ON CONTINUOUS FUNCTIONS DEFINABLE IN EXPANSIONS OF THE ORDERED REAL ADDITIVE GROUP

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ABSTRACT. Let $\mathcal{R}$ be an expansion of the ordered real additive group. Then one of the following holds: either every continuous function $[0,1] \to \mathbb{R}$ definable in $\mathcal{R}$ is C^2 on an open dense subset of $[0,1]$, or every C^2 function $[0,1] \to \mathbb{R}$ definable in $\mathcal{R}$ is affine, or every continuous function $[0,1] \to \mathbb{R}$ is definable in $\mathcal{R}$. If $\mathcal{R}$ is NTP_2 or more generally does not interpret a model of the monadic second order theory of one successor, the first case holds. It is due to Marker, Peterzil, and Pillay that whenever $\mathcal{R}$ defines a C^2 function $[0,1] \to \mathbb{R}$ that is not affine, it also defines an ordered field on some open interval whose ordering coincides with the usual ordering on $\mathbb{R}$. Assuming $\mathcal{R}$ does not interpret second-order arithmetic, we show that the last statement holds when C^2 is replaced by C^1.

1. INTRODUCTION

Throughout $\mathcal{R} = (\mathbb{R}, <, +, \dots)$ is an expansion of the ordered additive group of real numbers. The goal of this paper is to understand continuous functions definable in $\mathcal{R}$. This research question and the results we produce are relevant to two programs in model theory. On the one hand, this inquiry is part of the program of studying tameness of definable sets in expansions of the real line, as outlined by Miller in [35]. On the other hand, many of the results we produce strengthen and extend classical results that appeared in work on the classification of o-minimal expansions of $(\mathbb{R}, <, +)$.

We fix some notation before stating our results. First of all, **definable** means "$\mathcal{R}$-definable, possibly with parameters", and I, J, L will always range over nonempty bounded open subintervals of $\mathbb{R}$. We say $\mathcal{R}$ is of **field-type** if there are definable $\oplus, \otimes : I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(\mathbb{R}, <, +, \cdot)$. It is easy to see that such $\oplus, \otimes$ must be continuous.

Fact 1.1. *If $\mathcal{R}$ defines a C^2 non-affine $f : I \to \mathbb{R}$, then $\mathcal{R}$ is of field-type.*

Fact 1.1 is essentially due to Marker, Peterzil, and Pillay [34, Section 3]. Their proof is only written to cover the case when $\mathcal{R}$ is o-minimal, but goes through in general (see our proof of Theorem E below). The construction is based on earlier ideas appearing in work of Rabinovich on additive reducts of algebraically closed

fields [38] (see also Hasson and Sustretov [23]). In this paper we will produce a variety of results that strengthen Fact 1.1 both in general and in special cases. We will first present our general result and then present the various stronger statements in special cases.

We say $f : I \to \mathbb{R}$ is **repetitious** if for every open subinterval $J \subseteq I$ there are $\delta > 0$, $x, y \in J$ such that $\delta < y - x$ and

$$f(x + \epsilon) - f(x) = f(y + \epsilon) - f(y) \quad \text{for all } 0 \leq \epsilon < \delta$$

Theorem A. *The following are equivalent:*

(1) $\mathcal{R}$ *is of field-type.*
(2) $\mathcal{R}$ *defines a continuous non-repetitious* $f : I \to \mathbb{R}$.
(3) $\mathcal{R}$ *defines a* C^2 *non-affine* $f : I \to \mathbb{R}$.

A simple convexity argument (Lemma 4.3 below) shows that (3) implies (2). Our main contribution lies in showing that (2) implies (1). Theorem A relies on the following fundamental trichotomy for expansions of $(\mathbb{R}, <, +)$.

An ω-**orderable set** is a definable set that is either finite or admits a definable ordering with order type ω. We say such a set is **dense** if it is dense in some open subinterval of $\mathbb{R}$. Expansions of $(\mathbb{R}, <, +)$ are divided into three distinct types:

(A) $\mathcal{R}$ does not define a dense ω-orderable set,
(B) $\mathcal{R}$ defines a dense ω-orderable set, but does not define some compact set,
(C) $\mathcal{R}$ defines every compact subset of every $\mathbb{R}^k$.

We say $\mathcal{R}$ is **type A** if it satisfies statement (A). In the same way we define what it means for $\mathcal{R}$ to be **type B** or **type C**.

We will prove Theorem A by establishing it independently for the three different types of expansions. In the case of type C expansions it is easy to see that Theorem A holds, simply because every type C expansion is of field-type. Therefore it is only left to prove Theorem A for type A and type B structures. Indeed, in both cases we will produce substantial strengthenings of Theorem A. Before doing so, we want to give the reader more information about the above trichotomy.

1.1. **The Trichotomy.** Since a type C expansion defines a dense ω-orderable set (see [17, Theorem 3.9(i)]), the above trichotomy is indeed a trichotomy. Although not stated in precisely this way, the argument in [24] already shows that any expansion of $(\mathbb{R}, <, +, \cdot)$ that defines a dense ω-orderable set, defines $\mathbb{Z}^1$. Such an expansion defines all closed subsets of all $\mathbb{R}^k$ (see Kechris [29, 37.6]). The following fact follows easily.

Fact 1.2. *If $\mathcal{R}$ is of field-type and admits a dense ω-orderable set, then $\mathcal{R}$ is type C. If $\mathcal{R}$ expands $(\mathbb{R}, <, +, \cdot)$, then $\mathcal{R}$ defines every closed set.*

This shows that a type B expansions cannot be of field-type. This is not the only restriction on type B structures. No expansion of $(\mathbb{R}, <, +)$ that defines multiplication by λ for uncountably many $\lambda \in \mathbb{R}$ is type B by [17, Theorem C]. Furthermore, by [28, Theorem A] an expansion of $(\mathbb{R}, <, +)$ that admits a dense ω-orderable set

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[1]This statement can also be deduced easily from [17, Theorem 3.9(ii)]

defines[2] the standard model of the monadic second order theory of one successor $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. For that reason a type B expansion violates all known Shelah-style combinatorial tameness properties such as NIP or NTP_2 (see e.g. Simon [40] for definitions). So all NTP_2 expansions of $(\mathbb{R}, <, +)$ are type A.

So what do type B structures look like? While such expansions do not lie in the world of Shelah-tameness, there are several well-behaved examples. For example, $(\mathbb{R}, <, +, x \mapsto \sqrt{2}x, \mathbb{Z})$ (see [26]) and $(\mathbb{R}, <, +, C)$, where C is the middle-thirds Cantor set (see Balderrama and Hieronymi [2]) are type B expansions with decidable theories. We describe another interesting example in Section 6.4. Type B expansions have received little attention within tame geometry and model theory, but have appeared in theoretical computer science (Boigelot, Rassart, and Wolper [6]) and fractal geometry (Charlier, Leroy, and Rigo [8]). One reason might be that all known examples of type B expansions are bi-interpretable with $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. The theory of the latter structure was shown to be decidable by Büchi [7] using automata-theoretic rather then model-theoretic methods.

As already indicated above, being type B is a rather restrictive property. Throughout this paper we will use these restrictions, in particular Fact 1.3 and Fact 1.4 below, to obtain strong statements about continuous definable functions in type B expansions.

Fact 1.3 ([17, Proposition 3.8])**.** *If $\mathcal{R}$ defines an order $(D, \prec)$, an open interval $I \subseteq \mathbb{R}$, and a function $g \colon \mathbb{R}^3 \times D \to D$ such that*

(i) *$(D, \prec)$ has order type ω and D is dense in I,*
(ii) *for every $a, b \in U$ and $e, d \in D$ with $a < b$ and $e \preceq d$,*

$$\{c \in \mathbb{R} : g(c, a, b, d) = e\} \cap (a, b) \text{ has nonempty interior,}$$

then $\mathcal{R}$ is type C.

This should be compared to a similar result of Elgot and Rabin [14, Theorem 1] on decidable expansions of the monadic second theory of one successor. See [2, Section 3.2] for a more detailed discussion. The following corollary to Fact 1.3 is often easier to apply.

Fact 1.4 ([2, Lemma 3.7])**.** *If there exist two dense ω-orderable subsets C and D of $[0,1]$ satisfying $(C - C) \cap (D - D) = \{0\}$, then $\mathcal{R}$ is type C.*

1.2. **Proof of Theorem A.** We now return to the proof of Theorem A. Since we already know Theorem A for type C expansions, it is only left to establish Theorem A when $\mathcal{R}$ is either type A or type B. Let us consider the case when $\mathcal{R}$ is type A first. In this case we will prove a significant strengthening of Theorem A (see Theorem B below). In order to state this result, we need recall a few necessary definitions, following those in Peterzil and Starchenko [37].

Unless otherwise stated, **dimension** will refer to topological dimension (see the remarks following Fact 1.7). For our purposes a **curve** is a one-dimensional subset

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[2]When we say an structure $\mathcal{M}$ defines another structure $\mathcal{N}$, we mean that $\mathcal{M}$ defines an isomorphic copy of $\mathcal{N}$.

of $\mathbb{R}^2$. Let $\mathcal{A} = \{A_x : x \in \mathbb{R}^l\}$ be a family of curves. The dimension of $\mathcal{A}$ is the dimension of $\{x \in \mathbb{R}^l : A_x \neq \emptyset\}$, $\mathcal{A}$ is **closed** if $\{(x,y) \in \mathbb{R}^l \times \mathbb{R}^2 : y \in A_x\}$ is closed, and $\mathcal{A}$ is **normal** if $A_x \cap A_y$ is zero-dimensional for all distinct $x, y \in \mathbb{R}^l$.

Theorem B. *Suppose $\mathcal{R}$ is type A. Then the following are equivalent:*

(1) *$\mathcal{R}$ is of field-type.*
(2) *$\mathcal{R}$ defines a C^2 non-affine $f : I \to \mathbb{R}$.*
(3) *$\mathcal{R}$ defines a continuous non-repetitious $f : I \to \mathbb{R}$.*
(4) *$\mathcal{R}$ defines a continuous nowhere locally affine $f : I \to \mathbb{R}$.*
(5) *$\mathcal{R}$ admits a normal 2-dimensional closed definable family of curves.*
(6) *$\mathcal{R}$ defines a continuous $f : I \times J \times L \to \mathbb{R}$ such that*

$$\dim\{z \in L : f(x,y,z) = f(x',y',z)\} = 0$$

for all distinct $(x,y), (x',y') \in I \times J$.

Theorem B gives evidence to the thesis that being type A is the ultimate generalization of o-minimality in the setting of expansions of $(\mathbb{R}, <, +)$. The proof of Theorem B rests on topological tameness properties of definable sets in type A expansions established in [17, 28], on Fact 1.1, and on Theorem C below. Letting U be a definable open subset of $\mathbb{R}^k$, we say that a property holds **almost everywhere**, or **generically**, on U if there is a dense definable open subset of U on which it holds. It follows from [17, Theorem D] that if $\mathcal{R}$ is type A, then a nowhere dense definable subset of $\mathbb{R}^k$ has null k-dimensional Lebesgue measure. Therefore if a property holds almost everywhere in our sense, it holds almost everywhere in the usual measure-theoretic sense.

Theorem C. *Suppose $\mathcal{R}$ is type A. Let $f : I \to \mathbb{R}$ be definable and continuous. Then f is C^k almost everywhere for all k.*

Laskowski and Steinhorn [31] prove Theorem C in the o-minimal setting. Our proof makes crucial use of their ideas. In particular we also use a classical theorem of Boas and Widder [4]. The assumption of continuity in Theorem C is necessary, as $(\mathbb{R}, <, +, \mathbb{Q})$ is type A and the characteristic function of $\mathbb{Q}$ is nowhere C^1. Further observe that Theorem C cannot be strengthened to assert that a continuous function definable in a type A expansion is C^∞ on a dense definable open set. Such a result fails already in the o-minimal setting by Rolin, Speissegger, and Wilkie [39]. In the case of expansions of $(\mathbb{R}, <, +, \cdot)$, Theorem C is due to Fornaserio [16]. However, this special case is substantially easier because of definability of division.

It is easy to see that a generically locally affine $f : I \to \mathbb{R}$ is repetitious. As a non-affine C^2-function is not repetitious, we can deduce from Theorem C that a continuous $f : I \to \mathbb{R}$ definable in a type A expansion is repetitious if and only if it is generically locally affine.

We now describe the proof of Theorem A for type B expansions. In this case, it is easy to decduce from earlier work that statements (1)-(2) of Theorem A fail in a type B expansion. Indeed, by Fact 1.2 a type B expansion is not of field-type, and thus by Fact 1.1 every C^2 definable $f : I \to \mathbb{R}$ in a type B expansion is affine. The following theorem gives the failure of statement (3) of Theorem A for type B expansions.

Theorem D. *Suppose $\mathcal{R}$ is type B. Then every continuous definable $f : I \to \mathbb{R}$ is repetitious.*

The proof of Theorem D relies crucially on Fact 1.4 and the Baire Category Theorem. This completes the proof of Theorem A. As a consequence of Theorem C and Theorem D we obtain the following:

Corollary A. *One of the following statements holds:*

  (1) *every continuous definable function $I \to \mathbb{R}$ is C^2 almost everywhere.*
  (2) *every definable C^2 function $I \to \mathbb{R}$ is affine.*
  (3) *every continuous function $I \to \mathbb{R}$ is definable.*

1.3. **Optimality.** A few questions about the optimality of Theorem A arise immediately. First of all, one can ask whether "continuous non-repetitious" in Theorem A may be replaced with "continuous nowhere locally affine", that is:

If $f : I \to \mathbb{R}$ is continuous and nowhere locally affine, is $(\mathbb{R}, <, +, f)$ of field-type?

