3.2) from (3.1), we obtain
\[
\frac{dw}{dt} + \nu A w + B(u, w) + B(w, u) - B(w, w) = f(t) - g(t).
\]
Taking the inner product with \(A \Delta\) and using (2.5), (2.6), we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta\|^2 + \nu \|A \Delta\|^2 \leq \langle B(w, w), A \Delta \rangle + \|B(u, w) + B(w, u) - B(w, w), A \delta \| + \|f(t) - g(t)\| \|A \Delta\|.
\]
For the second term on the right side of the inequality we use (2.3) and \(|A^{3/2} \delta| \leq \lambda_m^{3/2} \delta\|\|\) to obtain
\[
\|B(u, w) + B(w, u) - B(w, w), A \delta\| \leq 2c_1 |u(t)|^{1/2} |w(t)|^{1/2} + c_1 |w(t)||w(t)||\lambda_m^{3/2} \delta(t)| = M_1(t)|\delta(t)|.
\]
Since \(|u(t)|\), \(|v(t)|\) remain bounded as \(t \to \infty\) (this follows immediately from Equations (3.1), (3.2), using \(B(u, u), A u = 0\)), \(M_1(t)\) is bounded as \(t \to \infty\).

For the first term we use the equation \(B(w, w) = B(\Delta, \Delta) + B(\delta, w) + B(\Delta, \delta)\) and (2.4) to obtain
\[
|B(w, w), A u)\| \leq |B(\Delta, \Delta), A u)\| + M_2(t)|\delta(t)|\|A u\| + c_2|\delta|\|A \Delta\|\|A u\|,
\]

where $M_2(t)$ may be chosen to be $c_3 \lambda_{m+1}^1/2 (\|u\| + \|v\|)$. Using (2.4), we also have
\[
\|(\tilde{B}(\Delta, \nabla), Au)\| \leq c_3 |A\Delta|^{1/2} |\Delta u|^{1/2} |Au| \leq \frac{c_3 |\Delta u|^2}{\mu \lambda_{m+1}^{1/2}} \|\Delta\| |Au|.
\]
Applying Young’s inequality we conclude
\[
\frac{d}{dt} \|\Delta\|^2 + \|\Delta\|^2 \left( \nu \lambda_{m+1} - \frac{3c_3^2 |Au|^2}{\nu \lambda_{m+1}} \right) \leq \beta(t),
\]
where $\beta(t) = 2M_1(t) |\delta| + 2M_2(t) |Au| |\delta| + \frac{2}{\nu} c_3^2 |Au|^2 |\delta|^2 + \frac{|f(t) - g(t)|^2}{\nu}$. It follows from a priori estimates on the time average of $|Au|$ (see [5], [9]),
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |Au|^2 d\tau \leq \frac{F^2}{Tv^3 \lambda_1} + \frac{F^2}{\nu^2},
\]
for every $T > 0$ (here we take $T = (\nu \lambda_1)^{-1}$). The assumptions on $f, g, \delta$, and $u, v$ imply that $\lim_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0$. Set
\[
\alpha = \nu \lambda_{m+1} - \frac{3c_3^2 |Au|^2}{\nu \lambda_{m+1}}.
\]
Also from the estimate on the time average of $|Au|^2$ (see (3.4)) we have
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha^- (\tau) d\tau < \infty.
\]
Similarly,
\[
\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau > 0
\]
holds provided $\lambda_{m+1}/\lambda_1 \geq \sqrt{3} c_3 Gr$. Hence, from the above and Lemma 3.1 we conclude that $\lim_{t \to \infty} \|u(t)\| = 0$. 

Let us remark that if we do not require additional regularity of the forcing function $f(t)$, then $|Au(t)|^2$ need not to be uniformly bounded for all $t > 0$. However, it is always bounded in average (see (3.4)). This means that the function $\beta(t)$ need not to converge to zero, but its average does. For this very reason we use Lemma 3.1, the modified version of Gronwall’s Lemma.

4 Determining Nodes

Let \[ \mathcal{E} = \{x^1, x^2, \ldots, x^N\} \]
be a collection of points in $\Omega$. The set $\mathcal{E}$ is called a set of determining nodes if for any two solutions $u$ and $v$ solving Equations (3.1), (3.2) respectively, and satisfying
\[
\lim_{t \to \infty} (u(x^j, t) - v(x^j, t)) = 0, \quad j = 1, \ldots, N,
\]
we have
\[ \lim_{t \to \infty} |u(t) - v(t)| = 0. \]

The existence of a set of determining nodes was first proven in [7]. Later in [9] an upper bound for the number of determining nodes for the NSE described in Section 2 was found to be proportional to \( Gr^2(1 + \log Gr) \). Rather than finding an estimate on the maximum distance the nodes can be separated (as in [7], [9]), we simply estimate the number of nodes needed to be determining directly. For this we divide the domain \( \Omega \) into \( N \) equal squares of side \( l = L/\sqrt{N} \). Further we place one of the points \( x_j, j = 1, \ldots, N \) in each subsquare.

