INTERPOLATIVE FUSIONS II: PRESERVATION RESULTS

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Abstract. We study interpolative fusion, a method of combining theories T_1 and T_2 in distinct languages in a "generic" way over a common reduct $T_\cap$, to obtain a theory $T_\cup^*$. When each T_i is model-complete, $T_\cup^*$ is the model companion of the union $T_1 \cup T_2$. Our goal is to prove preservation results, i.e., to find sufficient conditions under which model-theoretic properties of T_1 and T_2 are inherited by $T_\cup^*$.

We first prove preservation results for quantifier elimination, model-completeness, and related properties. We then apply these tools to show that, under mild hypotheses, including stability of $T_\cap$, the property NSOP_1 is preserved. We also show that simplicity is preserved under stronger hypotheses on algebraic closure in T_1 and T_2. This generalizes many previous results; for example, simplicity of ACFA and the random n-hypergraph are both non-obvious corollaries. We also address preservation of stability, NIP, and $\aleph_0$-categoricity, and we describe examples which witness that these results are sharp.

1. Introduction

It is a well-known idea that simplicity can be seen as "stability + randomness". The results in this paper suggest a slight variation of this idea: the property NSOP_1 should be seen as "stability + randomness", while simplicity should be seen as "stability + controlled randomness". In particular, we show that a generic (or "random") fusion of stable theories (over a common reduct) is NSOP_1. This provides us with a conceptual explanation for the fact that several known theories are NSOP_1 by also observing that these theories can be decomposed as generic fusions of stable theories. The above result is a special case of the following preservation phenomenon (Theorem 1.1): Under some mild hypotheses, a generic fusion of NSOP_1 theories over a common stable reduct is NSOP_1. Simplicity, on the other hand, is preserved under the more restrictive setting of "relatively disintegrated fusions" (Theorem 1.2). Several important examples do fit into this setting: the simplicity of the random graph and simplicity of ACFA can both be seen as special cases of Theorem 1.2. We also show that other model-theoretic properties (stability, NIP, $\aleph_0$-categoricity, etc.) of many well-known examples can be seen as special cases of results on preservation of these properties under generic fusions.

We first describe the basic setting and set some global conventions. Throughout I is an index set, $L_\cap$ is a language, $(L_i)_{i\in I}$ is a family of languages, all with the same set of sorts, such that $L_i \cap L_j = L_\cap$ when $i \neq j$, T_i is a (possibly incomplete) consistent L_i-theory for each i, and each T_i has the same set $T_\cap$ of $L_\cap$-consequences. In most examples each T_i is complete, in this case our assumptions ensure that $T_i \cap T_j = T_\cap$ when $i \neq j$. We declare $L_\cup \coloneqq \bigcup_{i\in I} L_i$, $T_\cup \coloneqq \bigcup_{i\in I} T_i$, and suppose that

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$T_\cup$ is consistent. (When $T_\cap$ is complete this follows by Robinson joint consistency.) Throughout $\mathcal{M}_\cup$ is an $L_\cup$-structure and $\mathcal{M}_\square$ is the $L_\square$-reduct of $\mathcal{M}_\cup$ for $\square \in I \cup \{\cap\}$.

In [KTW21] we began to study the interpolative fusion $T^*_\cup$ of $(T_i)_{i\in I}$ over $T_\cap$ (which exists under some conditions); this is the precise version of the generic (or "random") fusion described earlier. When each T_i is model complete $T^*_\cup$ is simply the model companion (if it exists) of $T_\cup$ and in general one can view $T^*_\cup$ as the model companion of $T_\cup$ *relative* to $(T_i)_{i\in I}$. (We will often reduce to the case when each T_i is model complete by Morleyizing.) We prefer to work with a language-independent definition of $T^*_\cup$ described in Section 2.1. Our first paper [KTW21] was largely devoted to existence conditions for $T^*_\cup$. In this paper we will generally assume that $T^*_\cup$ exists. The reader of this paper will only need certain things from [KTW21] which are recalled in Section 2.1.