Observe that the answer is positive if the assumption is added that $(\mathbb{R}, <, +, f)$ is either type A or type C. However, in general this question remains open.

It is also natural to ask if "C^2 non-affine" in Theorem A may be replaced with "C^1 non-affine", that is:

$$\text{If } f : I \to \mathbb{R} \text{ is } C^1 \text{ non-affine, is } (\mathbb{R}, <, +, f) \text{ of field-type?}$$

We do not know the answer to this question in general. We obtain a positive answer under an absolutely minimal model-theoretic assumption on $(\mathbb{R}, <, +, f)$.

Theorem E. *Suppose $\mathcal{R}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$. Then the following are equivalent:*

  (1) *$\mathcal{R}$ defines a non-affine C^1 function $f : I \to \mathbb{R}$.*
  (2) *$\mathcal{R}$ is of field-type.*

Of course, the two-sorted structure $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$ mentioned in Theorem E is just second-order arithmetic, and its non-definability is arguably the weakest tameness condition one could imagine. We do not know whether this assumption can be dropped. Combining Theorem C and [28, Theorem A] with Theorem E allows us to see how stronger assumptions on the model-theoretic tameness of $\mathcal{R}$ imply stronger statements about definable functions.

Corollary B. *Suppose $\mathcal{R}$ is not of field-type. Let $f : I \to \mathbb{R}$ be definable and continuous.*

  (1) *If $\mathcal{R}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$, then f is generically locally affine.*
  (2) *If $\mathcal{R}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$ and f is C^1, then f is affine.*

In particular if $\mathcal{R}$ is type B and does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$, then every definable C^1 function $f : I \to \mathbb{R}$ is affine.

Thus every C^1 function definable in a type B structure with a decidable theory is affine. This covers the examples of type B structures described above. We collect consequences of Corollary B to automata theory in Section 6.4.

1.4. **Applications.** We anticipate that applications of the results presented in this paper are numerous. In Section 6 we already collect the most immediate consequences of our work related to descriptive set theory, metric geometry and automata theory. While these results are interesting in their own right, we do not wish to further extend the introduction. We refer the reader to Section 6 for a precise description of these results.

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Notations. Let $X \subseteq \mathbb{R}^n$. We denote by $\mathrm{Cl}(X)$ the closure of X, by $\mathrm{Int}(A)$ the interior of X, and by $\mathrm{Bd}(X)$ the boundary $\mathrm{Cl}(X) \setminus \mathrm{Int}(X)$ of X. Whenever $X \subseteq \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$, then X_x denotes the set $\{y \in \mathbb{R}^n : (x, y) \in X\}$.

We always use i, j, k, l, m, n, N for natural numbers and $r, s, t, \lambda, \epsilon, \delta$ for real numbers. We let $\|x\| := \max\{|x_1|, \ldots, |x_n|\}$ be the l_∞ norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

1.5. **D_Σ sets in type A expansions.** Our work here relies crucially on earlier results about D_Σ sets, a definable analogue of F_σ sets. The D_Σ sets form a collection of definable sets that are topologically better behaved then general definable sets and still satisfy certain useful closure properties. A set $X \subseteq \mathbb{R}^n$ is **D_Σ** if there is a definable family $\{Y_{r,s} : r, s > 0\}$ of compact subsets of $\mathbb{R}^n$ such that $X = \bigcup_{r,s} Y_{r,s}$, and $Y_{r,s} \subseteq Y_{r',s'}$ if $r \leq r'$ and $s \geq s'$, for all $r, r', s, s' > 0$. The family $\{Y_{r,s} : r, s > 0\}$ **witnessses** that X is D_Σ. See Dolich, Miller and Steinhorn [10], where this definition of D_Σ sets was first given, for the following basic facts about such sets.

Fact 1.5. *Open and closed definable subsets of $\mathbb{R}^k$ are D_Σ, finite unions and finite intersections of D_Σ sets are D_Σ, and the image or pre-image of a D_Σ set under a continuous definable function is D_Σ.*

In particular, the projection of a D_Σ set is D_Σ. However, the complement of a D_Σ set need not be D_Σ.

We say that a family $\{A_x : x \in \mathbb{R}^l\}$ of subsets of $\mathbb{R}^m$ is D_Σ if the set of $(x, y) \in \mathbb{R}^l \times \mathbb{R}^m$ satisfying $y \in A_x$ is D_Σ.

We make extensive use of the **Strong Baire Category Theorem**, or short **SBCT**, established in [17].

Fact 1.6 (SBCT [17, Theorem D]). *Suppose $\mathcal{R}$ is type A. Then every D_Σ subset of $\mathbb{R}^k$ either has interior or is nowhere dense.*

Another result from [17] we use is the following D_Σ-**selection** result.

Fact 1.7 ([17, Proposition 5.5]). *Suppose $\mathcal{R}$ is type A. Let $A \subseteq \mathbb{R}^k$ be D_Σ such that $\pi(A)$ has interior, where $\pi : \mathbb{R}^k \to \mathbb{R}^l$ is the projection onto the first l coordinates. Then there is a definable open $V \subseteq \pi(A)$ and a continuous definable $f : V \to \mathbb{R}$ such that $(p, f(p)) \in A$ for all $p \in V$.*

In a few place through this paper we will refer to the dimension of a D_Σ set in a type A expansion. It is necessary to explain what dimension we refer to. Given $X \subseteq \mathbb{R}^n$ we let $\dim(X)$ be the topological dimension of X. **Topological dimension** here refers to either small inductive dimension, large inductive dimension, or Lebesgue covering dimension. These three dimensions coincide on all subsets of $\mathbb{R}^k$ (see Engelking [15] for details and definitions). Model-theorists usually consider as a dimension of a subset X of $\mathbb{R}^n$ the maximal k for which there is a coordinate projection $\rho : \mathbb{R}^n \to \mathbb{R}^k$ such that $\rho(X)$ has nonempty interior. In [17] this is called the **naive dimension** of X. In general, this naive dimension is not well-behaved for arbitrary subsets of $\mathbb{R}^n$ and does not coincide with the topological dimension. However, this does not happen for D_Σ sets.

Fact 1.8 ([17, Proposition 5.7, Theorem E]). *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^n$ be D_Σ. Then $\dim(X)$ is equal to the maximal k for which there is a coordinate projection $\rho : \mathbb{R}^n \to \mathbb{R}^k$ such that $\rho(X)$ has nonempty interior. Moreover, $\dim \mathrm{Cl}(X) = \dim(X)$.*

2. Defining a field

In this section we show how to recover a field from a non-affine C^1 function. As pointed out in the introduction the idea behind our construction goes back to Rabinovich and in the case of a C^2 function our central argument has already appeared in [34]. Our main contribution is the extension to C^1 functions, in particular statement (2) in the following theorem.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be definable C^1 and non-affine.*
  (1) *If f' is strictly increasing or strictly decreasing on some open subinterval of I, then $\mathcal{R}$ is of field-type.*
  (2) *If f' is not strictly increasing or strictly decreasing on any open subinterval of I, then $\mathcal{R}$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.*
In particular, if f is C^2, then $\mathcal{R}$ is of field-type.

Theorem E follows. Note that a structure defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$ if and only if it defines an isomorphic copy of $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. Friedman, Kurdyka, Miller, and Speissegger [19] gave an example of type A structure that defines an isomorphic copy of $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. Therefore the conclusion of statement (2) in Theorem 2.1 does not rule out that $\mathcal{R}$ is type A.

Before proving Theorem 2.1 we establish a few lemmas used in the proof. We fix one further notation: A **complementary interval** of $A \subseteq \mathbb{R}$ is a connected component of the complement of the closure of A.

Lemma 2.2. *Let $F \subseteq \mathbb{R}$ be such that $(F, <)$ is isomorphic to $(\mathbb{R}, <)$. Then F either has interior or is nowhere dense. Furthermore, if I is a bounded complementary interval of F, then either the left endpoint or the right endpoint of I is in F.*

Proof. Let $\iota : (\mathbb{R}, <) \to (F, <)$ be an isomorphism. Let J be an open interval. We suppose F is dense in J and show $J \subseteq F$. The first claim then follows. Let $t \in J$ and $X := \{x \in \mathbb{R} : \iota(x) < t\}$. The density of F in J yields an $x \in \mathbb{R}$ satisfying $\iota(x) > t$. Thus X is bounded from above. Let u be the supremum of X in $\mathbb{R}$. As F is dense in J, we must have $\iota(u) = t$. Therefore $t \in F$. We proceed to the second claim. Let I be a bounded complementary interval of F. The density of $(F, <)$ shows that F

contains at most one endpoint of I. Let $z \in I$ and $Y := \{x \in \mathbb{R} : \iota(x) < z\}$. As I is a bounded complementary interval there is an $x \in \mathbb{R}$ such that $\iota(x) > z$. Hence Y is bounded above in $\mathbb{R}$. Let $u \in \mathbb{R}$ be the supremum of Y. If $u \in Y$, then $\iota(u)$ is the left endpoint of J. If $u \notin Y$, then $\iota(u)$ is the right endpoint of I. $\square$

Lemma 2.3. *Let $A \subseteq \mathbb{R}$ be definable and bounded. Then the set D of endpoints of bounded complementary intervals of A is ω-orderable.*

Proof. The lemma trivially holds when D is finite. We suppose D is infinite and define an ω-order $\prec$ on D. Note that each element of D is the endpoint of at most two bounded complementary intervals. Let $\delta : D \to \mathbb{R}$ be the definable function that maps each d to the minimal length of a complementary interval with endpoint d. We declare $d \prec d'$ if either $\delta(d') < \delta(d)$ or $(\delta(d') = \delta(d)$ and $d < d')$. It is easy to see that $\prec$ is an ω-order on D (see Section 2 of [28] for details). $\square$

Lemma 2.4. *Let $F \subseteq \mathbb{R}$ be definable and bounded, and $\oplus, \otimes : F^2 \to F$ be definable such that $(F, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$. Then F either has interior or is nowhere dense and,*

(1) *if F has interior, then $\mathcal{R}$ is of field-type.*
(2) *if F is nowhere dense, then there is a definable $Z \subseteq F$ such that the structure $(F, <, \oplus, \otimes, Z)$ is isomorphic to $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.*

Proof. Lemma 2.2 shows that F either has interior or is nowhere dense. Item (1) above follows easily from the fact that for any open interval I there are $(\mathbb{R}, <, +, \cdot)$-definable $\oplus', \otimes' : I^2 \to I$ such that $(I, <, \oplus', \otimes')$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$. We leave the details of (1) to the reader and prove (2). Suppose F is nowhere dense. Let D be the set of endpoints of bounded complementary intervals of F and $D' = D \cap F$.

We first show that D' is dense in F. Let $x, y \in F$ and $x < y$. Since F is nowhere dense there is a complementary interval I of F such that $x < z < y$ for every $z \in I$. By Lemma 2.2 one of the endpoints of I lies in F. Thus D' is dense in F.

Let $\prec$ be the ω-order on D given by Lemma 2.3 and denote its restriction to D' by $\prec'$. Note that $(D', \prec')$ has order-type ω. Consider $\mathcal{F} := (F, <, \oplus, \otimes, D', \prec')$. Clearly $\mathcal{F}$ is definable. Note that ι is an isomorphism between $\mathcal{F}$ and an expansion of $(\mathbb{R}, <, +, \cdot)$ that admits a dense ω-orderable set. An expansion of the real field that admits a dense ω-orderable set defines $\mathbb{Z}$ by Fact 1.2. Thus $\mathcal{F}$ defines $Z := \iota^{-1}(\mathbb{Z})$ and $(F, <, \oplus, \otimes, Z)$ is isomorphic to $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. $\square$

Lemma 2.5. *Let $f : [a, b] \to \mathbb{R}$ be C^1 and definable, let $\{(g_x : [0, c_x] \to \mathbb{R}) : x \in X\}$ be a definable family of functions such that*

(1) *$f'(a) = 0$ and $f'(t) > 0$ for all $a < t \leq b$, and*
(2) *$c_x > 0$, $g_x(0) = 0$, and $g'_x(0)$ exists for all $x \in X$.*

Then the relations $g'_y(0) < g'_x(0)$, $g'_x(0) \leq g'_y(0)$, and $g'_x(0) = g'_y(0)$ are definable on X.

Proof. We only prove the first claim, the latter two follow. Fix $x, y \in X$. We show that (i) and (ii) below are equivalent:

(i) $g'_x(0) < g'_y(0)$,
(ii) there is $z \in (a, b)$ such that

$$(\star) \qquad g_x(\epsilon) + [f(z + \epsilon) - f(z)] < g_y(\epsilon) \quad \text{for sufficiently small } \epsilon > 0.$$

Suppose (ii) holds. Let $z \in (a, b)$ be such that ($\star$) holds. Dividing by ϵ and taking the limit as $\epsilon \to 0$, we have

$$g'_x(0) + f'(z) \leq g'_y(0).$$

As $z > a$, $f'(z) > 0$. Thus $g'_x(0) < g'_y(0)$ and (i) holds.

