AN NIP STRUCTURE WHICH DOES NOT INTERPRET AN INFINITE GROUP BUT WHOSE SHELAH COMPLETION INTERPRETS AN INFINITE FIELD

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Abstract. We describe one.

1. Introduction

All structures are first order and "definable" means "first order definable, possibly with parameters". Let $\mathcal{M}$ be a structure. The structure **induced** on $A \subseteq M^m$ by $\mathcal{M}$ is the structure with domain A whose primitive n-ary relations are all sets of the form $X \cap A^n$ for $\mathcal{M}$-definable $X \subseteq M^{nm}$. Let $\mathcal{N}$ be a highly saturated elementary extension of $\mathcal{M}$. The **Shelah completion** $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ is the structure induced on M by $\mathcal{N}$. A subset of M^n is **externally definable** if it is of the form $X \cap M^n$ for some $\mathcal{N}$-definable $X \subseteq N^n$. Saturation shows that the collection of externally definable sets does not depend on choice of $\mathcal{N}$, so $\mathcal{M}^{\mathrm{Sh}}$ essentially does not depend on choice of $\mathcal{N}$. Fact 1.1 is due to Shelah [7], see also Chernikov and Simon [2].

Fact 1.1. *Suppose $\mathcal{M}$ is* NIP*. Then every $\mathcal{M}^{\mathrm{Sh}}$-definable set is externally definable.*

Fact 1.1 implies that $\mathcal{M}^{\mathrm{Sh}}$ is NIP when $\mathcal{M}$ is NIP. More generally and informally, it shows that $\mathcal{M}^{\mathrm{Sh}}$ has the same combinatorial properties as $\mathcal{M}$. We show that new algebraic structure can appear in $\mathcal{M}^{\mathrm{Sh}}$. Geometric stability theory contains dichotomies between combinatorial simplicity and algebraic structure. Our construction suggests that if one seeks to obtain such dichotomies in the NIP setting then one should look for the algebraic structure in the Shelah completion. If $\mathcal{M}$ is stable then every externally definable set is definable, so this phenomenon cannot occur in the stable setting.

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2. Notation and Conventions

Throughout s, t are real numbers. An open set in a topological space is **regular** if it is the interior of its closure. A subset of an o-minimal structure is **independent** if it is independent in the sense of algebraic closure (equivalently: definable closure). We let $\mathrm{Cl}(X)$ be the closure in $\mathbb{R}^n$ of $X \subseteq \mathbb{R}^n$.

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3. The structure

Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +)$, $\mathcal{R} \prec \mathcal{N}$ be highly saturated, H be a dense independent subset of N, and $\mathcal{H}$ be the structure induced on H by $\mathcal{N}$. Dolich, Miller, and Steinhorn [3] study the expansion of an o-minimal structure by a unary predicate defining a dense independent set. Fact 3.1 is [3, 2.16].

Fact 3.1. *Any subset of H^n definable in $(\mathcal{N}, H)$ is of the form $X \cap H^n$ for some $\mathcal{N}$-definable $X \subseteq N^n$. It follows that the theory of $\mathcal{H}$ is weakly o-minimal.*

Fact 3.1 shows that $\mathcal{H}$ is NIP, furthermore dp-minimal and distal. Informally: $\mathcal{H}$ should have the same combinatorial properties as $\mathcal{R}$.

Proposition 3.2. *$\mathcal{H}$ does not interpret an infinite group.*

Proof. A theorem of Eleftheriou [5, Theorem C] shows that $\mathcal{H}$ eliminates imaginaries, so it suffices to show that $\mathcal{H}$ does not define an infinite group. Fact 3.1 and the fact that H is independent together show that the algebraic closure in $\mathcal{H}$ of any $A \subseteq H$ is A. It is now routine to show that $\mathcal{H}$ does not define an infinite group. $\square$

Proposition 3.4 below shows that $\mathcal{H}^{\mathrm{Sh}}$ interprets $(\mathbb{R}, <, +)$ in general, and interprets $(\mathbb{R}, <, +, \cdot)$ when $\mathcal{R}$ expands $(\mathbb{R}, <, +, \cdot)$. To obtain interpretability of $\mathcal{R}$ we will need Lemma 3.3. It is easier to show that $(\mathbb{R}, <, +)$ or $(\mathbb{R}, <, +, \cdot)$ is interpretable, as

$$\{(s, s', t) \in \mathbb{R}^3 : s + s' < t\} \quad \text{and} \quad \{(s, s', t) \in \mathbb{R}^3 : ss' < t\}$$

are both regular open.

Lemma 3.3. *Every $\mathcal{R}$-definable set is a boolean combination of regular open $\mathcal{R}$-definable sets.*

The proof is complicated by the fact that a definable open set need not be a union of finitely many open cells. We say that $\mathcal{R}$ defines a global field structure if there are definable $\oplus, \otimes : \mathbb{R}^2 \to \mathbb{R}$ such that $(\mathbb{R}, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$. If $\mathcal{R}$ defines a global field structure then the topological study of $\mathcal{R}$-definable sets entirely reduces to the study of definable sets in o-minimal expansions of $(\mathbb{R}, <, +, \cdot)$.

Proof. An application of o-minimal cell decomposition shows that every $\mathcal{R}$-definable set is a boolean combination of open $\mathcal{R}$-definable sets, so it suffices to suppose $U \subseteq \mathbb{R}^n$ is open and $\mathcal{R}$-definable and show that U is a finite union of regular open $\mathcal{R}$-definable sets. Note that an open cell is regular.