Our notion of fusion generalizes many "generic" model-theoretic constructions. Interesting theories are often (up to bi-interpretation) fusions of strictly simpler theories, e.g. the theory of the random graph is bi-interpretable with a fusion of two theories which are both bi-interpretable with the theory of equality and ACFA is bi-interpretable with a fusion of two theories which are both bi-interpretable with ACF. We survey many more examples in Section 6; these provid us with a testing ground for the preservation results we will discuss next.

Section 4 studies the preservation of classification theoretic properties. Stability and NIP are only rarely preserved (see Section 4.1). This is natural, as the fusion typically introduces "randomness". We do find sufficient conditions for their preservation which cover a number of interesting examples. At present, NSOP_1 is the only classification theoretic property which we know to be broadly preserved (see Section 4.2). An extra necessary hypothesis is a slight strengthening of the independence theorem, which we call the $T_\cap$-generic independence theorem. This hypothesis holds in all examples that we know of an NSOP_1 theory with a stable reduct. Our main result for NSOP_1 is Theorem 1.1 (following from Theorem 4.6, Corollary 4.7, and Corollary 4.8).

Theorem 1.1. *Assume $T^*_\cup$ exists. Suppose $T_\cap$ is stable, each T_i is NSOP_1, and Kim-independence in each T_i satisfies the $T_\cap$-generic independence theorem. Then $T^*_\cup$ is NSOP_1. It follows that:*

(1) If each T_i is stable, then $T^*_\cup$ is NSOP_1.
(2) If $T_\cap$ is stable with disintegrated forking and each T_i is NSOP_1, then $T^*_\cup$ is NSOP_1.

We say a theory T has **disintegrated forking** if $A \not\perp^f_C B$ implies $A \not\perp^f_C b$ for some element $b \in B$. Some authors refer to this as "trivial forking". An NSOP_1 theory is either simple or has the independence property. Hence it follows from (1) above that if each T_i is stable and $T^*_\cup$ is NIP then $T^*_\cup$ is stable.

NSOP_1 theories come equipped with a canonical independence relation, Kim-forking. Under the assumptions of Theorem 1.1, we give an explicit description of Kim-forking in $T^*_\cup$ in terms of Kim-forking in the various T_i. (In a simple theory Kim-forking agrees with forking.)

Using results about preservation of $\aleph_0$-categoricity discussed in Section 5.1, we show in Section 5.2 that stability of $T_\cap$ in Theorem 1.1 is necessary. In particular, we construct two simple theories T_1 and T_2 whose fusion over a simple theory $T_\cap$

interprets the Henson graph, the best-known example of a theory which is NSOP and SOP_1. In this example $T_\cap$ is the theory of the random 3-hypergraph.

We will see that Theorem 1.1 generalizes several known results: the model companion of the empty theory in an arbitrary language is NSOP_1, the generic Skolemization of an NSOP_1 theory is NSOP_1, the theory of an algebraically closed field of positive characteristic with a generic additive subgroup is NSOP_1, etc.

Simplicity is only preserved under more restrictive hypotheses (see Section 4.3). However, some of the main examples of simple theories can be realized as fusions of stable theories satisfying these hypotheses. Theorem 1.2 is our main result on simple theories (following from Theorem 4.13). We say that T_i is disintegrated relative to $T_\cap$ (or just **relatively disintegrated**) if for any $\mathcal{M}_i \models T_i$ we have

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap\left(\bigcup_{a\in A} \mathrm{acl}_i(a)\right) \quad \text{for all subsets } A \text{ of } M.$$

Theorem 1.2. *Suppose $T^*_\cup$ exists, each T_i is simple, $T_\cap$ is stable, and either:*

(1) every T_i is disintegrated relative to $T_\cap$, or

(2) there is $i^* \in I$ such that Kim-independence in T_{i^*} satisfies the $T_\cap$-generic independence theorem, and if $i \neq i^*$ then algebraic closure and forking in T_i agrees with algebraic closure and forking in $T_\cap$.

Then $T_\cup$ is simple.