Suppose (i) holds. Let $\delta > 0$ be such that $g'_x(0) + \delta < g'_y(0)$. Since f' is continuous and $f'(a) = 0 < f'(b)$, there is $z \in (a, b)$ such that $f'(z) < \delta$. Fix such z. Then $g'_x(0) + f'(z) < g'_y(0)$. Thus

$$\lim_{\epsilon \to 0} \frac{g_x(\epsilon)}{\epsilon} + \lim_{\epsilon \to 0} \frac{f(z + \epsilon) - f(z)}{\epsilon} < \lim_{\epsilon \to 0} \frac{g_y(\epsilon)}{\epsilon}.$$

Hence

$$\frac{g_x(\epsilon)}{\epsilon} + \frac{f(z + \epsilon) - f(z)}{\epsilon} < \frac{g_y(\epsilon)}{\epsilon} \quad \text{for sufficiently small } \epsilon > 0.$$

Therefore ($\star$) holds. $\qquad\square$

Proof of Theorem 2.1. There are $a, b \in \mathbb{R}$ with $a < b$ such that $f'(a) \neq f'(b)$ and one of the following two cases holds:

(I) f' is strictly increasing or strictly decreasing on $[a, b]$.
(II) there is no open subinterval of $[a, b]$ on which f' is strictly increasing or strictly decreasing

Now replace f by its restriction to $[a, b]$. So from now on, f is a function from $[a, b]$ to $\mathbb{R}$ satisfying $f'(a) \neq f'(b)$ and either condition (I) or (II). After replacing f with $-f$ if necessary, we suppose $f'(a) < f'(b)$. In case (I) these assumptions imply f' is strictly increasing.

Let $q \in \mathbb{Q}$ be such that $f'(a) < q < f'(b)$. By continuity of f' there is an $x \in (a, b)$ such that $f'(x) = q$. Continuity of f' further implies that the set of such x is closed. Let c be the maximal element of $[a, b]$ such that $f'(c) = q$. Note that $c < b$. By the intermediate value theorem, $f'(x) > q$ for all $x \in (c, b]$. After replacing a with c if necessary, we may suppose $f'(a) = q$ and $f'(x) > q$ for all $a < x \leq b$.

Let $h : [a, b] \to \mathbb{R}$ be given by $h(x) = f(x) - q(x - a)$. Note that $h'(x) = f'(x) - q$ for all $x \in [a, b]$. Thus $h'(a) = 0$ and $h'(x) > 0$ for all $a < x \leq b$. Moreover, h' is strictly increasing if f' is. Thus h' is strictly increasing in case (I). Since q is rational, h is definable. After replacing f with h if necessary, we may suppose that $f'(a) = 0$.

Let $N \in \mathbb{N}$ satisfy $f'(b) \geq \frac{1}{N}$. After replacing f with Nf if necessary, we can assume $f'(b) \geq 1$. Let d be the minimal element of $[a, b]$ such that $f'(d) = 1$. After replacing b with d if necessary, we may suppose that $f'(b) = 1$ and $0 < f'(x) < 1$ for all $a < x < b$.

Applying Lemma 2.5 to the definable family $g_x(t) = f(x + t) - f(x)$ we see that the relations $f'(x) < f'(y)$, $f'(x) \leq f'(y)$, and $f'(x) = f'(y)$ are definable on I. Let $E \subseteq [a, b]$ be the set of x such that $f'(y) < f'(x)$ for all $y \in [a, x)$. Observe that E is definable. For every $t \in [0, 1]$ the set $\{z \in [a, b] \ : \ f'(z) \geq t\}$ is closed and nonempty. Therefore this set has a minimal element w. Since $f'(a) = 0$ and f' is continuous, this minimal element w must satisfy $f'(w) = t$. In particular, $w \in E$.

Thus for every $t \in [0,1]$ there is an $x \in E$ such that $f'(x) = t$. Note that $a, b \in E$. Furthermore, if $x, y \in E$ and $x < y$, then $f'(x) < f'(y)$. It follows that $x \mapsto f'(x)$ gives an isomorphism between $(E, <)$ and $([0,1], <)$.

In case (I) we trivially have $E = [a, b]$, because in this case f' is strictly increasing. If E contains an open interval, then f' must be strictly increasing on that interval. Thus E has empty interior in case (II).

For $x \in E \setminus \{b\}$ and $t \in [0, b-x]$ we set $f_x(t) = f(x+t) - f(x)$. We declare $f_b(t) = t$ for all $t > 0$. Then $f'_x(0) = f'(x)$ for all $x \in E$. As f is strictly increasing, each f_x is strictly increasing. We suppose $a = 0$ after translating $[a, b]$ if necessary. Then E is a subset of $[0, b]$. We declare

$$E_1 = -(E \setminus \{0, b\}) + 2b, \quad E_2 = -(E \setminus \{0\}), \quad E_3 = -E_1.$$

Then E, E_1, E_2, E_3 are pairwise disjoint as they are subsets of $[0, b], (b, 2b), [-b, 0)$, and $(-2b, -b)$ respectively. Set $F = E \cup E_1 \cup E_2 \cup E_3$. Note that in case (I) we have $F = (-2b, 2b)$. So in this case F is an interval. Furthermore, in case (II) E has empty interior as each E_i has empty interior. We now construct a definable family of functions $\{h_x : x \in F\}$ with the following two properties:

(i) For all $t \in \mathbb{R}$ there is a unique $x \in F$ such that $h'_x(0) = t$.
(ii) If $x, y \in F$ and $x < y$, then $h'_x(0) < h'_y(0)$.

When $x \in E$, we set $h_x = f_x$. When $x \in E_1$, set h_x to be the compositional inverse of f_{2b-x}. Since each f_x is strictly increasing, each f_x has a compositional inverse. Observe that $h'_x(0) = f'_{2b-x}(0)^{-1}$ for all $x \in E_1$. It follows that for every $t > 1$ there is a unique $x \in E_1$ such that $h'_x(0) = t$. When $x \in E_2$, we let $h_x = -f_{-x}$. Then $h'_x(0) = -f'_{-x}(0)$ for all $x \in E_2$. Again we directly deduce that for every $t \in [-1, 0]$ there is a unique $x \in E_2$ such that $h'_x(0) = t$. Finally, if $x \in E_3$, we set $h_x = -h_{-x}$. Also in this situation we get that for all $t < -1$ there is a unique $x \in E_3$ such that $h'_x(0) = t$. Conditions (i) and (ii) above follow.

We are ready to define the field structure on F. For this, we need to define two functions $\oplus, \otimes : F^2 \to F$. Given $x, y \in F$, we let $x \oplus y$ be the unique element of F such that

$$h'_{x \oplus y}(0) = (h_x + h_y)'(0)$$

and $x \otimes y$ be the unique element of F such that

$$h'_{x \otimes y}(0) = (h_x \circ h_y)'(0).$$

It follows easily from Lemma 2.5 that $\oplus$ and $\otimes$ are definable. By our construction, we immediately get that for all $x, y \in F$

$$h'_{x \oplus y}(0) = h'_x(0) + h'_y(0) \text{ and } h'_{x \otimes y}(0) = h'_x(0)h'_y(0).$$

So $x \mapsto h'_x(0)$ gives an isomorphism $(F, <, \oplus, \otimes) \to (\mathbb{R}, <, +, \cdot)$. As observed above, F is an interval in case (I) and has empty interior in case (II). Now apply Lemma 2.4. $\square$

We record some corollaries.

Corollary 2.6. *Let $f : I \to \mathbb{R}$ be a non-affine C^1 and generically locally affine function. Then*

(1) $(\mathbb{R}, <, +, f)$ *defines* $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.

(2) $(\mathbb{R}, <, +, \cdot, f)$ *is type C.*

Proof. The derivative of f is locally constant almost everywhere and therefore is not strictly increasing or strictly decreasing on any open subinterval of I. Thus $(\mathbb{R}, <, +, f)$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$ by Theorem 2.1. Thus (1) holds.

For (2), first observe that f' is definable in $(\mathbb{R}, <, +, \cdot, f)$. Let $(F, <, \oplus, \otimes, Z)$ be constructed from f as in the proof of Theorem 2.1. An inspection of the proof of Theorem 2.1 shows that the isomorphism $(F, <, \oplus, \otimes, Z) \to (\mathbb{R}, <, +, \cdot, \mathbb{Z})$ given by $x \mapsto h'_x(0)$ is $(\mathbb{R}, <, +, \cdot, f)$-definable. It follows that $(\mathbb{R}, <, +, \cdot, f)$ is type C. $\square$

The following corollary sharpens the third claim of Corollary B.

Corollary 2.7. *Let K be a subfield of $\mathbb{R}$ such that $\mathcal{R}$ defines a dense ω-orderable subset of K. Then*

(1) *if $\mathcal{R}$ is not type C, then every definable C^2 function $f : I \to \mathbb{R}$ is affine with slope in K.*

(2) *if $\mathcal{R}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$, then every definable C^1 function $f : I \to \mathbb{R}$ is affine with slope in K.*

Proof. First observe that if does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$, then $\mathcal{R}$ is not type C. Since $\mathcal{R}$ defines a dense ω-orderable set, it has to be type B. Therefore $\mathcal{R}$ can not be of field-type by Fact 1.2. Thus by Theorem 2.1 every definable C^2 function $f : I \to \mathbb{R}$ is affine. Moreover, if in addition $\mathcal{R}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$, then every definable C^1 function $f : I \to \mathbb{R}$ is affine by Theorem 2.1. It is left to show that the slope of a definable affine function $f : I \to \mathbb{R}$ is in K. Towards a contradiction suppose there is such a function with slope $\alpha \notin K$. Its definability immediately implies definability of $x \mapsto \alpha x$ on $[0, 1]$. Let D be a dense ω-orderable subset of K. Note that αD is also dense in $\mathbb{R}$. Observe $D - D \subseteq K$ and $\alpha(D - D) \subseteq \alpha K$ Since $\alpha \notin K$, we have $\alpha K \cap K = \{0\}$. This yields

$$(D - D) \cap (\alpha D - \alpha D) = \{0\}.$$

Thus $\mathcal{R}$ is type C by Fact 1.4. A contradiction. $\square$

Let C be the middle-thirds Cantor set, or one of the generalized Cantor sets discussed in [2]. It is observed the proof of [17, Corollary 3.10] and the introduction of [2] that $(\mathbb{R}, <, +, C)$ defines a dense ω-orderable subset of $\mathbb{Q}$. Since $(\mathbb{R}, <, +, C)$ has a decidable theory, it can not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$. Therefore Corollary 2.7 shows that if $f : I \to \mathbb{R}$ is C^1 and $(\mathbb{R}, <, +, C, f)$ is not type C, then f is affine with rational slope.

3. THE PROOF OF THEOREM C

In this section we prove Theorem C. The reader will find it helpful to have copies of [17, 28] handy, as we repeatedly make use of results from these papers. We need to include a remark about the work in [28]. By [28, Theorem A], an expansion of $(\mathbb{R}, <, +)$ that does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$ is type A. In Sections 3 and 4 of [28] all results were stated for expansions that do not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. However, the proofs only made use of the fact that such structures are type A. This should have been made clear, but the authors did not anticipate the relevance of the weaker assumption.

3.1. **Prerequisites.** Throughout this subsection $\mathcal{R}$ is type A. Before diving into the proof of Theorem C we establish a few basic facts for later use.

Lemma 3.1. *Let $X \subseteq I \times \mathbb{R}_{>0}$ be D_Σ such that X_x is finite for every $x \in I$. Then there is an open subinterval $J \subseteq I$ and an $\epsilon > 0$ such that $J \times [0, \epsilon]$ is disjoint from X.*

Proof. Let $\pi : I \times \mathbb{R}_{>0} \to \mathbb{R}$ be the projection onto the first coordinate. By Fact 1.5 $\pi(X)$ is D_Σ. Therefore $\pi(X)$ either has interior or is nowhere dense by SBCT. If $\pi(X)$ is nowhere dense, then there is an open subinterval $J \subseteq I$ that is disjoint from $\pi(X)$. For this subinterval J we get that $J \times \mathbb{R}_{\geq 0}$ is disjoint from X.
Now suppose $\pi(X)$ has interior. Let I' be an open subinterval of I contained in $\pi(X)$. After replacing I with I' and X with $X \cap [I' \times \mathbb{R}]$, we may suppose that $\pi(X) = I$. Let $\{B_{s,t} : s, t \in \mathbb{R}_{>0}\}$ be a definable family of compact sets witnessing that X is D_Σ. Let

$$C_{s,t} = \pi(X \setminus B_{s,t}) \quad \text{and} \quad D_{s,t} = I \setminus C_{s,t} \quad \text{for all } s, t > 0.$$

As $\pi(X) = I$, we have $x \in D_{s,t}$ if and only if $X_x \subseteq (B_{s,t})_x$. Since each X_x is finite, every $x \in I$ is contained in some $D_{s,t}$. Thus $\bigcup_{s,t} D_{s,t} = I$. By the classical Baire Category Theorem there are $s, t \in \mathbb{R}_{>0}$ such that $D_{s,t}$ is somewhere dense. Fix such s and t. Then $X \setminus B_{s,t}$ is the intersection of a D_Σ set by an open set and is thus D_Σ. Thus $C_{s,t}$ is D_Σ as well. Because $D_{s,t}$ is somewhere dense and the complement of a D_Σ set, $D_{s,t}$ has interior by SBCT. Let J be an open subinterval whose closure is contained in the interior of $D_{s,t}$. Then

$$X \cap [\mathrm{Cl}(J) \times \mathbb{R}_{>0}] = B_{s,t} \cap [\mathrm{Cl}(J) \times \mathbb{R}_{>0}].$$

As $\mathrm{Cl}(J) \times \{0\}$ and $B_{s,t}$ are disjoint compact subsets of $\mathbb{R}^2$, there is an $\epsilon > 0$ such that no point in $B_{s,t}$ lies within distance ϵ of any point in $\mathrm{Cl}(J) \times \{0\}$. For such an ε the set $J \times [0, \epsilon]$ is disjoint from $B_{s,t}$, and thus disjoint from X. $\qquad\square$

Definition 3.2. A subset D of $\mathbb{R}_{>0}$ is a **sequence set** if it is bounded and discrete with closure $D \cup \{0\}$.

