FRACTALS AND THE MONADIC SECOND ORDER THEORY OF ONE SUCCESSOR

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Abstract. We show that if X is virtually any classical fractal subset of $\mathbb{R}^n$, then $(\mathbb{R}, <, +, X)$ interprets the monadic second order theory of $(\mathbb{N}, +1)$. This result is sharp in the sense that the standard model of the monadic second order theory of $(\mathbb{N}, +1)$ is known to interpret $(\mathbb{R}, <, +, X)$ for various classical fractals X including the middle-thirds Cantor set and the Sierpinski carpet. Let $X \subseteq \mathbb{R}^n$ be closed and nonempty. We show that if the C^k-smooth points of X are not dense in X for some $k \geq 1$, then $(\mathbb{R}, <, +, X)$ interprets the monadic second order theory of $(\mathbb{N}, +1)$. The same conclusion holds if the packing dimension of X is strictly greater than the topological dimension of X and X has no affine points.

1. Introduction

This paper is a contribution to a larger research enterprise (see [13, 9, 18, 17, 19, 10, 1]) motivated by the following fundamental question:

What is the logical/model-theoretic complexity generated by fractal objects?

Here we will focus on fractal objects defined in first-order expansions of the ordered real additive group $(\mathbb{R}, <, +)$. Throughout this paper $\mathcal{R}$ is a first-order expansion of $(\mathbb{R}, <, +)$, and "definable" without modification means "$\mathcal{R}$-definable, possibly with parameters from $\mathbb{R}$". The main problem we want to address here is:

If $\mathcal{R}$ defines a fractal object, what can be said about the logical complexity of $\mathcal{R}$?

The first result in this direction is [19, Theorem B], stating that whenever $\mathcal{R}$ defines a Cantor set (that is, a nonempty compact subset of $\mathbb{R}$ without interior or isolated points), then $\mathcal{R}$ defines an isomorphic copy of the two-sorted first-order structure $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$. The latter structure is the standard model of the monadic second-order theory of $(\mathbb{N}, +1)$. Let $\mathcal{B}$ denote this structure. As pointed out in [19], while the theory of $\mathcal{B}$ is decidable by Büchi [3], the structure does not enjoy any Shelah-style combinatorial tameness properties, such as NIP or NTP2 (see e.g. Simon [28] for definitions). Thus every structure that defines an isomorphic copy of $\mathcal{B}$, can not satisfy these properties either, and for that reason has to be regarded as complicated or wild in the sense of these combinatorial/model-theoretic tameness notions. In this paper, we extend such results to fractal subsets of $\mathbb{R}^n$.

Let $X \subseteq \mathbb{R}^n$ be nonempty. Given $k \geq 0$, a point p on X is **$\mathbf{C^k}$-smooth** if $U \cap X$ is a C^k-submanifold of $\mathbb{R}^n$ for some nonempty open neighbourhood U of p. A point p on X is **affine** if there is an open neighbourhood U of p such that $U \cap X = U \cap H$

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for some affine subspace H. We say that $\mathcal{R}$ is **field-type** if there is an open interval I, definable functions $\oplus, \otimes : I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \times)$.

Theorem A. *Let X be a nonempty closed definable subset of $\mathbb{R}^n$. If the C^k-smooth points of X are not dense in X for some $k \geq 0$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. If the affine points of X are not dense in X, then $\mathcal{R}$ either defines an isomorphic copy of $\mathcal{B}$ or is field-type.*

When $\mathcal{R}$ is o-minimal [1], the first statement of Theorem A can be deduced from the o-minimal cell decomposition and a theorem of Laskowski and Steinhorn [21, Theorem 3.2], stating that definable functions in such expansions are C^k outside a definable lower-dimensional set. The second statement of Theorem A follows in the o-minimal setting from work by Marker, Peterzil and Pillay [24], who essentially show that an o-minimal expansion of $(\mathbb{R}, <, +)$ that defines a nowhere locally affine set is field-type. While not necessary for the proof of Theorem A, we will show in Section 6 that for every expansion $\mathcal{R}$ the C^k-smooth points of a definable set are definable again.

There is no precise definition of a fractal subset of $\mathbb{R}^n$, but according to Mandelbrot a set X is a fractal if the topological dimension of X is strictly less than the Hausdorff dimension of X. Topological dimension here refers to either small inductive dimension, large inductive dimension, or Lebesgue covering dimension. On subsets of $\mathbb{R}^n$ these three dimensions coincide (see Engelking [6] for details and definitions). Given Mandelbrot's definition, it is natural to explore situations in which metric dimensions and topological dimensions do not coincide. Here, we will discuss three important and well-known metric dimensions: Hausdorff, packing, and Assouad dimension. We refer the reader to Heinonen [16], Mattila [25], or Fraser [12] for the definitions of these dimensions and the basic facts we apply. It is well-known that

$$\dim X \leq \dim_{\text{Hausdorff}} X \leq \dim_{\text{Packing}} X \leq \dim_{\text{Assouad}} X$$

for all nonempty subsets X of $\mathbb{R}^n$. Here and below $\dim X$ is the topological dimension of X. Essentially all metric dimensions are bounded below by the topological dimension and above by the Assouad dimension.

In this paper, we will obtain the following theorem as a corollary of Theorem A.

Theorem B. *Let X be a nonempty closed definable subset of $\mathbb{R}^n$. Then*

(i) *If X is bounded,* $\dim X < \dim_{\text{Assouad}} X$*, and the affine points of X are not dense in X, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*
(ii) *If X does not have affine points and* $\dim X < \dim_{\text{Packing}} X$*, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

Statement (i) in Theorem B does not generalize to unbounded sets, and in statement (ii) packing dimension can not be replaced by Assouad dimension. The structure $(\mathbb{R}, <, +, \sin)$ does not define an isomorphic copy of $\mathcal{B}$, see Section 7.1. However, the reader can check that the $(\mathbb{R}, <, +, \sin)$-definable set

$$\{(x, t + \sin(x)) : t \in \pi\mathbb{Z}, x \in \mathbb{R}\}$$

[1] Recall that $\mathcal{R}$ is **o-minimal** if every nonempty definable subset of $\mathbb{R}$ is a finite union of open intervals and singletons, and that an o-minimal structure cannot define an isomorphic copy of $(\mathbb{N}, +1)$ by [30, Remark 2.14].

has Assouad dimension two, topological dimension one, and no affine points. We discuss $(\mathbb{R}, <, +, \sin)$ and related structures in Section 7.1 below.

Theorem B is not the first result that establishes model-theoretic wildness in expansions of the real line in which metric dimensions do not coincide with the topological dimension. For $a \in \mathbb{R}$, let $\lambda_a : \mathbb{R} \to \mathbb{R}$ map x to ax. We denote by $\mathbb{R}_{\mathrm{Vec}}$ the ordered vector space $(\mathbb{R}, <, +, (\lambda_a)_{a\in\mathbb{R}})$.

Fact 1.1 (Fornasiero, Hieronymi and Walsberg [10, Theorem A])**.** *Suppose $\mathcal{R}$ expands $\mathbb{R}_{\mathrm{Vec}}$. Let $X \subseteq \mathbb{R}^n$ be nonempty, closed, and definable. If the topological dimension of X is strictly less than the Hausdorff dimension of X, then $\mathcal{R}$ defines every bounded Borel subset of every $\mathbb{R}^n$.*

Observe that whenever $\mathcal{R}$ defines every bounded Borel subset of every $\mathbb{R}^n$, it also defines an isomorphic copy of $\mathcal{B}$. Thus Theorems A and B can be seen as an analogue of Fact 1.1 when $\mathcal{R}$ does not necessarily expand $\mathbb{R}_{\mathrm{Vec}}$. Note that there exists a compact subset X of $\mathbb{R}$ with topological dimension zero and positive packing dimension such that $(\mathbb{R}_{\mathrm{Vec}}, X)$ does not define all bounded Borel sets (see [10, Section 7.2]). There are even stronger results for expansions of the real field.