Edmundo, Eleftheriou, and Prelli [4] show that if $\mathcal{R}$ does not define a global field structure then U is a finite union of open cells. Suppose $\mathcal{R}$ defines a global field structure. Without loss of generality we suppose $\mathcal{R}$ expands $(\mathbb{R}, <, +, \cdot)$. Let B_t be the open ball in $\mathbb{R}^n$ with center the origin and radius $t > 0$. It suffices to show that $U \cap B_2$ and $U \setminus \mathrm{Cl}(B_1)$ are both finite unions of definable regular open sets. Let $\iota : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the definable homeomorphism given by

$$\iota(t_1, \ldots, t_n) = (t_1^{-1}, \ldots, t_n^{-1}).$$

Wilkie [8] shows that any bounded definable open set is a finite union of open cells. So $U \cap B_2$ is a finite union of open cells. Furthermore $\iota(U \setminus \mathrm{Cl}(B_1)) \subseteq B_1$ is a union of open cells $V_1, \ldots, V_m$. So $U \setminus \mathrm{Cl}(B_1)$ is the union of the $\iota(V_k)$. As each V_k is regular open it follows that each $\iota(V_k)$ is regular open. So $U \setminus \mathrm{Cl}(B_1)$ is a finite union of definable regular open sets. $\square$

Proposition 3.4. $\mathcal{H}^{\mathrm{Sh}}$ *interprets* $\mathcal{R}$.

We will need to apply the easy fact that if $\mathcal{M}$ is an NIP expansion of a linear order then every convex subset of M is externally definable.

Proof. Let O be the set of $a \in H$ such that $|a| < t$ for some $t > 0$. Let Q be the set of $a \in H$ such that $s < a < t$ for some $s, t > 0$. Then O and Q are both convex, hence definable in $\mathcal{H}^{\mathrm{Sh}}$.

Let E be the equivalence relation on H where $(a, b) \in E$ when $|a - b| < t$ for all $t > 0$. We show that E is definable in $\mathcal{H}^{\mathrm{Sh}}$. Let X be the set of $(a, b, c) \in N^3$ such that $|a - b| < c$. Then $C := X \cap H^3$ is definable in $\mathcal{H}$. Observe that

$$E = \bigcap_{c \in Q} \{(a, b) \in H^2 : (a, b, c) \in C\}$$

so E is definable in $\mathcal{H}^{\mathrm{Sh}}$.

Each E-class is convex so we put a $\mathcal{H}^{\mathrm{Sh}}$-definable linear order on H/E by declaring the class of a to be less than the class of b when $a < b$. Observe that the E-class of any element of O contains a unique real number and every real number is contained in the E-class of some element of O. We therefore identify O/E with $\mathbb{R}$, observe that the $\mathcal{H}^{\mathrm{Sh}}$-definable ordering on O/E agrees with the usual order on $\mathbb{R}$, and let $\mathrm{st} : O \to \mathbb{R}$ be the quotient map. As $\mathcal{H}^{\mathrm{Sh}}$ defines the usual order on $\mathbb{R}$, it defines a basis for the topology on $\mathbb{R}^n$. We show that $\mathcal{R}$ is a reduct of the structure induced on $\mathbb{R}$ by $\mathcal{H}^{\mathrm{Sh}}$. By Lemma 3.3 it suffices to suppose that $U \subseteq \mathbb{R}^n$ is regular, open, and $\mathcal{R}$-definable and show that U is definable in $\mathcal{H}^{\mathrm{Sh}}$. Let U' be the subset of N^n defined by any formula defining U. It is easy to see that $\mathrm{Cl}(U) = \mathrm{st}(U' \cap O^n)$, so $\mathrm{Cl}(U)$ is definable in $\mathcal{H}^{\mathrm{Sh}}$. So U is $\mathcal{H}^{\mathrm{Sh}}$-definable as U is the interior of $\mathrm{Cl}(U)$ and $\mathcal{H}^{\mathrm{Sh}}$ defines a basis for $\mathbb{R}^n$. $\square$

The proof of Proposition 3.4 shows that $\mathcal{R}$ is interpretable in the expansion of $\mathcal{H}$ by two convex sets, O and Q.

4. Triviality?

There should be a notion of triviality for NIP structure satisfying the following.

- (A1) A trivial structure cannot interpret an infinite group.
- (A2) Any linear order is trivial.
- (A3) An o-minimal structure is trivial if and only if it is trivial in the sense of the Peterzil-Starchenko trichotomy [6].
- (A4) Suppose $\mathcal{M}$ is a trivial NIP structure, and $A \subseteq M^m$ is such that every subset of A^m definable in the induced structure on A is of the form $X \cap A^n$ for some $\mathcal{M}$-definable $X \subseteq M^{mn}$. Then the induced structure on A is trivial. (So the Shelah completion of a trivial NIP structure is trivial.)

If there is a notion of triviality satisfying (A1)-(A4) then one would follow that if $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, <)$, $\mathcal{N}$ is a highly saturated elementary extension of $\mathcal{R}$, H is a dense independent subset of N, and $\mathcal{H}$ is the induced structure on H, then $\mathcal{H}$ is trivial if and only if $\mathcal{R}$ is trivial. (Note: Dolich, Miller, and Steinhorn only study o-minimal expansions of ordered groups, in the more general case one has to apply Berenstein and Vassiliev [1], in particular [1, Proposition 3.5]).

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