It is easy to see that $(D, >)$ has order type ω when D is sequence set. By [28, Lemma 3.2] $\mathcal{R}$ either defines a sequence set or every bounded nowhere dense definable subset of $\mathbb{R}$ is finite.

Lemma 3.3. *Let D be a definable sequence set and $X \subseteq D \times \mathbb{R}$ be definable such that X_d is nowhere dense for each $d \in D$. Then $\bigcup_{d \in D} X_d$ is nowhere dense.*

Proof. Set

$$Y := \{(d, x) \in D \times \mathbb{R} \ : \ \exists e \in D \ e \geq d \wedge (e, x) \in X\}.$$

As $(D, >)$ has order type ω, the set $\{e \in D \ : \ d \leq e\}$ is finite for every $d \in D$. Therefore Y_d is nowhere dense for every $d \in D$. Because $Y_d \subseteq Y_e$ when $d \geq e$, the family $\{Y_d \ : \ d \in D\}$ is increasing. By [28, Lemma 3.3] $\bigcup_{d \in D} Y_d$ is nowhere dense. It follows directly that $\bigcup_{d \in D} X_d$ is nowhere dense. $\qquad\square$

3.2. **Proof of Theorem C.** Throughout this subsection $\mathcal{R}$ is type A. Let $I = (a, b)$, $f : I \to \mathbb{R}$, and $h = (h_1, \ldots, h_k) \in \mathbb{R}^k$. We define the **generalized k-th difference of** f as follows:

$$\Delta^0 f(x) := f(x),$$

and for $k \geq 1$

$$\Delta_h^k f(x) := \Delta_{(h_1,\ldots,h_{k-1})}^{k-1} f(x + h_k) - \Delta_{(h_1,\ldots,h_{k-1})}^{k-1} f(x).$$

Observe that $\Delta_{(l_1,l_2)}^k f(x) = \Delta_{l_2}^1 \Delta_{l_1}^{k-1} f(x)$ whenever $l_1 \in \mathbb{R}^{k-1}$ and $l_2 \in \mathbb{R}$. Note that for given h, the function $\Delta_h^k f$ is defined on the interval $(a, b - k\|h\|)$.

In the proof of the o-minimal case of Theorem C in [31], one only has to consider the usual k-th difference (that is the case when $h_1 = \cdots = h_n$). Our proof of Theorem C however depends crucially on allowing the h_i to differ. The reason for this difference between the two proofs is that we do not have an analogue of the o-minimal cell decomposition theorem for type A expansions.

Let J be an open subinterval of I and $k \in \mathbb{N}$. A tuple $(u, x) \in \mathbb{R}_{\geq 0}^k \times J$ is (J, k)-**suitable** if $x + k\|u\| \in J$. We denote the set of such pairs by $S_{J,k}$. Note that $S_{J,k}$ is open and $\Delta_h^k f(x)$ is defined for each $(h, x) \in S_{J,k}$.

The following fact about generalized k-th differences follows easily by applying induction to k. We leave the details to the reader.

Lemma 3.4. *Let $k \in \mathbb{N}$, $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{k-1}$ and $x \in \mathbb{R}$ such that $(h, x) \in S_{I,k}$. Let $f, g : I \to \mathbb{R}$ be two functions. Then*

(1) $\Delta_{(h_1,h_2)}^k f(x) = \Delta_{h_2}^{k-1} \Delta_{h_1}^1 f(x)$, *and*
(2) $\Delta_h^k (f + g)(x) = \Delta_h^k f(x) + \Delta_h^k g(x)$.

Definition 3.5. We say H_k^f **holds on** J if either

- $\Delta_{(h,\ldots,h)}^k f(x) \geq 0$ for all $(x, h) \in J \times \mathbb{R}_{>0}$ with $((h,\ldots,h), x) \in S_{J,k}$ or
- $\Delta_{(h,\ldots,h)}^k f(x) \leq 0$ for all $(x, h) \in J \times \mathbb{R}_{>0}$ with $((h,\ldots,h), x) \in S_{J,k}$.

If $I = (a, b)$ and $(x, h) \in I \times \mathbb{R}_{>0}$, then $((h,\ldots,h), x) \in S_{I,k}$ if and only if $a < x < x + kh < b$.

As in [31] our proof of Theorem C is based on the following theorem of Boas and Widder.

Fact 3.6 ([4, Theorem]). *Let $f : I \to \mathbb{R}$ be continuous and $k \geq 2$. If H_k^f holds on I, then $f^{(k-2)}$ exists and is continuous on I.*

Before proving Theorem C we establish Lemma 3.7. Loosely speaking, it states that in order to show that the generalized k-th difference is non-negative on a given set, it is enough to prove that the k-th difference is non-negative on a subset whose projections onto the first coordinate is a sequence set.

Lemma 3.7. *Let $f : J \to \mathbb{R}$ be a continuous definable function and D be a definable sequence set. If $\Delta_{(d,h)}^k f(x) \geq 0$ for all $((d, h), x) \in S_{J,k} \cap (D \times \mathbb{R}^{k-1}) \times J$, then $\Delta_u^k f(x) \geq 0$ for all $(u, x) \in S_{J,k}$.*

Proof. By continuity of f and openness of $S_{J,k}$, it is enough to show that $\{(u, x) \in S_{J,k} : \Delta_u^k f(x) \geq 0\}$ is dense in $S_{J,k}$. Let $U \subseteq S_{J,k}$ be open. Let $(u_1, u_2, x) \in U$, where $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}^{k-1}$. Because D is a sequence set, there are $n \in \mathbb{N}$ and $d_1, \ldots, d_n \in D$ such that $(\sum_{i=1}^n d_i, u_2, x) \in U$. It is left to show the following claim:

For every $j \in \{1, \ldots, n\}$, $(\sum_{i=1}^{j} d_i, u_2, x) \in S_{J,k}$ and $\Delta_{(\sum_{i=1}^{j} d_i, u_2)}^{k} f(x) \geq 0$.

First observe that since $\sum_{i=1}^{j} d_i < \sum_{i=1}^{n} d_i$ and $(\sum_{i=1}^{n} d_i, u_2, x) \in S_{J,k}$, we have $(\sum_{i=1}^{j} d_i, u_2, x) \in S_{J,k}$. We now show the second statement of the claim by applying induction to j. For $j = 1$, $\Delta_{(d_1, u_2)}^{k} f(x) \geq 0$ by our assumptions on D. So now let $j > 1$ and suppose $\Delta_{(\sum_{i=1}^{j-1} d_i, u_2)}^{k} f(x) \geq 0$. Since $(\sum_{i=1}^{j} d_i, u_2, x) \in S_{J,k}$, it follows immediately that $(d_j, u_2, x + \sum_{i=1}^{j-1} d_i) \in S_{J,k}$. Thus $\Delta_{(d_j, u_2)}^{k} f(x + \sum_{i=1}^{j-1} d_i) \geq 0$ by our assumption on D. Applying Lemma 3.4 and using our induction hypothesis we obtain

$$
\begin{aligned}
\Delta_{(\sum_{i=1}^{j} d_i, u_2)}^{k} f(x) &= \Delta_{u_2}^{k-1} \Delta_{\sum_{i=1}^{j} d_i}^{1} f(x) = \Delta_{u_2}^{k-1} \left(f\left(x + \sum_{i=1}^{j} d_i \right) - f(x) \right) \\
&= \Delta_{u_2}^{k-1} \left(f\left(x + \sum_{i=1}^{j} d_1 \right) - f\left(x + \sum_{i=1}^{j-1} d_i \right) + f\left(x + \sum_{i=1}^{j-1} d_i \right) - f(x) \right) \\
&= \Delta_{u_2}^{k-1} \left(\Delta_{d_j}^{1} f\left(x + \sum_{i=1}^{j-1} d_i \right) + \Delta_{\sum_{i=1}^{j-1} d_i}^{1} f(x) \right) \\
&= \Delta_{u_2}^{k-1} \Delta_{d_j}^{1} f\left(x + \sum_{i=1}^{j-1} d_i \right) + \Delta_{u_2}^{k-1} \Delta_{\sum_{i=1}^{j-1} d_i}^{1} f(x) \\
&= \Delta_{(d_j, u_2)}^{k} f\left(x + \sum_{i=1}^{j-1} d_i \right) + \Delta_{(\sum_{i=1}^{j-1} d_i, u_2)}^{k} f(x) \geq 0.
\end{aligned}
$$

$\square$

Proof of Theorem C. Let $f : I \to \mathbb{R}$ be definable and continuous. Let $a, b \in \mathbb{R}$ be such that $I = (a, b)$. We show that for every $k \in \mathbb{N}$ there is a definable open dense subset $U \subseteq I$ such that f is C^k on U.

We first treat the case when $\mathcal{R}$ defines a sequence set D. By Fact 3.6 it is enough to show that for every $k \in \mathbb{N}$ there is a definable open dense subset $U \subseteq I$ such that for every connected component J of U

- $\Delta_h^k f(x) \geq 0$ for all $(h, x) \in S_{J,k}$, or
- $\Delta_h^k f(x) \leq 0$ for all $(h, x) \in S_{J,k}$.

We proceed by induction on k. The case $k = 0$ follows immediately from the weak monotonicity theorem for type A structures (see [17, Fact 3.3]).

Let $k > 0$. Observe that for $d \in D$, $\Delta_{d,h}^{k} f = \Delta_h^{k-1} \Delta_d^1 f$ and that $\Delta_d^1 f$ is defined on the interval $(a, b - d)$. By the induction hypothesis we find for every $d \in D$ a dense definable open set $U_d \subseteq (a, b - d)$ such that for each connected component J of U_d, either $\Delta_h^{k-1} \Delta_d^1 f(x) \geq 0$ for all $(h, x) \in S_{J,k-1}$, or $\Delta_h^{k-1} \Delta_d^1 f(x) \leq 0$ for all $(h, x) \in S_{J,k-1}$. For $d \in D$ set

$$
X_d = \left((a, b - d) \setminus U_d \right) \cup \{b - d\}.
$$

By Lemma 3.3 $\bigcup_{d \in D} X_d$ is nowhere dense. Set $U := I \setminus \text{Cl}(\bigcup_{d \in D} X_d)$ and observe that U is definable, open, and dense in I. Let J be a connected component of U. Then for each $d \in D$, either

(i) $(a, b - d) \cap J = \emptyset$ or
(ii) $J \subseteq (a, b - d)$ and one of the following is true:
    (a) $\Delta_h^{k-1} \Delta_d^1 f(x) \geq 0$ for all $(h, x) \in S_{J,k-1}$, or
    (b) $\Delta_h^{k-1} \Delta_d^1 f(x) \leq 0$ for all $(h, x) \in S_{J,k-1}$.

Since D is a sequence set, there are infinitely many $d \in D$ for which (ii) holds. Denote the set all such $d \in D$ by D'. Let

$$D'' := \{d \in D' \ : \ \Delta_h^{k-1} \Delta_d^1 f(x) \geq 0 \text{ for all } (h, x) \in S_{J,k-1}\}.$$

Then either $D' \setminus D''$ is infinite or D'' is infinite. Suppose D'' is infinite. We now want to show that $\Delta_u^k f(x) \geq 0$ for all $(u, x) \in S_{J,k}$. By Lemma 3.7 it is enough to show that $\Delta_{d,h}^k f(x) \geq 0$ for all $((d, h), x) \in S_{J,k} \cap (D'' \times \mathbb{R}^{k-1}) \times J$. Let $(d, h, x) \in S_{J,k} \cap (D'' \times \mathbb{R}^{k-1}) \times J$. By definition of $S_{J,k}$, we get that $x + k\|(d, h)\| \in J$. Thus $x + (k-1)\|h\| \in J$ and hence $(h, x) \in S_{J,k-1}$. Since $d \in D''$, we get

$$\Delta_{d,h}^k f(x) = \Delta_h^{k-1} \Delta_d^1 f(x) \geq 0.$$

The case when $D' \setminus D''$ is infinite may be handled similarly.