Fact 1.2 (Hieronymi, Miller [18, Theorem A])**.** *Suppose $\mathcal{R}$ expands $(\mathbb{R}, +, \times)$ and $X \subseteq \mathbb{R}^n$ is nonempty, closed, and definable. If the topological dimension of X is strictly less than the Assouad dimension of X, then $\mathcal{R}$ defines every Borel subset of every $\mathbb{R}^n$.*

Theorems A and B show that if $\mathcal{R}$ defines an object that can be called a fractal, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. Hence any model-theoretic tameness condition that is preserved by interpretability (such as NIP and NTP2), fails for such $\mathcal{R}$ whenever it fails for $\mathcal{B}$. It is natural to wonder whether such expansions fail any such tameness condition that holds for $\mathcal{B}$. It is known (see the next paragraph) that there exists expansions of $(\mathbb{R}, <, +)$ that define fractal subsets of $\mathbb{R}^n$ and are bi-interpretable with $\mathcal{B}$. Since every combinatorical/model-theoretic tameness condition à la Shelah should be preserved by bi-interpretability, every such condition satisfied by *all* expansions of $(\mathbb{R}, <, +)$ that define fractal objects in sense of Theorems A and B, has to be satisfied by $\mathcal{B}$ as well. Thus it can be argued that our theorems are optimal in this sense.

We describe an example of such an expansion $\mathcal{R}$ that is bi-interpretable with $\mathcal{B}$ and defines fractal subsets of $\mathbb{R}^n$. Fix a natural number $r \geq 2$. Let $V_r(x, u, d)$ be the ternary predicate on $\mathbb{R}$ that holds whenever $u = r^n$ for $n \in \mathbb{N}, n \geq 1$ and there is a base r expansion of x with nth digit d. Let $\sigma_r : r^{\mathbb{N}} \to r^{-\mathbb{N}}$ be the function that maps r^n to r^{-n} for all $n \in \mathbb{N}$. We let $\mathcal{R}_r$ be $(\mathbb{R}, <, +, V_r, \sigma_r)$. It is easy to see that $\mathcal{R}_3$ defines the middle-thirds Cantor set, the Sierpinski triangle, and the Menger carpet. Adjusting the work in Boigelot, Rassart and Wolper [2] to account for σ_r, one can show that $\mathcal{B}$ and $\mathcal{R}_r$ are bi-interpretable.

We finish with a few open questions. We do not know if Theorem A remains true when "C^k" is replaced with "C^∞". Note that by Rolin, Speissegger and Wilkie [27], there is an o-minimal expansion of $(\mathbb{R}, +, \times)$ that defines a function $f : \mathbb{R} \to \mathbb{R}$ that is not C^∞ on a dense definable open subset of $\mathbb{R}$. However, this function is still C^∞ on a dense open subset of $\mathbb{R}$. These considerations lead to the following question:

Question 1.3. *Is there an o-minimal expansion of* $(\mathbb{R}, +, \times)$ *that defines a function* $f : \mathbb{R} \to \mathbb{R}$ *which is not* C^∞ *on a dense open subset of* $\mathbb{R}$*?*

The author of [22] indicated to us that the it might be possible to adapt ideas from that paper to construct such an expansion.

Questions 1.4. *Let* X *be a nonempty closed definable subset of* $\mathbb{R}^n$*. If the topological dimension of* X *is strictly less than the Hausdorff dimension of* X*, then must* $\mathcal{R}$ *define an isomorphic copy of* $\mathcal{B}$*?*

Observe that Theorem B gives an affirmative answer to Question 1.4 under the additional assumption that X does not have any affine points. We do not even know the answer to the following weaker question.

Questions 1.5. *If* $\mathcal{R}$ *defines an uncountable nowhere dense subset of* $\mathbb{R}$*, must* $\mathcal{R}$ *define an isomorphic copy of* $\mathcal{B}$*? Weaker: if* $\mathcal{R}$ *defines an uncountable nowhere dense subset of* $\mathbb{R}$*, must* $\mathcal{R}$ *have* IP*?*

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2. Conventions, notation and background

2.1. **Conventions and notations.** Throughout m, n are natural numbers, i, j, k, l are integers and $s, t, \delta, \varepsilon$ are real numbers. Throughout "dimension" is topological dimension unless stated otherwise. Let X be a subset of $\mathbb{R}^n$. Then $\dim X$ is the dimension of X, $\mathrm{Cl}(X)$ and $\mathrm{Int}(X)$ are the closure and interior of X, and $\mathrm{Bd}(X) := \mathrm{Cl}(X) \setminus \mathrm{Int}(X)$ is the boundary of X. Given $A \subseteq \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$ we let

$$A_x := \{y \in \mathbb{R}^n : (x, y) \in A\}.$$

We let $\Gamma(f)$ be the graph of a function f and let $f|_Z$ be the restriction of f to a subset Z of its domain. A family $(A_t)_{t>0}$ of sets is **increasing** if $s < t$ implies $A_s \subseteq A_t$, and **decreasing** if $s < t$ implies $A_t \subseteq A_s$.

Throughout $\|\cdot\|$ is the ℓ_∞-norm and an "open ball" is an open ℓ_∞-ball. For $x \in \mathbb{R}^n$, we denote by $B_\varepsilon(x)$ the open ball of radius ε around x. We use the ℓ_∞-norm as opposed to the ℓ_2-norm, since ℓ_∞ is $(\mathbb{R}, <, +)$-definable. All dimensions of interest are bi-lipschitz invariants and therefore unaffected by our choice of norm.

2.2. **Background.** We review definitions and results from the theory of first-order expansions of $(\mathbb{R}, <, +)$. An **ω-orderable set** is a definable set that is either finite or admits a definable order of order-type ω. One should think of "ω-orderable sets" as "definably countable sets". A **dense ω-order** is an ω-orderable subset of $\mathbb{R}$ that is dense in some nonempty open interval. We say $\mathcal{R}$ is **type A** if it does not admit a dense ω-order, **type C** if it defines every bounded Borel subset of every $\mathbb{R}^n$, and **type B** if it is neither type A nor type C. It is easy to see that these three classes of structures are mutually exclusive. We refer the reader to [20] for a more detailed discussion of this trichotomy and its relevance. In the following fact we collect several of main theorem from [19, 20].

Fact 2.1. *Let $U \subseteq \mathbb{R}^m$ be a definable open set.*

1. *If $\mathcal{R}$ is not type A, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ (see* [19, Theorem A]*).*
2. *If $\mathcal{R}$ is type B, then $\mathcal{R}$ is not field-type (see* [20, Theorem C]*).*
3. *If $\mathcal{R}$ is type A, $k \geq 1$, $U \subseteq \mathbb{R}^m$, and $f : U \to \mathbb{R}^n$ is continuous and definable, then there is a dense open definable subset V of U on which f is C^k (see* [20, Theorem B]*).*
4. *If $\mathcal{R}$ is type A and not field-type, and $f : U \to \mathbb{R}^n$ is definable and continuous, then there a is dense open definable subset V of U on which f is locally affine (see* [20, Theorem A]*).*
5. *If $\mathcal{R}$ is not of field-type, then every C^2-function $f : U \to \mathbb{R}^n$ on a nonempty open box $U \subseteq \mathbb{R}^m$ is affine (see* [20, Theorem 8.1]*).*

A subset X of $\mathbb{R}^n$ is $\mathbf{D_\Sigma}$ if $X = \bigcup_{s,t>0} X_{s,t}$ for a definable family $(X_{s,t})_{s,t>0}$ of compact subsets of $\mathbb{R}^n$ such that $X_{s,t} \subseteq X_{s,t'}$ when $t \leq t'$ and $X_{s',t} \subseteq X_{s,t}$ when $s \leq s'$. We say that such a family **witnesses** that X is D_Σ. Note that a D_Σ set is definable and that every D_Σ set is F_σ.