The key to our estimates is the following lemma (cf. [9], [10]) which is proven in the appendix.

**Lemma 4.1** For every \( w \in D(A) \) set
\[
\eta(w) = \max_{1 \leq j \leq N} |w(x_j)|. 
\]

Then
\[
|w|^2 \leq 4L^2 \eta^2(w) + \frac{c_8 L^4}{N^2} |A w|^2, \tag{4.1}
\]
\[
\|w\|^2 \leq c_9 N \eta^2(w) + \frac{c_{11} L^2}{N} |A w|^2, \tag{4.2}
\]
\[
\|w\|_\infty^2 = \sup_{x \in D} |w(x)| \leq c_9 N \eta^2(w) + \frac{c_{11} L^2}{N} |A w|^2. \tag{4.3}
\]

Let \( u(t), v(t) \) solve (3.1), (3.2) respectively, and suppose \( |f(t) - g(t)| \to 0 \) as \( t \to \infty \) as before.

**Theorem 4.2** Let \( \Omega \) be divided into \( N \) equal squares with the points \( \mathcal{E} = \{x^1, x^2, \ldots, x^N\} \) distributed one in each square. Then \( \mathcal{E} \) is a set of determining nodes provided
\[
N \geq c_{11} Gr,
\]
where \( c_{11} = 4 \sqrt{2} c_8 c_{10} \pi^2 \).

Proof. With the exception of the use of Lemma 4.1 and a minor modification described below, the proof is the same as in [9]. We therefore only provide a sketch of the proof here.

Again set \( w(t) = u(t) - v(t) \). Then \( \eta(w) \to 0 \) as \( t \to \infty \). Subtracting Equation (3.2) from (3.1) and using (2.5), (2.6), (2.4), we have
\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu |A w|^2 \leq |(B(w, w), Au)| + |f - g||A w|
\leq c_8 \|w\|_\infty \|w\| \|Au\| + |f - g||A w|.
\]

Now we use (4.3) to estimate \( \|w\|_\infty \). Further, instead of using the estimate \( |A w| \geq \lambda_1^{1/2} \|w\| \), as in [9], we use (4.2) to obtain
\[
|A w|^2 \geq \frac{N}{c_{11} L^2} \|w\|^2 - \frac{c_9 N^2 \eta^2(w)}{c_{11} L^2},
\]

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After using Young's inequality and some algebra, we find
\[
\frac{d}{dt} \|w(t)\|^2 + \|w(t)\|^2 \left( \frac{\nu N}{c_{18}L^2} - \frac{2c_3^2c_{16}L^2}{N\nu} |Au(t)|^2 \right) \leq \beta(t),
\]
where
\[
\beta(t) = \frac{c_3\nu N^2 \eta^2(w)}{c_{18}L^2} + 2c_3\sqrt{\frac{\nu}{N}} \eta(w) \|Au(t)\| + \frac{2}{\nu} |f(t) - g(t)|.
\]
As in the proof of Theorem 3.2, we have \( \lim_{t\to-\infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau)d\tau = 0 \). Set
\[
\alpha = \frac{\nu N}{c_{18}L^2} - \frac{2c_3^2c_{16}L^2}{N\nu} |Au(t)|^2.
\]
Again from the estimate on the time average of \( |Au(t)|^2 \), inequality (3.4), we have
\[
\limsup_{t\to-\infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau)d\tau < \infty.
\]
Similarly,
\[
\liminf_{t\to-\infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau)d\tau > 0
\]
holds provided \( N \geq \sqrt{2c_3c_{16}\lambda_1} L^2 Gr \). Hence, from the above and Lemma 3.1 we conclude that \( \lim_{t\to-\infty} \|w(t)\| = 0 \). □

5 Determining Finite Volume Elements

We divide \( \Omega \) in to \( N \) equal squares of side \( l = L/\sqrt{N} \), and label the squares by \( Q_1, \ldots, Q_N \). Here we suppose that the average values of solutions on each of the \( Q_j \)'s is known. For this we set
\[
\langle u \rangle_{Q_j} = \frac{N}{L^2} \int_{Q_j} u(x)dx
\]
for every \( 1 \leq j \leq N \). A set of volume elements is said to be determining if for any two solutions \( u \) and \( v \) solving Equations (3.1), (3.2) respectively, and satisfying
\[
\lim_{t\to-\infty} (\langle u \rangle_{Q_j} - \langle v \rangle_{Q_j}) = 0,
\]
we have
\[
\lim_{t\to-\infty} |u(t) - v(t)| = 0.
\]

The notion of determining volume elements was introduced in [8]. The existence of determining volume elements for the NSE was shown in [10]. Moreover, it was found there that the number of volume elements need to be determine the solutions was of the order \( Gr^2 \).