We now suppose $\mathcal{R}$ does not define a sequence set. Let $f : I \to \mathbb{R}$ be continuous and definable, and let $k \in \mathbb{N}$. We will show that f is C^k outside a definable nowhere dense subset of I. By [28, Lemma 3.2] every bounded nowhere dense definable subset of $\mathbb{R}$ is finite. Set

$$S := \{(x, h) \in I \times \mathbb{R}_{>0} \ : \ (h, \ldots, h, x) \in S_{I,k+2}\}.$$

and

$$V_1 := \{(x, h) \in S \ : \ \Delta_{(h,\ldots,h)}^{k+2} f(x) \geq 0\}$$
$$V_2 := \{(x, h) \in S \ : \ \Delta_{(h,\ldots,h)}^{k+2} f(x) \leq 0\}.$$

Observe that S is open and both V_1 and V_2 are closed in S. Let $W := S \setminus (\text{Int}\, V_1 \cup \text{Int}\, V_2)$. Then W is D_Σ. Since $W \subseteq (V_1 \setminus \text{Int}\, V_1) \cup (V_2 \setminus \text{Int}\, V_2)$, W is nowhere dense and therefore has no interior. It follows immediately from [17, Fact 2.9(1) & Proposition 5.7] that $\dim W \leq 1$. Let $\pi : I \times \mathbb{R}_{>0} \to I$ be the coordinate projection onto I. Consider

$$Y := \{x \in I \ : \ \dim W_x \geq 1\}.$$

By [17, Fact 2.14(2)] Y is D_Σ. By [17, Theorem 3] $\dim Y = 0$, so Y is nowhere dense. Now let U be the complement of $\text{Cl}(Y)$ in I. From the definition of Y we get that $\dim W_x = 0$ for all $x \in U$. In particular, each W_x is nowhere dense and hence finite. Consider

$$Z := \{x \in U \ : \ \forall \delta, \epsilon > 0 \ (x - \delta, x + \delta) \times (0, \epsilon) \cap W \neq \emptyset\}.$$

We will show that Z is nowhere dense. Suppose J is an open subinterval of I in which Z is dense. Observe that $(J \times \mathbb{R}_{>0}) \cap W$ is D_Σ. Applying Lemma 3.1 to this set we get a subinterval $J' \subseteq J$ and an $\epsilon > 0$ such that $J' \times (0, \epsilon)$ is disjoint from W. This contradicts the density of Z in J. Thus Z is nowhere dense.

Let U' be the complement of $\text{Cl}(Z)$. Let $x \in U'$. As $x \notin Z$, there are $\delta, \epsilon > 0$ such that $(x - \delta, x + \delta) \times (0, \epsilon) \cap W = \emptyset$. It follows from connectedness that $(x - \delta, x + \delta) \times (0, \epsilon)$ is contained in $\text{Int}(V_1)$ or $\text{Int}(V_2)$. If necessary decrease δ such that $2\delta < (k + 2)\epsilon$. Then it is easy to check that H^f_{k+2} holds on $(x - \delta, x + \delta)$. By Fact 3.6 the function f is C^k on $(x - \delta, x + \delta)$. $\square$

3.3. **Corollaries.** In this subsection we no longer suppose $\mathcal{R}$ is type A. Corollary 3.8 generalizes a result in [35] from expansions of $(\mathbb{R}, <, +, \cdot)$ to expansions of $(\mathbb{R}, <, +)$.

Corollary 3.8. *The following are equivalent:*

(1) *for every $k \in \mathbb{N}$, every definable $f : I \to \mathbb{R}$ is generically C^k,*
(2) *every definable $f : I \to \mathbb{R}$ is generically continuous,*
(3) *every definable subset of $\mathbb{R}$ has interior or is nowhere dense.*

Proof. Note that (1) directly implies (2). If A is dense and co-dense in $[0, 1]$, then the characteristic function of A is nowhere continuous, so (2) implies (3). We show that (3) implies (1). Suppose $\mathcal{R}$ satisfies (3). As a dense ω-orderable set is dense and co-dense in an interval, (3) implies that $\mathcal{R}$ is type A. The proof of Theorem 3.3 in [35] shows that (2) holds. An application of Theorem C yields (1). $\square$

The next corollary states that when given a nowhere C^k function f, the complexity of definable sets in $(\mathbb{R}, <, +, f)$ depends on how differentiable f is.

Corollary 3.9. *Let $f : I \to \mathbb{R}$ be nowhere C^k for some k.*

(1) *If f is continuous, then $(\mathbb{R}, <, +, f)$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$.*
(2) *If f is C^1, then $(\mathbb{R}, <, +, f)$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.*
(3) *If f is C^2, then $(\mathbb{R}, <, +, f)$ is type C.*

Proof. Suppose f is continuous. Let $\mathcal{S} := (\mathbb{R}, <, +, f)$. Because f is nowhere C^k, $\mathcal{S}$ admits a dense ω-orderable set by Theorem C. If $\mathcal{S}$ is of field-type, then $\mathcal{R}$ is type C by Fact 1.2. Thus in this case the conclusions of (1)-(3) all hold. We assume $\mathcal{S}$ is not of field-type. As f is nowhere C^k it is nowhere locally affine. Then (1) and (2) follow from Corollary B. If f is nowhere locally affine and C^2, then $\mathcal{R}$ is of field-type by Theorem E. A contradiction. $\square$

4. Proof of Theorem A

In this section we will prove Theorem A. As outlined in the introduction we handle the cases of type A and type B expansions separately, because in both situation we will prove different strengthenings of Theorem A.

4.1. **The type A case.** The following theorem implies Theorem A for type A expansions.

Theorem 4.1. *Suppose $\mathcal{R}$ is type A. The following are equivalent:*

(1) *$\mathcal{R}$ defines a C^2 non-affine function $f : I \to \mathbb{R}$.*
(2) *$\mathcal{R}$ is of field-type.*
(3) *$\mathcal{R}$ defines a closed family of curves that is normal of dimension ≥ 2.*
(4) *$\mathcal{R}$ admits a D_Σ family of curves that is normal of dimension ≥ 2.*

(5) $\mathcal{R}$ defines a continuous function $f : I \times J \times L \to \mathbb{R}$ such that
$$\dim\{z \in L : f(x, y, z) = f(x', y', z)\} = 0$$
for all distinct $(x, y), (x', y') \in I \times J$.
(6) $\mathcal{R}$ defines a continuous nowhere locally affine $f : I^k \to \mathbb{R}$ for some k.
(7) $\mathcal{R}$ defines a continuous nowhere locally affine $f : I \to \mathbb{R}$.
(8) $\mathcal{R}$ defines a strictly convex function $f : I \to \mathbb{R}$.
(9) $\mathcal{R}$ defines a continuous non-repetitious function $f : I \to \mathbb{R}$.

The proof is broken up into a series of lemmas. Recall $f : I \to \mathbb{R}$ is **strictly convex** if
$$\frac{f(y) - f(x)}{y - x} < \frac{f(y') - f(x')}{y' - x'} \quad \text{for all} \quad x, y, x', y' \in I, x < y \le x' < y'.$$
If f is differentiable and f' is strictly increasing, then strict convexity of f follows by the mean value theorem. Strictly convex functions are well known to be continuous. The next lemma follows by basic analysis, we leave the details to the reader.

Lemma 4.2. If $f : I \to \mathbb{R}$ is C^2 and non-affine, then there is an open subinterval $J \subseteq I$ such that f or $-f$ is strictly convex on J.

Lemma 4.3. Let $f : I \to \mathbb{R}$ be either strictly convex, or C^2 and non-affine. Then f is not repetitious.

Proof. By Lemma 4.2 we can directly reduce to the case when f is strictly convex. Let $a, b \in \mathbb{R}$ be such that $I = [a, b]$. Since f is strictly convex, then for all elements $x, y \in I$ with $x < y$ and $\epsilon > 0$
$$f(x + \epsilon) - f(x) < f(y + \epsilon) - f(y) \quad \text{whenever } 0 < \epsilon < y - x, b - y.$$
Thus f is not repetitious. $\qquad\square$

Lemma 4.4. Suppose $\mathcal{R}$ is type A. Let $f : I \to \mathbb{R}$ be continuous and definable. Then f is repetitious if and only if it is generically locally affine.

Proof. Because any affine function is repetitious, it follows easily that a generically locally affine $f : I \to \mathbb{R}$ is repetitious. Suppose f is not generically locally affine. By Theorem C there is an non-empty open subinterval J of I such that that f is C^2 and non-affine on J. By Lemma 4.3 the function f is not repetitious. $\qquad\square$

Lemma 4.5. Suppose that $\mathcal{R}$ is type A and that $\mathcal{R}$ admits a normal D_Σ family of curves of dimension at least 2. Then there is a continuous definable $f : I \times J \times L \to \mathbb{R}$ such that
$$\dim\{z \in L : f(x, y, z) = f(x', y', z)\} = 0$$
for all distinct $(x, y), (x', y') \in I \times J$.

Proof. Let $\{A_x : x \in \mathbb{R}^l\}$ be a D_Σ family of curves that is normal of dimension at least 2. Let
$$B = \{x \in \mathbb{R}^l : A_x \ne \emptyset\} \quad \text{and} \quad A = \{(x, y) \in B \times \mathbb{R}^2 : y \in A_x\}.$$
We first show that we can reduce to the case when B is an open subset of $\mathbb{R}^2$. Note that B is the projection of A onto $\mathbb{R}^l$ and is hence D_Σ. As $\dim(B) \ge 2$ there is a coordinate projection $\rho : \mathbb{R}^l \to \mathbb{R}^2$ such that $\rho(B)$ has interior by Fact 1.8. Without loss of generality we can assume that ρ is the projection onto the first two

coordinates. By D_Σ-selection (see Fact 1.7) we obtain a definable open $V \subseteq \rho(B)$ and a continuous definable $g : V \to \mathbb{R}^{l-2}$ such that $(p, g(p)) \in B$ for all $p \in V$. Set

$$A' := \{(p, q) \in V \times \mathbb{R}^2 \; : \; (p, g(p), q) \in A\}.$$

Then $A'_p = A_{(p,g(p))}$ for all $p \in V$. Because A' is the preimage of A under a continuous definable map, A' is D_Σ by Fact 1.5. After replacing A with A' and B with V, we may suppose that B is an open subset of $\mathbb{R}^2$.

Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate projections onto the first and second coordinates, respectively. As $\dim A_x = 1$, we have that for every $x \in B$ either $\pi_1(A_x)$ or $\pi_2(A_x)$ has interior by Fact 1.8. For $i = 1, 2$, we set

$$A_i := \{x \in B : \dim \pi_i(A_x) \geq 1\}.$$

It follows from [17, Fact 2.14] that A_i is D_Σ for $i \in \{1, 2\}$. As $B = A_1 \cup A_2$, either A_1 or A_2 is somewhere dense in B. By SBCT either A_1 or A_2 has interior in B. Let us assume that A_1 has interior. The case that A_2 has interior can be handled similarly. After replacing B with a definable nonempty open subset of A_1 we may suppose that every $\pi_1(A_x)$ has interior for every $x \in B$. Let $\rho : \mathbb{R}^4 \to \mathbb{R}^3$ be the projection onto the first three coordinates. Note that $\rho(A)_x = \pi_1(A_x)$ for all $x \in \mathbb{R}^2$. Thus $\rho(A)_x$ has interior for all $x \in B$. Since $\rho(A)$ is D_Σ, we that $\dim \rho(A) = 3$ by [17, Theorem E(3)]. Thus $\rho(A)$ has interior by Fact 1.8.

Let $I, J, L \subseteq \mathbb{R}$ be nonempty open intervals such that $I \times J \times L \subseteq \rho(A)$. Thus for all $(x, y, z) \in I \times J \times L$ there is a $q \in \mathbb{R}$ such that $(z, q) \in A_{(x,y)}$. Applying D_Σ-selection and replacing I, J, L with smaller nonempty open intervals if necessary we obtain a continuous definable $f : I \times J \times L \to \mathbb{R}$ such that $(z, f(x, y, z)) \in A_{(x,y)}$ for all $(x, y, z) \in I \times J \times L$. By normality of $\{A_{(x,y)} : (x, y) \in I \times J\}$ we immediately have that for all distinct $(x, y), (x', y') \in I \times J$

$$\dim\{z \in L : f(x, y, z) = f(x', y', z)\} = 0.$$

$\square$

In what follows a **box** is a subset of $\mathbb{R}^k$ given as a product of k nonempty open intervals. Let $\pi_{-i} : \mathbb{R}^k \to \mathbb{R}^{k-1}$ be the projection away from the i-th coordinate. Given a box $W = J_1 \times \ldots \times J_k$, $f : W \to \mathbb{R}$, $1 \leq i \leq k$, and $x = (x_1, \ldots, x_{k-1}) \in \pi_{-i}(W)$ we let $f_x^i : J_i \to \mathbb{R}$ be given by

$$f_x^i(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots x_{k-1}) \quad \text{for all } t \in J_i.$$

We recall two basic facts from analysis.