Fact 2.2 (Dolich, Miller and Steinhorn [5, 1.10])**.** *Open and closed definable sets are D_Σ, a finite union or finite intersection of D_Σ sets is D_Σ, and the image of a D_Σ set under a continuous definable function is D_Σ.*

A key result about D_Σ sets in type A structures is the following Strong Baire Category Theorem, or **SBCT**.

Fact 2.3 ([10, Theorem 4.1])**.** *Suppose $\mathcal{R}$ is type A. Let X be a D_Σ subset of $\mathbb{R}^n$ witnessed by the definable family $(X_{s,t})_{s,t>0}$. Then X either has interior or is nowhere dense. If X has interior, then $X_{s,t}$ has interior for some $s, t > 0$. Furthermore, if $(X_t)_{t>0}$ is an increasing family of D_Σ sets and $\bigcup_{t>0} X_t$ has interior, then X_t has interior for some $t > 0$. Finally, if X is dense in a definable open set U, then the interior of X is dense in U.*

The latter two claims follow by applying the Baire Category Theorem to the first claim. Corollary 2.4 below follows from SBCT and the fact that the closure and the interior of a D_Σ set are also D_Σ.

Corollary 2.4. *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^n$ be D_Σ. Then $\mathrm{Bd}(X)$ is nowhere dense.*

We also need the following $\mathbf{D_\Sigma}$**-selection** result in type A structures.

Fact 2.5 ([10, Proposition 5.5])**.** *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^{m+n}$ be D_Σ, and $U \subseteq \mathbb{R}^m$ be a nonempty open set contained in the coordinate projection of X onto $\mathbb{R}^m$. Then there is a nonempty definable open $V \subseteq U$ and a continuous definable $f : V \to \mathbb{R}^n$ such that the graph $\Gamma(f)$ is contained in X.*

Fact 2.6 and Fact 2.7 below are special cases of more general results on additivity of dimension [10, Theorem E].

Fact 2.6. *Suppose $\mathcal{R}$ is type A. Let $d \in \mathbb{N}$ and let $A \subseteq \mathbb{R}^n$ be D_Σ of dimension d with $1 \leq d \leq n - 1$. Then there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and a nonempty open $U \subseteq \mathbb{R}^d$ contained in $\pi(A)$ such that for all $x \in U$*

$$\dim(\pi^{-1}(\{x\})) = 0.$$

Fact 2.7. *Suppose $\mathcal{R}$ is type A. Let $X \subseteq \mathbb{R}^n$ be D_Σ, and let $f : X \to \mathbb{R}^m$ be a continuous definable function. Then* $\dim f(X) \leq \dim X$.

One corollary of Fact 2.7 is that type A expansions cannot define space-filling curves [10, Theorem E]. For our purposes a **Cantor set** is a nonempty compact nowhere dense subset of $\mathbb{R}$ without isolated points.

Fact 2.8 ([19, Theorem B])**.** *If $\mathcal{R}$ defines a Cantor set, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

If $X \subseteq \mathbb{R}$, then $p \in X$ is C^k-smooth (for any $k \geq 0$) if and only if p is either isolated in X or lies in the interior of X. Therefore Fact 2.8 yields Theorem A for definable subsets of $\mathbb{R}$.

3. Hausdorff continuity of definable families

Throughout this section $\mathcal{R}$ is assumed to be type A and U is a fixed nonempty definable open subset of $\mathbb{R}^m$. The goal of this section is to show that definable families of D_Σ sets indexed by U are Hausdorff continuous on a dense open subset of U. We make this statement precise in Proposition 3.2.

We first recall some useful notions from metric geometry. Let $f : (X, d_X) \to (Y, d_Y)$ be a function between metric spaces. The **oscillation** of f at $x \in X$ is the supremum of all $\delta \geq 0$ such that for every $\varepsilon > 0$ there are $y, z \in X$ such that $d_X(x, y) < \varepsilon$, $d_X(x, z) < \varepsilon$ and $d_Y(f(y), f(z)) > \delta$. Recall that f is continuous at x if and only if the oscillation of f at x is zero. Furthermore, the set of $x \in X$ at which the oscillation of f is at least ε is closed for every $\varepsilon > 0$.

The **Hausdorff distance** $d_{\mathcal{H}}(A, B)$ between two bounded subsets A and B of $\mathbb{R}^n$ is the infimum of the set of all $\delta > 0$ such that for every $a \in A$ there is a $b \in B$ such that $\|a - b\| < \delta$ and for every $b \in B$ there is an $a \in A$ such that $\|a - b\| < \delta$. The Hausdorff distance between a bounded subset of $\mathbb{R}^n$ and its closure is zero. The Hausdorff distance restricts to a separable complete metric on the collection $\mathcal{C}$ of all compact subsets of $\mathbb{R}^m$. Lemma 3.1 below follows directly from the definition of $d_{\mathcal{H}}$.

Lemma 3.1. *Let W be a bounded open subset of $\mathbb{R}^n$, let $\mathcal{D}$ be a collection of open balls of diameter at least ε covering W, and let X and Y be subsets of W. If*

$$\{B \in \mathcal{D} : B \cap X \neq \emptyset\} = \{B \in \mathcal{D} : B \cap Y \neq \emptyset\},$$

then $d_{\mathcal{H}}(X, Y) \leq \varepsilon$.

Given $Z \subseteq \mathbb{R}^m$ and a family $\mathcal{A} = (A_x)_{x \in Z}$ of subsets of $\mathbb{R}^n$, we let $M_{\mathcal{A}} : Z \to \mathcal{C}$ be given by $M_{\mathcal{A}}(x) = \mathrm{Cl}(A_x)$. We say that $\mathcal{A}$ is **HD-continuous** if $M_{\mathcal{A}}$ is continuous. For $\varepsilon > 0$, let $\mathcal{O}_\varepsilon(\mathcal{A})$ be the set of points in Z at which $M_{\mathcal{A}}$ has oscillation at least ε. Let $\mathcal{O}(\mathcal{A})$ be the set of points at which $M_{\mathcal{A}}$ has positive oscillation. Observe that the complement of $\mathcal{O}(\mathcal{A})$ is the set of points at which $M_{\mathcal{A}}$ is continuous, and that each $\mathcal{O}_\varepsilon(\mathcal{A})$ is closed in Z.

We say that $A \subseteq \mathbb{R}^{m+n}$ is **vertically bounded** if there is an open ball $W \subseteq \mathbb{R}^n$ such that $A_x \subseteq W$ for all $x \in \mathbb{R}^m$.