Theorem 5.1 Let \( \Omega \) be divided into \( N \) equal squares. Suppose
\[
\lim_{t\to-\infty} (\langle u \rangle_{Q_j} - \langle v \rangle_{Q_j}) = 0,
\]
for \( 1 \leq j \leq N \). Then the volume elements are determining, that is,
\[
\lim_{t\to-\infty} \|u(t) - v(t)\| = 0,
\]
provided \( N \geq c_{13}Gr \).
Proof. Set \( w(t) = u(t) - v(t) \) and \( \gamma(w) = \max_{1 \leq i \leq N} |\{w\}_Q| \). With the exception of replacing the estimate \( |Aw| \geq \lambda \|w\| \), with the analog of (4.2),

\[
|Aw| \geq \frac{2\sqrt{N}}{L} \|w\| - \frac{2\sqrt{6}N}{L} \gamma(w),
\]

(see Lemma 3.1 of [10]), the proof is exactly the same as in [10] (and Theorem 4.2 above). We omit further details. \( \Box \)

6 Appendix

We now give the proof of Lemma 4.1. We begin with an auxiliary lemma.

**Lemma 6.1** Let \( \Omega_1 = [0, \Lambda] \times [0, d] \) and \( u(x, y) \in H^1(\Omega_1) \). Then

\[
\int_0^\Lambda |u(x, 0)|^2 dx \leq 2d^{-1} \|u\|_{L^2(\Omega_1)}^2 + d \|\partial u / \partial y\|_{L^2(\Omega_1)}^2.
\]

(6.1)

Proof. Without loss of generality we can assume that \( u \) is a smooth function. Then from by the fundamental theorem of calculus we have

\[
u^2(x, 0) = u^2(x, y) - \int_0^y \frac{\partial}{\partial y} u^2(x, s) ds.
\]

Integrating over \( \Omega_1 \), with respect to \( x \) and \( y \), and applying the Cauchy-Schwarz inequality, we obtain

\[
d \int_0^\Lambda |u(x, 0)|^2 dx \leq \|u\|_{L^2(\Omega_1)}^2 + 2d \|u\|_{L^2(\Omega_1)} \|\partial u / \partial y\|_{L^2(\Omega_1)}.
\]

The result follows after an application of Young's inequality. \( \Box \)

Now consider a square \( Q = [0, l] \times [0, l] \). For any two points in the square \((x_1, y), (x_2, y)\) we have

\[
|u(x_1, y) - u(x_2, y)|^2 \leq \int_{x_1}^{x_2} \left( \frac{\partial^2}{\partial x} \right) u(s, y) ds \leq l \left( \frac{\partial^2 u(s, y)}{\partial x} \right)_{L^2(0, l)}^2.
\]

We apply (6.1) to \( \partial u / \partial x \) with \( d \) replaced with the maximal distance of the \( y \) coordinate of the points \((x_1, y), (x_2, y)\) from the horizontal walls. That is, \( d = \max \{y, l - y\} \geq l/2 \). We arrive at

\[
|u(x_1, y) - u(x_2, y)|^2 \leq 4 \left( \frac{\partial^2 u}{\partial x} \right)_{L^2(Q)}^2 + l^2 \left( \frac{\partial^2 u}{\partial y} \right)_{L^2(Q)}^2.
\]

By symmetry a similar inequality holds for points of the form \((x_1, y_1), (x, y_2)\). Now for arbitrary points in \( Q \), \((x_1, y_1), (x_2, y_2)\), we have \( |u(x_1, y_1) - u(x_2, y_2)| \leq |u(x_1, y_1) - u(x_2, y_1)| + |u(x_2, y_1) - u(x_2, y_2)| \). Applying the above inequality to each part, we obtain

\[
|u(x_1, y_1) - u(x_2, y_2)| \leq 2 \left( 4 \left( \frac{\partial^2 u}{\partial x} \right)_{L^2(Q)}^2 + l^2 \left( \frac{\partial^2 u}{\partial y} \right)_{L^2(Q)}^2 \right)^{1/2}.
\]

(6.2)
Now divide the domain $\Omega$ into $N$ equal squares of side $l = L/\sqrt{N}$. Again label the squares $Q_j$, $j = 1, \ldots, N$, and set $\eta(w) = \max_{1 \leq j \leq N} |w(x_j)|$, where now $x_j$ is any point in $Q_j$. It follows from integrating (6.2) over $Q_j$ that
\[
\| w \|_{L^2(Q_j)}^2 \leq 2l^2 \eta^2(w) + 32l^2 \| \nabla w \|_{L^2(Q_j)}^2 + 8l^4 \left\| \frac{\partial^2 w}{\partial y \partial x} \right\|_{L^2(Q_j)}^2.
\]

Summing over $j$ and using $\| \nabla w \|^2 \leq \| w \| A w \|, \left| \frac{\partial^2 w}{\partial y \partial x} \right| \leq c_{1,3} |A w|$ (again, $\cdot$ means the $L^2(\Omega)$ norm), and an application of Young’s inequality, we obtain
\[
\| w \|^2 \leq 4Nl^2 \eta^2(w) + c_{s}l^4 |A w|^2
\]
which is Equation (4.1). The other equations in Lemma 4.1 follow from $\| w \|^2 \leq \| w \| A w \|$, Agmon’s inequality $\| w \|_{L^\infty} \leq c_{2} |w| |A w|$ and a repeated use of Young’s inequality (see [10]).

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