Fact 4.6. *Let $W, J_1, \ldots, J_k, f$ be as above. If $f_x^i : J_i \to \mathbb{R}$ is affine for all $1 \leq i \leq k$ and $x \in \pi_{-i}(W)$, then f is affine.*

Fact 4.7. *A continuous $g : J \to \mathbb{R}$ is affine if and only if*

$$\frac{g(x) + g(y)}{2} = g\left(\frac{x + y}{2}\right) \quad \text{for all } x, y \in J.$$

Lemma 4.8. *Suppose $\mathcal{R}$ is type A. If every continuous definable $f : I \to \mathbb{R}$ is generically locally affine, then every continuous definable $f : I^k \to \mathbb{R}$ is generically locally affine.*

Proof. By Fact 4.6 it is enough to find for every $1 \leq i \leq k$ an open dense definable subset U_i of I^k such that every $z \in U_i$ is contained in a box $W = J_1 \times \ldots \times J_k$ such that f_x^i is affine on J_i for all $x \in \pi_{-i}(W)$. We assume $i = 1$ as the general case follows in the same way. For $\delta > 0$ we define E_δ to be the set of all $(t, z) \in I \times I^{k-1}$ such that $(t - \delta, t + \delta)$ is a subset of I on which f_z^1 is affine. Note that $E_\delta \subseteq E_{\delta'}$ when $\delta' < \delta$. By Fact 4.7 the restriction f_z^1 to $(t - \delta, t + \delta)$ is affine if and only if

$$\frac{f_z^1(x) + f_z^1(y)}{2} = f_z^1\left(\frac{x+y}{2}\right) \quad \text{for all } x, y \in (t - \delta, t + \delta).$$

Thus $\{E_\delta : \delta > 0\}$ is a definable family. Continuity of f implies each E_δ is closed. Let E be the union of $\{E_\delta : \delta > 0\}$. Then E is D_Σ and E is the set of $(t, z) \in I \times I^{k-1}$ such that f_z^1 is locally affine at t. By our assumption, E is dense in $I \times \{z\}$ for all $z \in I^{k-1}$. Hence E is dense in I^k. By SBCT the interior of E is dense in I^k. Thus every $z \in \text{Int}(E)$ is contained in a box $W = J_1 \times \ldots \times J_k$ such that f_x^1 is affine on J_1 for all $x \in \pi_{-1}(W)$. □

Proof of Theorem 4.1. Fact 1.1 shows that (1) implies (2).

We show that (2) implies (3). Suppose $\mathcal{R}$ is of field type. Let $\oplus, \otimes : I^2 \to I$ be definable such that $(I, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$. Let $0_I, 1_I$ be the additive and multiplicative identities of $(I, <, \oplus, \otimes)$, respectively. Let

$$A = \{(a, b, x, y) \in I^2 \times I^2 : y = (a \otimes x) \oplus b, 0_I \leq x \leq 1_I\}.$$

It is easy to see that $\{A_{(a,b)} : (a, b) \in I^2\}$ is a closed family of curves that is normal of dimension 2.

Since every closed definable set is D_Σ, (3) implies (4). Lemma 4.5 shows that (4) implies (5).

We now establish that (5) implies (6). Suppose I, J, L and f satisfy the conditions of (5). Towards a contradiction, suppose f is somewhere locally affine. After shrinking I, J, L we suppose f is affine. Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$ be such that

$$f(x, y, z) = c_1 x + c_2 y + c_3 z + c_4 \quad \text{for all } (x, y, z) \in I \times J \times L.$$

We can easily find distinct $(x, y), (x', y') \in I \times J$ such that $c_1 x + c_2 y = c_1 x' + c_2 y'$. It follows that $f(x, y, z) = f(x', y', z)$ for all $z \in L$, contradicting our assumption on I, J, L and f.

Lemma 4.8 shows that (6) implies (7). We show that (7) implies (8). Suppose $f : I \to \mathbb{R}$ is definable and continuous nowhere locally affine. By Theorem C there is an open subinterval J of I such that the restriction of f to J is C^2. As the restriction of f to J is non-affine, either f or $-f$ is strictly convex on J by Lemma 4.2.

Lemma 4.3 shows that (8) implies (9). We finally show that (9) implies (1). Suppose f is continuous, non-reptitious, and definable. Then f is not generically locally affine by Lemma 4.4. Thus there is an open subinterval J of I such that f is not affine on any open subinterval of J. By Theorem C we may suppose, after further shrinking J if necessary, that f is C^2 on J. Thus the restriction of f to J is C^2 and non-affine. □

Various subclasses of the class of type A expansions satisfy stronger forms of
Theorem 4.1. As an example of such a subclass we consider all d-minimal expansions
of $(\mathbb{R}, <, +)$. An expansion $\mathcal{R}$ is **d-minimal** if every definable unary set in every
model of the theory of $\mathcal{R}$ is the union of an open set and finitely many discrete sets.
A d-minimal expansion cannot define a dense and co-dense subset of a nonempty
open interval and is thus type A. By [35, Theorem 3.3] for every function $f : U \to \mathbb{R}$
definable in a d-minimal expansion on an open set U there is an open dense definable
subset V of U such that f is continuous on V. It follows from [35, Theorem 3.4.1]
that if $\mathcal{R}$ is d-minimal, then every definable subset of $\mathbb{R}^k$ is a boolean combination
of open sets. Thus whenever $\mathcal{R}$ is d-minimal, every definable set is D_Σ by Fact 1.5.
With these facts and Theorem 4.1 it is easy to prove the following corollary. We
leave the details to the reader.

Corollary 4.9. *Suppose $\mathcal{R}$ is d-minimal. Then the following are equivalent:*

(1) $\mathcal{R}$ *defines a C^2 non-affine function $f : I \to \mathbb{R}$.*
(2) $\mathcal{R}$ *is of field-type.*
(3) $\mathcal{R}$ *defines a closed family of curves that is normal of dimension ≥ 2.*
(4) $\mathcal{R}$ *defines a family of curves that is normal of dimension ≥ 2.*
(5) $\mathcal{R}$ *defines a function $f : I \times J \times L \to \mathbb{R}$ such that*
$$\dim\{z \in L : f(x, y, z) = f(x', y', z)\} = 0$$
   for all distinct $(x, y), (x', y') \in I \times J$.
(6) $\mathcal{R}$ *defines a nowhere locally affine function $f : I^k \to \mathbb{R}$ for some k.*
(7) $\mathcal{R}$ *defines a nowhere locally affine function $f : I \to \mathbb{R}$.*
(8) $\mathcal{R}$ *defines a strictly convex function $f : I \to \mathbb{R}$.*
(9) $\mathcal{R}$ *defines a non-repetitious function $f : I \to \mathbb{R}$.*

The final corollary of this section is an instance of the Zil'ber trichotomy prin-
ciple over the real numbers that does not rely on any model-theoretic tameness
assumption.

Corollary 4.10. *Let $\{A_x : x \in \mathbb{R}^l\}$ be a normal family of curves of dimension at
least two, $A = \{(x, y) : y \in A_x\}$, and $\Lambda \subseteq \mathbb{R}$ be uncountable. Then $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \Lambda}, A)$ is of field-type.*

Proof. We declare $\mathcal{S} := (\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, A)$. If $\mathcal{S}$ is type A, then $\mathcal{S}$ is of
field-type by Theorem 4.1. If $\mathcal{S}$ is not type A, then it is type C by [17, Theorem C]
and thus of field-type. □

4.2. The type B case. In this section we prove Theorem D, a strengthening of
Theorem A for type B structures.

Lemma 4.11. *Suppose $\mathcal{R}$ is type B. Let D be an ω-orderable set that is dense in
I, let $f : I \to \mathbb{R}$ be definable and continuous, and let $J \subseteq I$ be an open interval on
which f is nonconstant. Then there are $d_1, d_2, d_3, d_4 \in J \cap D$ such that*
$$d_1 \neq d_2, d_3 \neq d_4 \quad and \quad d_1 - d_2 = f(d_3) - f(d_4).$$

Proof. Let $J \subseteq I$ be an open subinterval on which f is not constant. Let $a, b \in \mathbb{R}$
be such that $J = (a, b)$. The intermediate value theorem yields an open interval
$J' \subseteq f(J)$. Since D is dense in I, we have that $f(D \cap J) \cap J'$ is dense in J'. Let
$a', b' \in \mathbb{R}$ be such that $(a', b') = J'$. After decreasing b' we assume $b' - a' < b - a$.
Then both $(-a + D) \cap (0, b' - a')$ and $-a' + (f(D \cap J) \cap J')$ are dense ω-orderable

subsets of $(0, b' - a')$. By Fact 1.4, there are $d_1, d_2, d_3, d_4 \in J$ such that $d_1 \neq d_2$, $d_3 \neq d_4$ and

$$d_1 - d_2 = (-a + d_1) - (-a + d_2) = -a' + f(d_3) - \big(- a' + f(d_4)\big) = f(d_3) - f(d_4).$$

$\square$

Proof of Theorem D. Suppose $f : I \to \mathbb{R}$ is continuous and definable. We need to show that f is repetitious. If f is constant on some open subinterval of I, then f is repetitious. We may therefore assume that there is no open subinterval on which f is constant. Since $\mathcal{R}$ is type B, there is a dense ω-orderable subset D of I. We declare

$$C := \{(d_1, d_2, d_3, d_4) \in D^4 \ : \ d_1 \neq d_2, d_3 \neq d_4\}.$$

For $d = (d_1, d_2, d_3, d_4) \in C$, let $A_d \subseteq \mathbb{R}$ be the set of all $t \in \mathbb{R}$ such that

- $t + d_3, t + d_4 \in I$, and
- $d_1 - d_2 = f(t + d_3) - f(t + d_4)$.

For each $d \in C$, the set A_d is closed in I by continuity of f. Let $s > 0$ and J be an open subinterval of I such that $t + J \subseteq I$ for all $t \in (0, s)$. Let $r \in (0, s)$. Consider the function $g_r : I \to \mathbb{R}$ that maps $c \in I$ to $f(r + c)$. Applying Lemma 4.11 to g_r we obtain a $d = (d_1, d_2, d_3, d_4) \in C$ such that:

$$d_1 - d_2 = g_r(d_3) - g_r(d_4) = f(r + d_3) - f(r + d_4).$$

Thus $r \in A_d$. Therefore $(0, s) \subseteq \bigcup_{d \in C} A_d$. By the Baire Category Theorem A_d has interior for some $d \in C$. Fix such a $d = (d_1, d_2, d_3, d_4) \in C$ and let J' be an open interval in the interior of A_d. Then the function $J' \to \mathbb{R}$ given by $t \mapsto f(t + d_3) - f(t + d_4)$ is constant. The statement of the Theorem follows. $\square$

4.3. **Interpreting fields and defining groups.** In this section we will collect a few interesting corollaries about algebraic structures in expansions by continuous nowhere locally affine functions.

Corollary 4.12. *Let $f : I \to \mathbb{R}$ be continuous and nowhere locally affine. Then one of the following holds:*

(1) $(\mathbb{R}, <, +, f)$ *is of field-type.*
(2) $(\mathbb{R}, <, +, f)$ *defines* $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$.

Proof. If $(\mathbb{R}, <, +, f)$ is type B, then (2) holds by [28, Theorem A]. Thus we can assume that $\mathcal{R}$ is type A. By Theorem C there is an open interval $J \subseteq I$ such that the restriction of f to J is C^2. Since f is nowhere locally affine, $(\mathbb{R}, <, +, f)$ is of field-type by Fact 1.1. $\square$

It is natural to ask whether Corollary 4.12 can be strengthened.

Question 4.13. *Let $f : I \to \mathbb{R}$ be continuous and nowhere locally affine. Does $(\mathbb{R}, <, +, f)$ interpret an infinite field?*

By inspection of the proof of Corollary 4.12, Question 4.13 boils down to the question whether every type B expansions that defines a continuous nowhere locally affine function interprets an infinite field. So far we do not even know whether there is a type B expansion that defines such a function. In Section 6.4 we describe a type B expansion $\mathcal{T}_r$ that is bi-interpretable with $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. By Zaid, Grädel, Kaiser, and Pakusa [43] it is known that $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$ does not admit a parameter-free interpretation of an infinite field. Therefore showing the existence

of a continuous nowhere locally affine function definable in $\mathcal{T}_r$ would likely result in a negative answer to Question 4.13.

Eleftheriou and Starchenko [12, 13] showed that every $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$-definable group is virtually abelian. Thus an expansion can define all affine functions without defining any non-virtually abelian group. In contrast:

Proposition 4.14. *Let $f : I \to \mathbb{R}$ be continuous and nowhere locally affine. Then $(\mathbb{R}, <, +, f)$ defines a non-virtually solvable group.*

Proposition 4.14 requires two easy group-theoretic lemmas. We leave the proof of Lemma 4.15 to the reader.

Lemma 4.15. *Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of finite groups, G be the direct product of the G_i, and $\pi_i : G \to G_i$ be the projection onto the ith coordinate for all i. If H is a finite index subgroup of G, then $\pi_i(H) = G_i$ for all but finitely many $i \in \mathbb{N}$.*

For the following, let S be any finite non-solvable group such as the alternating group A_5.

Lemma 4.16. *The direct product of countably many copies of S is not virtually solvable.*

Proof. Let $(G_i)_{i \in \mathbb{N}}$ be given by $G_i = S$ for all i. Let G be the direct product of these G_i, and $\pi_i : G \to G_i$ be the projection onto the ith coordinate for all $i \in \mathbb{N}$. Suppose H is a solvable subgroup of G. Then $\pi_i(H)$ is solvable for all i. As S is not solvable, we have $\pi_i(H) \neq G_i$ for all i. Lemma 4.15 shows that H has infinite index in G. $\square$

Proof of Proposition 4.14. Let $\mathfrak{B} := (\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. By Corollary 4.12 it suffices to show that $(\mathbb{R}, <, +, \cdot)$ and $\mathfrak{B}$ each define a group that is not virtually solvable. It is well known that $GL_2(\mathbb{R})$ is $(\mathbb{R}, <, +, \cdot)$-definable and not virtually solvable. Recall that $\mathfrak{B}$, like any infinite structure, defines every finite group. Suppose G is a $\mathfrak{B}$-definable group isomorphic to S. It is easy to see that the set of all functions $\mathbb{N} \to G$ forms a $\mathfrak{B}$-definable family and that the pointwise group operations on this family are $\mathfrak{B}$-definable. Thus $\mathfrak{B}$ defines the direct product of countably many copies of S. Apply Lemma 4.16. $\square$

5. Weak Poles

In this section we give more restrictions on continuous functions definable in type B expansions. Our results apply to a more general class of expansions, those that do not admit weak poles. A **pole** is a definable homeomorphism between a bounded and an unbounded interval.