Proposition 3.2. *Let $A \subseteq U \times \mathbb{R}^n$ be* D_Σ *and vertically bounded. Let $\mathcal{A}$ be the definable family $(A_x)_{x\in U}$ of subsets of $\mathbb{R}^n$. Then there is a dense definable open subset V of U such that $(A_x)_{x\in V}$ is* HD*-continuous.*

Proof. We show that $\mathcal{O}(\mathcal{A})$ is nowhere dense and take V to be the interior of the complement of $\mathcal{O}(\mathcal{A})$ in U. Note that then V is definable, since $\mathcal{O}(\mathcal{A})$ is. Note that it suffices to show that every point in U has a neighbourhood V such that $\mathcal{O}(\mathcal{A}'$ is nowhere dense in U' where $\mathcal{A}'$ is the restricted family $(A_x)_{x\in U'}$. Therefore, we may assume without loss of generality that U is an open ball and in particular, connected. Since $(\mathcal{O}_\varepsilon(\mathcal{A}))_{\varepsilon>0}$ witnesses that $\mathcal{O}(\mathcal{A})$ is D_Σ, it suffices to show that $\mathcal{O}_\varepsilon(\mathcal{A})$ is nowhere dense for all $\varepsilon > 0$, and then apply SBCT to obtain nowhere density of $\mathcal{O}(\mathcal{A})$.

Fix $\varepsilon > 0$. Let W be an open ball in $\mathbb{R}^n$ such that $A_x \subseteq W$ for all $x \in U$. Let $\mathcal{D}$ be a finite collection of closed balls of diameter $\leq \varepsilon$ covering W. Let π be the coordinate projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ onto the first m coordinates. Set

$$E_B := \pi([\mathbb{R}^m \times B] \cap A) \quad \text{for each} \quad B \in \mathcal{D}.$$

That is, E_B is the set of $x \in U$ such that A_x intersects B. By Fact 2.2 each E_B is D_Σ, and hence $\mathrm{Bd}(E_B)$ is nowhere dense in U for each $B \in \mathcal{D}$ by Corollary 2.4. Recall that the boundary of a subset of a topological space is always closed. Therefore

$$Y := \bigcap_{B\in\mathcal{D}} U \setminus \mathrm{Bd}(E_B)$$

is dense and open in U. We show that Y is a subset of the complement of $\mathcal{O}_\varepsilon(\mathcal{A})$. The claim that $\mathcal{O}_\varepsilon(\mathcal{A})$ is nowhere dense, follows. Fix $p \in Y$. We prove that the oscillation of $M_{\mathcal{A}}$ at p is at most ϵ. Let R be an open ball with center p contained in Y. Since R and U are both connected, and $R \cap \mathrm{Bd}(E_B) = \emptyset$ for all $B \in \mathcal{D}$, we have that R is either contained in or disjoint from each E_B. Fix $q \in R$. For all $B \in \mathcal{D}$, we have that $q \in E_B$ if and only if $p \in E_B$. That is, for every $B \in \mathcal{B}$, A_q intersects if and only if A_p intersects B. By Lemma 3.1 we immediately get that $d_{\mathcal{H}}(A_p, A_q) \leq \varepsilon$.

□

We say that a point p in a subset X of $\mathbb{R}^n$ is **ε-isolated** if $\|p - q\| \geq \varepsilon$ for all $q \in X$ with $p \neq q$. We leave the verification of the following lemma, an exercise of metric geometry, to the reader.

Lemma 3.3. *Fix $\varepsilon > 0$. Let $A \subseteq U \times \mathbb{R}^n$ be a vertically bounded such that $(A_x)_{x\in U}$ is HD-continuous. Then the set of $(x, y) \in A$ such that y is ε-isolated in A_x is closed in A.*

If $X \subseteq \mathbb{R}^m$ is a D_Σ set and $Y \subseteq X$ is definable and closed in X then Y is D_Σ. Corollary 3.4 follows.

Corollary 3.4. *Let $A \subseteq U \times \mathbb{R}^n$ be a vertically bounded* D_Σ *set such that $(A_x)_{x\in U}$ is HD-continuous. Then the set of $(x, y) \in A$ such that y is ε-isolated in A_x is* D_Σ.

4. C^k-smooth points on D_Σ-sets

We prove Theorem A in this section. Recall that if $\mathcal{R}$ is not type A, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by Fact 2.1.1. If $\mathcal{R}$ defines a Cantor subset of $\mathbb{R}$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by Fact 2.8. Thus it suffices to show Theorem A (and

Theorem B) under the assumption that $\mathcal{R}$ is type A and does not define a Cantor subset of $\mathbb{R}$. We suppose throughout the remainder of this section that $\mathcal{R}$ is type A.

Lemma 4.1 is an elementary fact of real analysis, we leave the details to the reader. (Take V such that $\|f(x) - f(y)\| < \varepsilon/2$ for all $x, y \in V$).

Lemma 4.1. *Let $A \subseteq \mathbb{R}^{m+n}$, $U \subseteq \mathbb{R}^m$ be nonempty open, $\varepsilon > 0$, and $f : U \to \mathbb{R}^n$ be continuous such that $f(x)$ is an ε-isolated element of A_x for all $x \in U$. Then there are nonempty open $V \subseteq U$ and $W \subseteq \mathbb{R}^n$ such that $A \cap [V \times W] = \Gamma(f|_V)$.*

Lemma 4.2. *Let $k \geq 0$, let $A \subseteq \mathbb{R}^{m+n}$ be D_Σ, and let U be a nonempty definable open set contained in the coordinate projection of A onto $\mathbb{R}^m$ such that the isolated points of A_x are dense in A_x for all $x \in U$. Then there exist a nonempty definable open set $V \subseteq U$ and $W \subseteq \mathbb{R}^n$ and a definable C^k-function $f : V \to W$ such that*

$$A \cap [V \times W] = \Gamma(f).$$

If $\mathcal{R}$ is not field-type, then we can take f to be affine.

Proof. Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the coordinate projection onto the first m coordinates. We first reduce to the case when A is vertically bounded. Let

$$A(r) = \{(x, y) \in A : \|y\| < r\} \quad \text{for all} \quad r > 0.$$

Thus $\bigcup_{r>0} A(r) = A$. Then $(\pi(A(r)))_{r>0}$ is an increasing definable family of D_Σ sets and U is contained in $\bigcup_{r>0} \pi(A(r))$. By the SBCT, the projection $\pi(A(t))$ has interior for some $t > 0$. After replacing U with $\mathrm{Int}(A(t))$ and A with $A(t)$ if necessary, we may assume that A is vertically bounded. After applying Lemma 3.2 and replacing U with a smaller nonempty definable open set, we may assume that $(A_x)_{x \in U}$ is HD-continuous. For each $\varepsilon > 0$ we define $S_\varepsilon \subseteq A$ to be the set of (x, y) such that y is ε-isolated in A_x. By Corollary 3.4 each S_ε is D_Σ. Because A_x has an isolated point for each $x \in U$, we get that $U \subseteq \bigcup_{\varepsilon>0} \pi(S_\varepsilon)$. Since $(\pi(S_\epsilon))_{\epsilon>0}$ is an increasing family of D_Σ sets, the SCBT gives a $\delta > 0$ such that $\pi(S_\delta)$ has interior in U. After replacing U with a smaller nonempty definable open set if necessary, we may assume that U is contained in $\pi(S_\delta)$. Applying D_Σ-selection we obtain a nonempty definable open set $V \subseteq U$ and a continuous definable $f : V \to \mathbb{R}^n$ such that $(x, f(x)) \in S_\delta$ for all $x \in V$. Thus $f(x)$ is δ-isolated in A_x for all $x \in V$. After applying Fact 2.1.3 and shrinking V if necessary, we may assume that f is C^k. Now apply Lemma 4.1.

If $\mathcal{R}$ is not field-type, then after applying Fact 2.1.4 and shrinking V if necessary, we have that f is affine on V. □

Lemma 4.3. *Suppose $\mathcal{R}$ does not define a Cantor set. Let $A \subseteq \mathbb{R}^n$ be D_Σ. If $\dim A = 0$, then the isolated points of A are dense in A.*

It is easy to see that $\mathcal{R}$ defines a Cantor subset of $\mathbb{R}$ if and only if it defines a nowhere dense subset of $\mathbb{R}$ without isolated points (take closures).

Proof of Lemma 4.3. We proceed by induction on n. Suppose $n = 1$. Since $\dim A = 0$, we know that A is nowhere dense by SBCT. Let $U \subseteq \mathbb{R}$ be an open set that intersects A. Since nowhere dense definable subsets of $\mathbb{R}$ have isolated points, the intersection $A \cap U$ must contain an isolated point. Thus the isolated points of A are dense in A. Now suppose $n > 1$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection onto the first n coordinates. By Fact 2.7 we have that $\dim \pi(A) = 0$. Since $\pi(A)$ is D_Σ, the projection $\pi(A)$ contains an isolated point x by induction. Since A_x is a definable zero-dimensional subset of $\mathbb{R}$, it contains an isolated point t. It is easy to see that (x,t) is isolated in A. □

We are now ready to prove Theorem A. In the proof below we apply the fact that an arbitrary subset of $\mathbb{R}^n$ is n-dimensional if and only if it has nonempty interior (see [6, Theorem 1.8.10]).

Proof of Theorem A. As pointed out above, by Fact 2.1.1 and Fact 2.8 we can reduce to the case that $\mathcal{R}$ is type A and does not define a Cantor set. Let $A \subseteq \mathbb{R}^n$ be D_Σ and nonempty. We show that the C^k-smooth points of A are dense in A. It is enough to show that every definable open subset of $\mathbb{R}^n$ that intersects A, contains a C^k-smooth point. Let $U \subseteq \mathbb{R}^n$ be a definable open set that intersects A. Let d be the dimension of $U \cap A$. If $d = 0$, apply Lemma 4.3. If $d = n$, then $U \cap A$ has interior. Every interior point of A is C^k-smooth. Now suppose that $1 \leq d \leq n-1$. By Fact 2.6 there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and a nonempty definable open $V \subseteq \mathbb{R}^d$ such that $V \subseteq \pi(A)$ and $\dim(\pi^{-1}(\{x\})) = 0$ for all $x \in V$. Without loss of generality we can assume that π is the coordinate projection onto the first d coordinates. Since $\dim(\pi^{-1}(\{x\})) = 0$ for all $x \in V$, we have that $\dim A_x = 0$ for all $x \in V$. By Lemma 4.3, the isolated points of A_x are dense in A_x for all $x \in V$. Now apply Lemma 4.2. □

5. Coincidence of dimensions and Theorem B

In this section, we prove Theorem B, which we recall as a convenience for the reader.

Theorem B. *Let $X \subseteq \mathbb{R}^n$ be nonempty, closed and definable. Then*

(i) *If X is bounded, the affine points of X are not dense in X, and we have $\dim X < \dim_{\text{Assouad}} X$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*
(ii) *If X does not have affine points and $\dim X < \dim_{\text{Packing}} X$, then $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$.*

We obtain Theorem B as a corollary of Theorem A and an extension of Fact 1.2. Recall that by Fact 1.2 on every bounded D_Σ set X in a type A expansion of $(\mathbb{R}, +, \times)$, the Assouad dimension of X equals the topological dimension of X. We refer the reader to [18] and Luukkainen [23] for more on the relevance and significance of this phenomenon. We now extend this result to type A expansions that are field-type.

Theorem 5.1. [2] *Suppose $\mathcal{R}$ is type A and field-type. Let X be a bounded D_Σ set. Then the Assouad dimension of X is equal to the topological dimension of X.*

The coincidence of dimensions for D_Σ sets does not extend to type A expansions of $(\mathbb{R}, <, +)$. For example, $S = \{1/n : n \in \mathbb{N}, n \geq 1\}$ has Assouad dimension 1, topological dimension 0, and $(\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, S)$ is type A by [10, Theorem B]. The assumption that X is bounded is also necessary. The structure $(\mathbb{R}, <, +, \sin)$ is locally o-minimal and hence type A, see Section 7.1 below. This structure is also field-type, as the sine function is C^2 and non-affine, and defines $\pi\mathbb{Z} = \sin^{-1}(\{0\})$.

[2]This theorem first appeared in a preprint version of [20], but has been removed from the final version of [20].

The latter set has topological dimension 0 and Assouad dimension 1.

We want to clarify one notation before giving the proof of Theorem 5.1. Let e be the Euclidean metric on $\mathbb{R}^n$ and let $X \subseteq \mathbb{R}^n$. When we refer to the Assouad dimension of X, we mean the Assouad dimension of the metric space (X, e_X), where e_X is the restriction of the metric e to X.

Proof of Theorem 5.1. Since $\mathcal{R}$ is field-type, $\mathcal{R}$ defines a C^2 function $f : [0, 1] \to \mathbb{R}$ with nonconstant derivative by [20, Theorem A(4), Lemma 5.1] and Fact 2.1.3. By [20, Lemma 6.2] there is an interval $I \subseteq \mathbb{R}$, $\oplus, \otimes : I^2 \to I$ be continuous definable functions such that there is an isomorphism $\tau : (I, <, \oplus, \otimes) \to (\mathbb{R}, <, +, \times)$ and there is a subinterval $J \subseteq I$ such that the restriction of τ is a C^k-diffeomorphism. It is clear from the proof of [20, Lemma 6.2] that τ can be chosen to with f' on J. Since τ is strictly increasing, f' is strictly increasing. Thus f'' is positive on J. Since f'' is continuous, we may suppose (after further shrinking J if necessary) that f'' is bounded above and bounded away from zero on J. It follows from the mean value theorem that f' is bi-Lipschitz on J.

Let d be the natural metric on $(I, <, \oplus, \otimes)$ given by

$$d(x, y) = \tau(\|x \ominus y\|_I) \quad \text{for all } x, y \in I.$$

We now consider the metric space (J, d_J) where d_J is the restriction of d to J. As τ is an ordered field isomorphism which agrees with f' on J, we have

$$d(x, y) = \|f'(x) - f'(y)\| \quad \text{for all } x, y \in J.$$

Then the identity map $(J, e_J) \to (J, d_J)$ is a bi-Lipschitz equivalence because f' is bi-Lipschitz. Let d_n be the metric on J^n given by

$$d_n(x, y) = \max\{d(x_1, y_1), \ldots, d(x_n, y_n)\}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in J^n$. It is easy to see that the identity map $(J^n, d_n) \to (J^n, e_{J^n})$ is also a bi-Lipschitz equivalence.

Let $X \subseteq \mathbb{R}^n$ be a bounded D_Σ set. Let $q \in \mathbb{Q}_{>0}, t \in \mathbb{R}^n$ be such that $qX + t$ is a subset of J^n. Then $qX + t$ has the same Assouad dimension and topological dimension as X, because invertible affine maps are bi-Lipschitz and bi-Lipschitz equivalences preserve both Assouad and topological dimension. After replacing X with $qX + t$ if necessary we suppose that $X \subseteq J^n$. By [10, Theorem E] the topological dimension of the closure of X agrees with the topological dimension of X. It follows directly from the definition of Assouad dimension that taking closures does not raise Assouad dimension of subsets of $\mathbb{R}^n$. It therefore suffices to prove the theorem for closed definable sets. We assume that X is closed. Consider the structure $(I, <, \oplus, \otimes, X)$. This structure is isomorphic via τ to $\mathcal{S} := (\mathbb{R}, <, +, \times, \tau(X))$. Since $\mathcal{R}$ is type A, so is $\mathcal{S}$. In particular, $\mathcal{S}$ cannot define every Borel set. Therefore by Fact 1.2 Assouad dimension and topological dimension agree on $(\tau(X), e_{\tau(X)})$. Since τ is bi-Lipschitz on J, it follows from the definition of d_n that the Assouad and topological dimensions of $(X, (d_n)_X)$ agree. Since $\mathrm{id} : (J^n, d_n) \to (J^n, e_{J^n})$ is bi-Lipschitz, the Assouad and topological dimensions of X agree. □

The reduction to the case when X is closed is necessary as $(I, <, \oplus, \otimes, X)$ need not define a witness that X is D_Σ. Before proving Theorem B, we prove the following corollary to Theorem 5.1.