Definition 5.1. A **weak pole** is a definable family $\{h_d : d \in E\}$ of continuous maps $h_d : [0, d] \to \mathbb{R}$ such that

  (i) $E \subseteq \mathbb{R}_{>0}$ is closed in $\mathbb{R}_{>0}$ and $(0, \epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$,
  (ii) there is a $\delta > 0$ such that $[0, \delta] \subseteq h_d([0, d])$ for all $d \in E$.

It is to see (and also follows from Theorem 5.6 below) that $\mathcal{R}$ admits a weak pole whenever it defines a pole. To our knowledge weak poles have not been studied before. While we do not know of an expansion that admits a weak pole and is not of field-type, we believe such expansions exist. We first observe that type B expansions do not admit weak poles.

Theorem 5.2. *Suppose $\mathcal{R}$ is type B. Then $\mathcal{R}$ does not define a weak pole.*

Proof. Towards a contradiction, suppose $\mathcal{R}$ defines a dense ω-orderable set $(D, \prec)$ and a weak pole $\{h_d : d \in E\}$. Using Fact 1.3 we will show that $\mathcal{R}$ is type C, contradicting our assumption that $\mathcal{R}$ is type B. After rescaling we may assume that D is dense in $[0,1]$ and $[0,1] \subseteq h_d([0,d])$ for all $d \in E$. Set

$$Z := \{(a,b) \in [0,1]^2 \ : \ a < b\}.$$

Let $\lambda : \mathbb{R}_{>0} \to E$ map x to the maximal element of $(-\infty, x] \cap E$. Let $g : [0,1] \times Z \times D \to D$ map (c,a,b,d) to

$$\begin{cases} d, & \text{if } c - a > \lambda(b-a); \\ \prec\text{-minimal } e \in D_{\preceq d} \text{ s.t. } h_{\lambda(b-a)}(c-a) - e \text{ is minimal}, & \text{otherwise.} \end{cases}$$

We will now show that g satisfies the assumptions of Fact 1.3. For this, let $a, b \in Z$ and $d, e \in D$ with $e \preceq d$. As $[0,1] \subseteq h_{\lambda(b-a)}([0, \lambda(b-a)])$, there is $z \in [0, \lambda(b-a)]$ such that $h_{\lambda(b-a)}(z) = e$. Since $D_{\preceq d}$ is finite and $h_{\lambda(b-a)}$ is continuous, there is an open interval I around z such that for each $y \in I$, e is the only element in $D_{\preceq d}$ such that $h_{\lambda(b-a)}(y) - e$ is minimal. Let $c \in (a,b)$ be such that $c - a = z$. It follows immediately from the argument above that $g(x, a, b, d) = e$ for all $x \in (c+I) \cap (a,b)$. Thus (ii) of Fact 1.3 holds for our choice of g. Therefore $\mathcal{R}$ is type C. $\square$

We now prove several results about continuous definable functions in expansions that do not admit weak poles.

Proposition 5.3. *Suppose $\mathcal{R}$ does not admit a weak pole. Then every definable family $\{f_x : x \in \mathbb{R}^l\}$ of linear functions $[0,1] \to \mathbb{R}$ has only finitely many distinct elements.*

Proof. Let $\{f_x : x \in \mathbb{R}^l\}$ be a definable family of linear functions $[0,1] \to \mathbb{R}$ that has infinitely many distinct elements. After replacing each f_x with $|f_x|$, we may assume that each f_x takes nonnegative values. Let $B = \{f_x(1) : x \in \mathbb{R}^l\}$ and $g : B \times [0,1] \to \mathbb{R}$ be given by $g(\lambda, t) = f_y(t)$ for any $y \in \mathbb{R}^l$ with $f_y(1) = \lambda$. Note that g is definable and $g(\lambda, t) = \lambda t$ for all $(\lambda, t) \in B \times [0,1]$. We declare

$$\tilde{g}(\lambda, t) = \lim_{\lambda' \in B, \lambda' \to \lambda} g(\lambda', t) \quad \text{for all } (\lambda, t) \in \mathrm{Cl}(B) \times [0,1].$$

By continuity we have $\tilde{g}(\lambda, t) = \lambda t$ for all $(\lambda, t) \in \mathrm{Cl}(B) \times [0,1]$. After replacing g by $\tilde{g}$ and B by $\mathrm{Cl}(B)$, we may suppose that B is a closed and infinite subset of $\mathbb{R}_{\geq 0}$. One of the following holds:

- B is unbounded.
- B has an accumulation point.

First suppose B is unbounded. Let $\{h_d : d \in \mathbb{R}_{>0}\}$ be the definable family of functions $h_d : [0,d] \to \mathbb{R}$ given by declaring $h_d(t) = g(\lambda, t)$ where λ is the minimal element of B such that $g(\lambda, d) \geq 1$. Then $h_d(t) \geq d^{-1}t$ for all $t \in [0,d]$. It directly follows that $\{h_d : d \in \mathbb{R}_{>0}\}$ is a weak pole.

Now suppose (2) holds. Let μ be an accumulation point of B. We declare

$$\psi(\lambda, t) := |g(\mu, t) - g(\lambda, t)| = |\mu - \lambda|t \quad \text{for all } \lambda \in B, t \in [0,1].$$

Note that ψ is definable. Set

$$C := \{|\mu - \lambda| : \lambda \in B\} = \{\psi(\lambda, 1) : 0 \leq \lambda \leq 1\}.$$

Observe that C is closed, definable, and contains arbitrarily small positive elements as λ is an accumulation point of B. Let $\{h_d : d \in C\}$ be the definable family of functions $h_d : [0, d] \to \mathbb{R}$ such that h_d is the compositional inverse of $t \mapsto \psi(\lambda, t)$ where $\lambda \in B$ is such that $d = |\mu - \lambda| = \psi(\lambda, 1)$. Then h_d satisfies $h_d(t) = d^{-1}t$. It follows that $\{h_d : d \in C\}$ is a weak pole. $\qquad\square$

Proposition 5.4. *Suppose $\mathcal{R}$ does not define a weak pole. Then every continuous definable $f : I \to \mathbb{R}$ is uniformly continuous.*

Proof. Suppose $f : I \to \mathbb{R}$ is continuous, definable, and not uniformly continuous. We show that $\mathcal{R}$ defines a weak pole. Let $\delta > 0$ be such that for all $\epsilon > 0$ there are $t, t' \in I$ such that $|f(t) - f(t')| \geq \delta$ and $|t - t'| \leq \epsilon$. For every $\epsilon > 0$ let

$$A_\epsilon := \{t \in I : |f(t) - f(t')| \geq \delta \text{ for some } t \leq t' \leq t + \epsilon\}.$$

Note that each A_ϵ is closed in I and nonempty. Let p be a fixed element of I. Let $g_0(\epsilon)$ be the maximal element of $A_\epsilon \cap (\infty, p]$ if $A_\epsilon \cap (\infty, p] \neq \emptyset$ and the minimal element of $A_\epsilon \cap [p, \infty)$ otherwise. Note that $g_0 : \mathbb{R}_{>0} \to I$ is definable. Let $g_1(\epsilon)$ be the least $t' \in [g_0(\epsilon), g_0(\epsilon) + \epsilon]$ such that $|f(g_0(\epsilon)) - f(t')| \geq \delta$. Then $g_1 : \mathbb{R}_{>0} \to I$ is definable and for all $\epsilon > 0$:

$$0 < g_1(\epsilon) - g_0(\epsilon) \leq \epsilon \quad \text{and} \quad |f(g_1(\epsilon)) - f(g_0(\epsilon))| \geq \delta.$$

We consider the definable family of functions $h_\epsilon : [0, g_1(\epsilon) - g_0(\epsilon)] \to \mathbb{R}$ given by

$$h_\epsilon(t) := |f(g_0(\epsilon) + t) - f(g_0(\epsilon))|.$$

Each h_ϵ is continuous. It follows from the intermediate value theorem that $[0, \delta]$ is contained in the image of every h_ϵ. Thus $\{h_\epsilon : \epsilon \in \mathbb{R}_{>0}\}$ is a weak pole. $\qquad\square$

We leave the proof of the next lemma, an easy consequence of the triangle inequality, to the reader.

Lemma 5.5. *A uniformly continuous $f : I \to \mathbb{R}$ on a bounded open interval I is bounded. A uniformly continuous $f : \mathbb{R}_{>0} \to \mathbb{R}$ is bounded above by an affine function.*

Proposition 5.4 and Lemma 5.5 yield:

Corollary 5.6. *Suppose $\mathcal{R}$ does not admit a weak pole. Then every continuous definable function on a bounded interval is bounded, and every continuous definable $f : \mathbb{R}_{>0} \to \mathbb{R}$ is bounded above by an affine function.*

It is natural to ask if continuous functions definable in expansions without weak poles satisfy any strengthing of uniform continuity such as the following.

Question 5.7. *Suppose $\mathcal{R}$ does not admit a weak pole. Is every continuous definable $f : I \to \mathbb{R}$ generically locally Lipschitz?*

If $\mathcal{R}$ is type A and does not admit a weak pole then every continuous definable $f : I \to \mathbb{R}$ is generically locally affine, hence generically locally Lipschitz. Thus it suffices to answer the question for type B expansions. It follows by a result of Chaudhuri, Sankaranarayanan, and Vardi [9, Theorem 10] that every continuous $f : I \to \mathbb{R}$ definable without parameters in the structure $\mathcal{T}_r$ described in Section 6.4 below is Lipschitz. We do not know if this result extends to all $\mathcal{T}_r$-definable continuous functions.

6. Applications

6.1. **An application to descriptive set theory.** We start with an application to descriptive set theory. We assume that the reader is familiar with basic notions of the subject (see Kechris [29] for an introduction). Consider the Polish space $C^k([0,1])$ of all C^k functions $[0,1] \to \mathbb{R}$ equipped with the topology induced by the semi-norms $f \mapsto \max_{t \in [0,1]} |f^{(j)}(t)|$ for $0 \leq j \leq k$. Note that $C^0([0,1])$ is the space of continuous functions $[0,1] \to \mathbb{R}$ equipped with the topology of uniform convergence. We let $C^\infty([0,1])$ be the space of smooth functions with the topology induced by the semi-norms $f \mapsto \max_{t \in [0,1]} |f^{(j)}(t)|$ for $j \in \mathbb{N}$. Grigoriev [22] and later Le Gal [32] constructed a comeager $Z \subseteq C^\infty([0,1])$ such that $(\mathbb{R},<,+,\cdot,f)$ is o-minimal for all $f \in Z$. The corresponding result for $C^k([0,1])$ fails.

Theorem 6.1. *The set of all $f \in C^k([0,1])$ such that $(\mathbb{R},<,+,f)$ is type C, is comeager in $C^k([0,1])$ for any $k \in \mathbb{N}$.*

While it might not be surprising that expansions of $(\mathbb{R},<,+)$ by a generic bounded continuous function are not model-theoretically well behaved, Theorem 6.1 actually shows something stronger: a generic bounded continuous function defines all bounded continuous functions over $(\mathbb{R},<,+)$. Loosely speaking, this means that given two generic functions we can recover one from the other by using finitely many boolean operations, cartesian products, and linear operations.

Sketch of proof of Theorem 6.1. It is well-known that the set of somewhere $(k+1)$-differentiable functions in $C^k([0,1])$ is meager, the case $k=1$ being a classical result of Banach [3]. It therefore suffices to show that the collection of all C^k functions $[0,1] \to \mathbb{R}$ definable in type B expansions is meager. By Theorem D it is enough to prove that the set of reptitious $f \in C^k([0,1])$ is meager. For each $n \geq 1$ let A_n be the set of functions $f \in C^k([0,1])$ such that for some $x,y \in [0,1]$

- $\frac{1}{n} \leq y - x$, and
- $f(x+\epsilon) - f(x) = f(y+\epsilon) - f(y)$ for all $0 < \epsilon \leq \frac{1}{n}$.

Note that every reptitious C^k-function $[0,1] \to \mathbb{R}$ is in some A_n. We show that each A_n is nowhere dense. Let $n \geq 1$. As A_n is a closed subset of $C^k([0,1])$, we only need to show that A_n has empty interior in $C^k([0,1])$. For every $f \in C^k([0,1])$ and $\epsilon > 0$, it is easy to construct a smooth $g : [0,1] \to \mathbb{R}$ such that $g \notin A_n$ and $|f^{(j)}(t) - g^{(j)}(t)| < \epsilon$ for all $0 \leq j \leq k$ and $t \in [0,1]$. Thus A_n has empty interior. $\square$

6.2. **An application to metric dimensions.** The main result in Hieronymi and Miller [27] states that if $\mathcal{R}$ is an expansion of the ordered field of real numbers[3] that does not define $\mathbb{Z}$, then the Assouad dimension of any D_Σ set agrees with its topological dimension. See [33, Definition 3.2] for the definition of the Assouad dimension of a metric space. An important observation for this line of research is that topological dimension is bounded above by Assouad dimension and that all commonly encountered notions of metric dimension are bounded from below by topological dimension above by Assouad dimension. Therefore the coincidence of these two dimensions implies the equality of essentially all studied notions of dimension. We refer the reader to [27] and Luukkainen [33] for more on the relevance and significance of this phenomenon.