Corollary 5.2. [3] *Let* $f : (0,1] \to \mathbb{R}$ *be* $f(t) = \sin(1/t)$ *and* $\tan : \mathbb{R} \setminus [\pi\mathbb{Z} + \frac{\pi}{2}] \to \mathbb{R}$ *be the usual tangent function. Then* $(\mathbb{R}, <, +, f)$ *and* $(\mathbb{R}, <, +, \tan)$ *are type C.*

The second claim of Corollary 5.2 requires Fact 5.3. Fact 5.3 is due to García, Hare, and Mendivil [15, Proposition 4] (see also Fraser and Yu [11, Theorem 6.1]).

Fact 5.3. *Let* $g : \mathbb{R} \to \mathbb{R}_{>0}$ *be a differentiable function such that* g *and* $-g'$ *are strictly decreasing on* $[\delta, \infty)$ *for some* $\delta > 0$ *and that both* $g(t), -g'(t)$ *tend to* 0 *as* $t \to \infty$*. The one of the following holds:*

(1) $\{g(n) : n \in \mathbb{N}\}$ *has Assouad dimension one, or*
(2) there is $0 < \alpha < 1$ *such that* $g(n) \le \alpha^n$ *when* n *is sufficiently large.*

In the second case $\{g(n) : n \in \mathbb{N}\}$ *has Assouad dimenson zero.*

We now prove Corollary 5.2.

Proof. Both structures are field-type as they define non-affine C^2 functions. Observe that

$$\{x \in (0,1] \;:\; f(t) = 0\} = \{1/\pi n \;:\; n \in \mathbb{N}_{>0}\}.$$

Assouad dimension is invariant under inversion [23, Theorem XIII.5.2] and rescaling so

$$\dim_{\text{Assouad}}\{1/\pi n : n \in \mathbb{N}_{>0}\} = \dim_{\text{Assouad}} \pi\mathbb{N} = \dim_{\text{Assouad}} \mathbb{N} = 1.$$

By Theorem 5.1 $(\mathbb{R}, <, +, f)$ is type C. Let $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ be the usual arctangent. Note that $(\mathbb{R}, <, +, \tan)$ defines $\pi\mathbb{Z}$. Let $g : \mathbb{R}_{\ge 0} \to (0, \pi/2]$ be given by $g(t) = (\pi/2) - \arctan(\pi t)$. It is easy to see that g satisfies the conditions of Fact 5.3 and that $g(n)$ does not decrease exponentially. Hence by Fact 5.3 $\{g(n) : n \in \mathbb{N}\}$ has Assouad dimension one. Note that $\{g(n) : n \in \mathbb{N}\}$ is definable in $(\mathbb{R}, <, +, \tan)$. ☐

Proof of Theorem B. (i) Let $X \subseteq \mathbb{R}^n$ be nonempty, D_Σ, and bounded such that the affine points of X are not dense in X and the topological dimension of X is strictly less than the Assouad dimension of X. We have to show that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. By Theorem A, we have that either $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ or $\mathcal{R}$ is field-type. Suppose $\mathcal{R}$ is field-type. By Theorem 5.1, we get that $\mathcal{R}$ can not be of type A. However, every structure of type B or type C defines an isomorphic copy of $\mathcal{B}$.

(ii) Let $X \subseteq \mathbb{R}^n$ be D_Σ such that X has no affine points and the topological dimension of X is strictly less than the packing dimension of X. We need establish that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. It follows directly from the definition of the packing dimension that whenever $Y \subseteq \mathbb{R}^n$ is Borel, and $(Y_m)_{m\in\mathbb{N}}$ is a collection of Borel subsets of Y covering Y, then $\dim_{\text{Packing}} Y = \sup_m \dim_{\text{Packing}} Y_m$. The same statement holds for topological dimension provided each Y_m is F_σ by [6, Corollary 1.5.4].

Given $m \in \mathbb{Z}^n$ we let X_m be $([0,1]^n + m) \cap X$. As each X_m is F_σ, we have $\dim X = \sup_{m\in\mathbb{Z}^n} \dim X_m$. Thus if $\dim X_m = \dim_{\text{Packing}} X_m$ for all $m \in \mathbb{Z}^n$, we obtain $\dim X = \dim_{\text{Packing}} X$. If $\dim X_m < \dim_{\text{Packing}} X_m$ for some $m \in \mathbb{Z}^n$, then,

[3] We thank Harry Schmidt for asking about $(\mathbb{R}, <, +, \sin(1/t))$.

as each X_m has no affine points, Statement (i) of Theorem B gives that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$. $\square$

6. Definability of C^k-smooth points

Fix $k \in \mathbb{N}, k \geq 2$. Throughout this section, let X be a definable subset of $\mathbb{R}^n$. The main goal of this section is the following proposition.

Proposition 6.1. *The set of C^k-smooth points of X is definable. Equivalently: if Y is a subset of $\mathbb{R}^n$, then the set of C^k-smooth points of Y is $(\mathbb{R}, <, +, Y)$-definable.*

It is worth re-iterating that we do not assume $\mathcal{R}$ is type A in this section. Let $X \subseteq \mathbb{R}^n$ be definable and $x \in X$ be a C^k-smooth point of X. An application of the inverse function theorem shows that, after possibly permuting coordinates, there is $d \in \{0, \dots, n\}$, an open box $V \subseteq \mathbb{R}^d$ and a C^k-function $f : V \to \mathbb{R}^{n-d}$ such that $\Gamma(f) = U \cap X$ for some open box $U \subseteq \mathbb{R}^n$. Note that such an f must be definable.

We will use the following consequence of the fact that the image of a C^k-submanifold under a C^k-diffeomorphism is again a C^k-submanifold.

Fact 6.2. *Let $X, Y \subseteq \mathbb{R}^n$ be definable, let $x \in X$, and let $\tau : X \to Y$ be a definable C^k-diffeomorphism. Then x is a C^k-smooth point of X if and only if $\tau(x)$ is a C^k-smooth point of Y.*

6.1. When $\mathcal{R}$ is not field-type.

Suppose that $\mathcal{R}$ is not field-type. We now give a proof of Proposition 6.1 in this case. We need the following fact, which is a basic analysis exercise.

Fact 6.3. *Let $f : W \to \mathbb{R}^m$ be a function on a convex subset W of $\mathbb{R}^d$. Then f is affine if and only if*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \quad \text{for all } x, y \in W.$$

Proof of Proposition 6.1 when $\mathcal{R}$ is not of field-type. Let $X \subseteq \mathbb{R}^n$. We show that the set of C^k-smooth points of X is equal to the set of affine points of X, and that the set of affine points of X is definable. The latter statement follows immediately from Fact 6.3.

It is enough to show that every C^k-smooth point of X is an affine point of X. Let $x \in X$ be a C^k-smooth point of X. Thus there is $d \in \{0, \dots, n\}$, open boxes $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^d$ and a definable C^k-function $f : V \to \mathbb{R}^{n-d}$ such that after permuting coordinates if necessary, $x \in U$ and $\Gamma(f) = U \cap X$. By Theorem 2.1.5, f is affine. Thus x is an affine point of X. $\square$

6.2. When $\mathcal{R}$ is field-type.

We now treat the case when $\mathcal{R}$ is field-type. We leave Fact 6.4 as an exercise to the reader.

Fact 6.4. *Fix $k \geq 1$. Let L be the language of rings, P be an n-ary predicate, and L_P be the expansion of L by P. There is an L_P-formula $\varphi(x)$ with $x = (x_1, \dots, x_n)$ such that, for any subset X of $\mathbb{R}^n$, $\{b \in \mathbb{R}^n : (\mathbb{R}, <, +, \times, X) \models \varphi(b)\}$ is the set of C^k-smooth points of X.*

Proof of Proposition 6.1 when $\mathcal{R}$ is field-type. We will first show that there are open intervals I and J, definable functions $\oplus, \otimes : I^2 \to I$ and an isomorphism $\tau : I \to \mathbb{R}$ between $(I, <, \oplus, \otimes) \to (\mathbb{R}, +, \times)$ such that $J \subseteq I$ and the restriction of τ to J is a C^k-diffeomorphism. By Theorem 2.1.2, we only have to consider the case that $\mathcal{R}$ is type A or type C, because $\mathcal{R}$ is field-type. The type A case is shown in [20, Lemma 6.2].

Consider the case that $\mathcal{R}$ is type C. In this situation, $\mathcal{R}$ defines all bounded Borel sets. Set

$$\tau : (-2,2) \to \mathbb{R}$$
$$x \mapsto \begin{cases} -\frac{1}{x+2}, & -2 \le x < -1 \\ x, & -1 \le n \le 1 \\ \frac{1}{2-x}, & 1 < n \le 2 \end{cases}$$

Since $\mathcal{R}$ defines all bounded Borel sets, it is not hard see that $\mathcal{R}$ defines functions $\oplus$ and $\otimes$ such that τ is an isomorphism between $((-2,2), <, \oplus, \otimes)$ and $(\mathbb{R}, +, \times)$. Now set $J := (-1,1)$ and observe that the restriction of τ to J is a C^k-diffeomorphism.

Set $L := \tau(J)$. Let $\tau_m : J^m \to L^m$ be given by

$$\tau_m(x_1, \dots, x_m) = (\tau(x_1), \dots, \tau(x_m)) \quad \text{for all} \quad x_1, \dots, x_m \in J.$$

Note that τ_m is a C^k-diffeomorphism.

Let $(Z_a)_{a \in S}$ be a definable family of subsets of J^m. We now show that the sets of C^k-smooth points of this family are uniformly definable. By Fact 6.4 there is a definable family $(Y_a)_{a \in S}$ such that

$$Y_a := \{z \in Z_a \ : \ \tau_m(z) \text{ is a } C^k\text{-smooth point of } \tau_m(Z_a)\}.$$

Since τ_m is a C^k-diffeomorphism, we get that Y_a is the set of C^k-smooth points of Z_a.

Let $X \subseteq \mathbb{R}^n$ be definable. We are now ready to prove the definability of the set of C^k-smooth points of X. Fix $u \in J^m$ and $\varepsilon > 0$ such that $B_\varepsilon(u) \subseteq J^m$. For $x \in X$ and $\delta > 0$ with $\delta < \varepsilon$, we let $g_x : B_\delta(x) \to B_\delta(u)$ be the C^∞-diffeomorphism defined by

$$y \mapsto u + (y - x).$$

Now consider the set

$$Y := \{x \in X \ : \ \exists \delta > 0 \ \delta < \varepsilon \wedge u \text{ is a } C^k\text{-smooth point of } g_x(B_\delta(x) \cap X)\}.$$

Observe that the Y is definable by the argument in the previous paragraph. We finish the proof by establishing that Y is the set of C^k-smooth points of X.

Let $x \in X$ be a C^k-smooth point of X. Take $\delta > 0$ such that $\delta < \varepsilon$. Since x is a C^k-smooth point of X, it is also a C^k-smooth point of $X \cap B_\delta(x)$. Since g_x is a C^∞-diffeomorphism, we have that $g_x(x)$ is a C^k-smooth point of $g_x(B_\delta(x) \cap X)$. Since $g_x(x) = u$, we have that $x \in Y$.

Let $x \in Y$. Let $0 < \delta < \varepsilon$ be such that u is a C^k-smooth point of $g_x(B_\delta(x) \cap X)$. Since $g_x(x) = u$ and g_x is a C^∞-diffeomorphism, we deduce that x is a C^k-smooth point of $X \cap B_\delta(x)$ and hence a C^k-smooth point of X. □

7. Type A expansions without dimension coincidence

In this section we discuss two examples of type A expansions in which dimension coincidence fails. In Section 7.1 we discuss a natural family of type A expansions in which topology and Assouad dimensions do not agree and in Section 7.2 we describe a type A structure which defines a compact nowhere dense subset of $\mathbb{R}$ with positive Hausdorff dimension.

7.1. $(\mathbb{R}, <, +, \mathbb{Z})$ **and related structures.** Perhaps the most example natural of a subset with topological dimension zero and positive Assouad dimension is $\mathbb{Z}$, which has Assouad dimension one. More generally $\mathbb{Z}^m$ has topological dimension zero and Assouad dimension m. The structure $(\mathbb{R}, <, +, \mathbb{Z})$ is well-known to be tame. Miller [26] and Weispfenning [31] independently showed that $(\mathbb{R}, <, +, \mathbb{Z})$ has quantifier elimination in a natural expanded language. It follows that $(\mathbb{R}, <, +, \mathbb{Z})$ is locally o-minimal, hence type A. Similarly, Marker and Steinhorn showed in unpublished work that $(\mathbb{R}, <, +, \sin)$ is locally o-minimal, see [29]. Observe that $(\mathbb{R}, <, +, \sin)$ defines $\pi\mathbb{Z}$, a set of topological dimension zero and Assouad dimension one. These two examples belong to a general family characterized by Fact 7.1 below.

Given $\alpha > 0$ we let $+_\alpha$ be the function $[0, \alpha)^2 \to [0, \alpha)$ given by $t +_\alpha t^* = t + t^*$ when $t + t^* < \alpha$ and $t +_\alpha t^* = t + t^* - \alpha$ otherwise.

Fact 7.1. *The following are equivalent:*

1. *$\mathcal{R}$ is interdefinable with $(\mathbb{R}, <, +, \mathcal{B}, \alpha\mathbb{Z})$ for some $\alpha > 0$ and collection $\mathcal{B}$ of bounded subsets of Euclidean space such that $(\mathbb{R}, <, +, \mathcal{B})$ is o-minimal.*
2. *The reduct $\mathcal{R}^*$ of $\mathcal{R}$ generated by all bounded definable sets is o-minimal and $\mathcal{R}$ is interdefinable with $(\mathcal{R}^*, \alpha\mathbb{Z})$ for some $\alpha > 0$.*
3. *$\mathcal{R}$ is interdefinable with $(\mathcal{S}, \alpha\mathbb{Z})$ for some $\alpha > 0$ and o-minimal expansion $\mathcal{S}$ of $(\mathbb{R}, <, +)$ such that $(\mathcal{S}, \alpha\mathbb{Z})$ is locally o-minimal.*
4. *There is $\alpha > 0$ and an o-minimal expansion $\mathcal{O}$ of $([0, \alpha), <, +_\alpha)$ such that every definable subset of $\mathbb{R}^m$ is a finite union of sets of the form $X + Y$ for $\mathcal{O}$-definable $X \subseteq [0, \alpha)^m$ and $(\alpha\mathbb{Z}, <, +)$-definable $Y \subseteq (\alpha\mathbb{Z})^m$.*

For example $(\mathbb{R}, <, +, \sin)$ is interdefinable with $(\mathbb{R}, <, +, \sin|_{[0,\pi]}, \pi\mathbb{Z})$, recall that $(\mathbb{R}, <, +, \sin|_{[0,\pi]})$ is o-minimal. More generally if $f : \mathbb{R} \to \mathbb{R}$ is analytic and periodic with period $\alpha > 0$ then $(\mathbb{R}, <, +, f)$ is interdefinable with $(\mathbb{R}, <, +, f|_{[0,\alpha]}, \alpha\mathbb{Z})$ and $(\mathbb{R}, <, +, f|_{[0,\alpha]})$ is o-minimal, hence $(\mathbb{R}, <, +, f)$ satisfies (1)-(4) above.