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[3]An expansion of the real field is type C if and only if it does not define $\mathbb{Z}$.

The coincidence of dimensions for D_Σ sets does not extend to type A expansions of $(\mathbb{R}, <, +)$. For example, $S = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$ has Assouad dimension 1, topological dimension 0, and $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, S)$ is type A by [17, Theorem B]. However, using our new results we are able to obtain an extension to type A expansions of field-type.

We must clarify one notation before stating our main result. Let e be the Euclidean metric on $\mathbb{R}^n$ and let $X \subseteq \mathbb{R}^n$. When we refer to the Assouad dimension of X, we mean the Assouad dimension of the metric space (X, e_X), where e_X is the restriction of the metric e to X.

Theorem 6.2. *Suppose $\mathcal{R}$ is type A and of field-type. Let X be a bounded D_Σ set. Then the Assouad dimension of X is equal to the topological dimension of X.*

The assumption that X is bounded is necessary. Marker and Steinhorn observed (see [42] for a proof) that $(\mathbb{R}, <, +, \sin)$ is locally o-minimal and hence type A. This structure is also of field-type, as the sine function is C^2 and non-affine, and defines $\pi\mathbb{Z} = \sin^{-1}(\{0\})$. The latter set has topological dimension 0 and Assouad dimension 1.

Proof. Since $\mathcal{R}$ is of field-type, $\mathcal{R}$ defines a C^2 function $f : [0,1] \to \mathbb{R}$ with non-constant derivative. Let $I \subseteq \mathbb{R}$ be an interval and $\oplus, \otimes : I^2 \to I$ be continuous definable functions such that there is an isomorphism $\tau : (I, <, \oplus, \otimes) \to (\mathbb{R}, <, +, \cdot)$. We denote subtraction in the field $(I, \oplus, \otimes)$ by $\ominus : I^2 \to I$ and let $|\cdot|_I$ be the absolute value of the ordered field $(I, <, \oplus, \otimes)$. We suppose that $\oplus, \otimes$ are constructed from f in the same way as in the proof of Theorem 2.1. By inspection of the construction in the proof of Theorem 2.1 we observe that there is a subinterval $J \subseteq I$ such that the restriction of τ agrees with f' on J.

Since τ is strictly increasing, f' is strictly increasing. Thus f'' is positive on J. Since f'' is continuous, we may suppose (after further shrinking J if necessary) that f'' is bounded above and bounded away from zero on J. It follows from the mean value theorem that f' is bi-Lipschitz on J.

Let d be the natural metric on $(I, <, \oplus, \otimes)$ given by

$$d(x, y) = \tau(|x \ominus y|_I) \quad \text{for all } x, y \in I.$$

We now consider the metric space (J, d_J) where d_J is the restriction of d to J. As τ is an ordered field isomorphism which agrees with f' on J we have

$$d(x, y) = |f'(x) - f'(y)| \quad \text{for all } x, y \in J.$$

Then the identity map $(J, e_J) \to (J, d_J)$ is a bi-Lipschitz equivalence because f' is bi-Lipschitz. Let d_n be the metric on J^n given by

$$d_n(x, y) = \max\{d(x_1, y_1), \ldots, d(x_n, y_n)\}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in J^n$. It is easy to see that the identity map $(J^n, d_n) \to (J^n, e_{J^n})$ is also a bi-Lipschitz equivalence.

Let $X \subseteq \mathbb{R}^n$ be a bounded D_Σ set. Let $q \in \mathbb{Q}_{>0}, t \in \mathbb{R}^n$ be such that $qX + t$ is a subset of J^n. Then $qX + t$ has the same Assouad dimension and topological dimension as X, because invertible affine maps are bi-Lipschitz and bi-Lipschitz

equivalences preserve both Assouad and topological dimension. After replacing X with $qX + t$ if necessary we suppose that $X \subseteq J^n$. By Fact 1.8 the topological dimension of the closure of X agrees with the topological dimension of X. It follows directly from the definition of Assouad dimension that taking closures does not raise Assouad dimension of subsets of $\mathbb{R}^n$. It therefore suffices to prove the theorem for closed definable sets. We assume that X is closed. Consider the structure $(I, <, \oplus, \otimes, X)$. This structure is isomorphic via τ to $\mathcal{S} := (\mathbb{R}, <, +, \cdot, \tau(X))$. Since $\mathcal{R}$ is type A, so is $\mathcal{S}$. In particular, $\mathcal{S}$ cannot define $\mathbb{Z}$. Therefore by [27, Theorem A] Assouad dimension and topological dimension agree on $(\tau(X), e_{\tau(X)})$. Since τ is bi-Lipschitz on J, it follows from the definition of d_n that the Assouad and topological dimensions of $(X, (d_n)_X)$ agree. Since $id : (J^n, d_n) \to (J^n, e)$ is bi-Lipschitz, the Assouad and topological dimensions of X agree. $\square$

The reduction to the case when X is closed is necessary as $(I, <, \oplus, \otimes, X)$ need not define a witness that X is D_Σ.

Corollary 6.3. *Let* $f : (0, 1] \to \mathbb{R}$ *be given by* $f(t) = \sin(\frac{1}{t})$. *Then* $(\mathbb{R}, <, +, f)$ *is type C.*

Proof. Note first that $(\mathbb{R}, <, +, f)$ is of field-type as f is C^2 and non-affine. Let

$$Z := \{t \in (0, 1] : f(t) = 0\} = \left\{\frac{1}{n\pi} : n \geq 1\right\}.$$

Then Z is discrete and hence D_Σ. It follows easily from the definition of Assouad dimension that Z has Assouad dimension one. Apply Theorem 6.2. $\square$

Combining Theorem 6.2 with Corollary B and Fact 1.1 we directly obtain:

Corollary 6.4. *The Assouad dimension of any bounded* D_Σ *set agrees with its topological dimension if one of the following conditions holds:*

- $\mathcal{R}$ *defines a continuous nowhere locally affine* $f : I \to \mathbb{R}$ *and does not define* $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +)$.
- $\mathcal{R}$ *defines a* C^1 *non-affine* $f : I \to \mathbb{R}$ *and does not define* $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.
- $\mathcal{R}$ *defines a* C^2 *non-affine* $f : I \to \mathbb{R}$ *and is not type C.*

6.3. **Expansions by Sequences.** We now collect an interesting corollary about expansions by decreasing sequences. For our purposes a **decreasing sequence with decreasing gaps** is a strictly decreasing sequence $(s_n)_{n \in \mathbb{N}}$ with limit zero such that $(s_n - s_{n+1})_{n \in \mathbb{N}}$ is also strictly decreasing. We say that $(s_n)_{n \in \mathbb{N}}$ has **exponential decay** if for some $\lambda < 0$ we have $s_n < \exp(\lambda n)$ for sufficiently large n. Otherwise, we say that $(s_n)_{n \in \mathbb{N}}$ has **subexponential decay**. The natural examples of such sequences arise from the following Fact 6.5, whose proof is an application of the mean value theorem.

Fact 6.5. *Let* $f : \mathbb{R} \to \mathbb{R}_{>0}$ *be eventually differentiable such that* $f(t) \to 0$ *as* $t \to 0$ *and* f' *is eventually strictly decreasing. Then for sufficiently large* $n \in \mathbb{N}$ *the sequence* $(f(i))_{i \geq n}$ *is a decreasing sequence with decreasing gaps.*

Let $f : \mathbb{R} \to \mathbb{R}_{>0}$ be definable in an o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$ and satisfy $\lim_{t \to \infty} f(t) = 0$. Then f is eventually decreasing, eventually differentiable, and f' is also eventually decreasing by the o-minimal monotonicity theorem. Thus f satisfies the conditions of Fact 6.5.

For the remainder of this section, we fix a decreasing sequence $(s_n)_{n\in\mathbb{N}}$ with decreasing gaps and set $S = \{s_n : n \in \mathbb{N}\}$. In response to a question of Chris Miller, Fraser and Yu [18, Theorem 6.1] observed that the following fact follows from work of Garcia, Hare and Mendiv [21]:

Fact 6.6. *The following are equivalent:*

- *S has positive Assouad dimension,*
- *S has Assouad dimension one,*
- *$(s_n)_{n\in\mathbb{N}}$ has subexponential decay.*

Examples of sequences with subexponential decay are $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$, or more generally $(f(n))_{n\in\mathbb{N}}$ for a semi-algebraic $f : \mathbb{R} \to \mathbb{R}_{>0}$ satisfying $\lim_{t\to\infty} f(t) = 0$. Other classical examples are $(n^{-\log(n)})_{n\in\mathbb{N}}$ and $(2^{-\sqrt{n}})_{n\in\mathbb{N}}$.

Corollary 6.7. *Suppose $(s_n)_{n\in\mathbb{N}}$ has subexponential decay. Let $f : I \to \mathbb{R}$ be continuous.*

(1) *If f is nowhere locally affine, then $(\mathbb{R}, <, +, S, f)$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$.*
(2) *If f is C^1 and non-affine, then $(\mathbb{R}, <, +, S, f)$ defines $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.*
(3) *If f is C^2 and non-affine, then $(\mathbb{R}, <, +, S, f)$ is type C.*

Proof. Note that $S = \{s_n : n \in \mathbb{N}\}$ has topological dimension 0, but Assouad dimension 1 by Fact 6.6. The statement of the corollary follows from Corollary 6.4. $\square$

We make a few remarks about the optimality of statement (3) of Corollary 6.7. It follows from [20] that any expansion of $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda\in\mathbb{R}})$ by any countable compact subset of $\mathbb{R}^k$ is type A (see [17, Theorem B]). Therefore we cannot drop the assumption that f is non-affine in (3). It can easily be deduced from van den Dries [11] that the expansion of $(\mathbb{R}, <, +, \cdot)$ by $\{2^{-n} : n \in \mathbb{N}\}$ does not define $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. Therefore in (3) the requirement that $(s_n)_{n\in\mathbb{N}}$ has subexponential decay is necessary. Furthermore, the assumption that $(s_n)_{n\in\mathbb{N}}$ has decreasing gaps is necessary in (3). Using results from [20], Thamrongthanyalak [41] recently constructed a type A expansion of the real field that defines a decreasing sequence without decreasing gaps, but with subexponential decay.

6.4. **Applications to Automata Theory.** We finish with an application to automata theory. We first recall the terminology from [6]. Let $r \in \mathbb{N}_{\geq 2}$ and $\Sigma_r = \{0, \ldots, r-1\}$. Let $x \in \mathbb{R}$. A **base r expansion** of x is an infinite $\Sigma_r \cup \{\star\}$-word $a_p \cdots a_0 \star a_{-1} a_{-2} \cdots$ such that

$$(1) \qquad\qquad z = -\frac{a_p}{r-1} r^p + \sum_{i=-\infty}^{p-1} a_i r^i$$

with $a_p \in \{0, r-1\}$ and $a_{p-1}, a_{p-2}, \ldots \in \Sigma_r$. We will call the a_i's the **digits** of the base r expansion of x. The digit a_n is **the digit in the position corresponding to** r^n. We define $V_r(x, u, k)$ to be the ternary predicate on $\mathbb{R}$ that holds whenever there exists a base r expansion $a_p \cdots a_0 \star a_{-1} a_{-2} \cdots$ of x such that $u = r^n$ for some $n \in \mathbb{Z}$ and $a_n = k$. We denote by $\mathcal{T}_r$ the expansion of $(\mathbb{R}, <, +)$ by V_r. By [2, Lemma 3.1] $\mathcal{T}_r$ defines a dense ω-orderable set, and by [6, Theorem 6] the theory of $\mathcal{T}_r$ is decidable. Thus $\mathcal{T}_r$ is type B and does not interpret $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot)$.

The connection to automata theory arises as follows. A set $X \subseteq \mathbb{R}^n$ is r **- recognizable** if there is a Büchi automaton $\mathcal{A}$ over the alphabet $\Sigma_r^n \cup \{*\}$ which recognizes the set of all base-r encodings of elements of X. Such Büchi automata are also called **real vector automata** and were introduced in Boigelot, Bronne and Rasart [5]. By [6, Theorem 5] a subset of $\mathbb{R}^n$ is r-recognizable if and only if it is $\mathcal{T}_r$-definable without parameters. From Corollary B we immediately obtain:

Corollary 6.8. *If $f : I \to \mathbb{R}$ is C^1 and non-affine, then the graph of f is not r-recognizable.*

The base r-numeration system above may be replaced by other enumeration systems such as the β-numeration system used in [8] (when β is a Pisot number) or the Ostrowski numeration system based on a quadratic number used in [25]. These enumeration systems also give rise to type B structures with decidable theories. Thus analogues of Corollary 6.8 also hold for these enumeration systems. Results similar to Corollary 6.8 have been proven, for C^2 functions, or for more restricted classes of automata, by Anashin [1], Konečný [30], and Muller [36].

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