We now suppose that $\mathcal{R}$ satisfies one of the the equivalent conditions of Fact 7.1. We suppose that $\alpha = 1$ for the sake of simplicity. Item (4) shows that $\mathcal{R}$ is bi-interpretable with the disjoint union of an o-minimal expansion of $([0, 1), <, +_1)$ and $(\mathbb{Z}, <, +)$. In particular every $\mathcal{R}$-definable subset of $\mathbb{Z}^m$ is definable in $(\mathbb{Z}, <, +)$. There is a canonical dimension on $(\mathbb{Z}, <, +)$-definable sets which we denote by $\dim_\mathbb{Z}$, see [4]. Suppose that X is a definable subset of $\mathbb{R}^m$. We define the global dimension

$$\dim_{\text{Global}} X = \dim_\mathbb{Z}\{b \in \mathbb{Z}^m : (b + [0, 1)^m) \cap X \neq \emptyset\}$$

and the local dimension

$$\dim_{\text{Local}} X = \max\{\dim X \cap (b + [0, 1)^m) : b \in \mathbb{Z}^m\}$$

of X. It is easy to see that the topological, packing, and Hausdorff dimensions of X agree with $\dim_{\text{Local}} X$. One can also use the theory of weak tangents to show that the Assouad dimension of X agrees with $\max\{\dim_{\text{Local}} X, \dim_{\text{Global}} X\}$, see [12, 5.1] for a discussion of weak tangents.

7.2. **Topological and Hausdorff.** Whenever $\mathcal{R}$ is type B, we know that $\mathcal{R}$ defines an isomorphic copy of $\mathcal{B}$ by Fact 2.1.1. In this section we show that in Statement (ii) of Theorem B the statement "defines an isomorphic copy of $\mathcal{B}$" cannot be replaced by the stronger statement "is type B", even under the stronger assumption that $\dim X < \dim_{\text{Hausdorff}} X$. We do so by giving an example of a type A expansion that defines a compact zero-dimensional subset of $\mathbb{R}$ with positive Hausdorff dimension (and hence positive packing dimension). Our construction is based on an application of the following result of Friedman and Miller [14, Theorem A].

Proposition 7.2. *Let $\mathcal{M}$ be an o-minimal expansion of $(\mathbb{R}, <, +)$, and let $E \subseteq \mathbb{R}$ be closed and nowhere dense. Then the following are equivalent:*

1. *$(\mathcal{M}, E)$ is type A,*
2. *every $(\mathcal{M}, E)$-definable subset of $\mathbb{R}$ has interior or is nowhere dense,*
3. *$f(E^n)$ is nowhere dense for every $\mathcal{M}$-definable $f : \mathbb{R}^n \to \mathbb{R}$.*

Proof. By [14, Theorem A], (3) implies (2). A dense ω-orderable set is dense and co-dense in some nonempty open interval. Therefore (2) implies (1). It follows from Fact 2.7 that (1) implies (3). □

The following lemma can be deduced easily from the semi-linear cell decomposition (see van den Dries [30, Corollary 1.7.8]) for $(\mathbb{R}, <, +)$-definable sets. We leave the details to the reader.

Lemma 7.3. *Let $E \subseteq \mathbb{R}$. The following are equivalent:*

1. *$f(E^n)$ is nowhere dense for every $(\mathbb{R}, <, +)$-definable $f : \mathbb{R}^n \to \mathbb{R}$,*
2. *$T(E^n)$ is nowhere dense for every $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$.*

We now characterize compact nowhere dense subsets E of $\mathbb{R}$ such that $(\mathbb{R}, <, +, E)$ is type A.

Theorem 7.4. *Let $E \subseteq \mathbb{R}$ be compact and nowhere dense. Then the following are equivalent:*

1. *$(\mathbb{R}, <, +, E)$ is type A,*
2. *every $(\mathbb{R}, <, +, E)$-definable subset of $\mathbb{R}$ has interior or is nowhere dense,*
3. *$T(E^n)$ is nowhere dense for every $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$,*
4. *the subgroup of $(\mathbb{R}, +)$ generated by E is not equal to $\mathbb{R}$.*

Proof. The equivalence of (1), (2), and (3) follows immediately from Proposition 7.2 and Lemma 7.3. We show that (3) and (4) are equivalent.

Suppose that $T(E^n)$ is nowhere dense for all $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$. For $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$, set

$$E_u := \{u_1 e_1 + \dots + u_n e_n : e_1, \dots, e_n \in E\}.$$

Then $\bigcup_{n>0} \bigcup_{u \in \mathbb{Z}^n} E_u$ is the subgroup of $(\mathbb{R}, +)$ generated by E. Since $T(E^n)$ is nowhere dense for all $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$, we have that E_u is nowhere dense for every $u \in \mathbb{Z}^n$. Thus $\bigcup_{n>0} \bigcup_{u \in \mathbb{Z}^n} E_u$ is a countable union of nowhere dense sets, and hence by the Baire category theorem it is not equal to $\mathbb{R}$. Thus the subgroup of $(\mathbb{R}, +)$ generated by E is not equal to $\mathbb{R}$.

We show that (4) implies (3) by contrapositive. Suppose (4) fails. Then there is $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and a $\mathbb{Q}$-linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(E^n)$ is somewhere dense and

$$T(x) = q_1 x_1 + \ldots + q_n x_n \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $T(E^n)$ is compact as E^n is compact. Thus $T(E^n)$ has interior. Let $m \in \mathbb{Z}$ be such that $mq_i \in \mathbb{Z}$ for all $1 \le i \le n$, and set $mq := (mq_1, \ldots, mq_n)$. Then

$$E_{mq} = \{mq_1 e_1 + \ldots + mq_n e_n : e_1, \ldots, e_n \in E\} = mT(E^n)$$

has interior. So the subgroup of $(\mathbb{R}, +)$ generated by E has interior and therefore equals $\mathbb{R}$. $\square$

When E is closed, the subgroup of $(\mathbb{R}, +)$ generated by E is F_σ and therefore Borel. There are examples, first due to Erdös and Volkmann [7], of proper Borel subgroups of $(\mathbb{R}, +)$ with positive Hausdorff dimension. If G is a proper Borel subgroup of $(\mathbb{R}, +)$ with positive Hausdorff dimension, then inner regularity of Hausdorff measure (see [25, Corollary 4.5]) yields a compact subset of G with positive Hausdorff dimension. Such subsets necessarily have empty interior and are therefore topological dimension 0 (See Falconer [8, Example 12.4] for specific examples of compact subsets of $\mathbb{R}$ that generate proper subgroups of $(\mathbb{R}, +)$). By adding one of these compact set with positive Hausdorff dimension to $(\mathbb{R}, <, +)$ we obtain by Theorem 7.4 a type A expansion that satisfies the assumptions of Theorem B (ii).

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