We give a number of examples in Section 6 which show that the strong assumptions of Theorem 1.2 are sharp (fusions are typically TP_2). We view (1) as lift of the fact that ACFA is simple and (2) as a lift of the fact that that the expansion of a simple theory by a generic unary predicate is simple. Case (2) generalizes a result of Tsuboi [Tsu01], see Fact 4.12 below. The assumptions in (2) are very strong but they are satisfied in more examples then one might expect, in particular the random n-hypergraph and the generic tournament.

The main tool required for Theorems 1.1 and 1.2 is Theorem 1.3 (following from Proposition 3.9). It is of independent interest.

Theorem 1.3. *Suppose $T_\cap$ is stable with weak elimination of imaginaries and $T^*_\cup$ exists. Let $\mathcal{M}_\cup \models T^*_\cup$ and A be a subset of $\mathcal{M}$. Then*

(1) every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas

$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$

where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J}\varphi_i(x,y)$ is bounded in y,

(2) a tuple b is in $\mathrm{acl}_\cup(A)$ if and only if

$$b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1,\dots,i_n \in I,$$

(i.e. $\mathrm{acl}_\cup(A)$ is the smallest superset of A which is acl_i-closed for all $i \in I$),

(3) and if A is $\mathrm{acl}_\cup$-closed then

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{tp}_{L_i}(A) \models \mathrm{tp}_{L_\cup}(A).$$

Theorem 1.3 may fail without weak elimination of imaginaries, see Section 6.8. In applications of Theorem 1.3 we can usually avoid assuming weak elimination of imaginaries by passing to the eq of a structure.

1.1. **Conventions and notation.** All languages and theories are first-order. Throughout, I, $(L_i)_{i\in I}$, $L_\cap$, $L_\cup$, $(T_i)_{i\in I}$, $T_\cap$, $T_\cup$, and $T^*_\cup$ are as in the introduction. We let acl_i be algebraic closure in T_i and $\mathrm{acl}_\cup$ be algebraic closure in $T_\cup$. We let $\boldsymbol{\mathcal{M}}$ be the monster model. All models other than the monster are small and all sets of parameters are small.

We work in a multi-sorted setting. Let S be the set of sorts of L. Suppose $\mathcal{M}$ is an L-structure. We use the corresponding capital letter M to denote the S-indexed family $(M_s)_{s\in S}$ of underlying sets of the sorts of $\mathcal{M}$. By $A \subseteq M$, we mean $A = (A_s)_{s\in S}$ with $A_s \subseteq M_s$ for each $s \in S$. If $A \subseteq M$, then a tuple of elements (possibly infinite) in A is a tuple whose each component is in A_s for some $s \in S$. If $x = (x_j)_{j\in J}$ is a tuple of variables (possibly infinite), we let $A^x = \prod_{j\in J} A_{s(x_j)}$ where $s(x_j)$ is the sort of the variable x_j. If $\varphi(x,y)$ is an L-formula and $b \in M^y$, we let $\varphi(\mathcal{M}, b)$ be the set defined in $\mathcal{M}$ by the $L(b)$-formula $\varphi(x,b)$. We call such $\varphi(\mathcal{M}, b)$ a definable set in $\mathcal{M}$ or an $\mathcal{M}$-definable set. Hence, "definable" means "definable, possibly with parameters". If we wish to exclude parameters, we write "$\varnothing$-definable".

2. Preliminaries

2.1. **Basic properties of interpolative fusions.** We work with the following language-independent definition of $T^*_\cup$, from [KTW21]. We say that $\mathcal{M}_\cup \models T_\cup$ is **interpolative** if whenever $J \subseteq I$ is finite, X_i is an $\mathcal{M}_i$-definable set for all $i \in J$ and $\bigcap_{i\in J} X_i = \varnothing$, then for each $i \in J$ there is an $\mathcal{M}_\cap$-definable set Y_i such that $X_i \subseteq Y_i$ and $\bigcap_{i\in J} Y_i = \varnothing$. If the interpolative $T_\cup$-models form an elementary class then we let $T^*_\cup$ be their theory and refer to $T^*_\cup$ as the **interpolative fusion** of $(T_i)_{i\in I}$ (over $T_\cap$). We also say "$T^*_\cup$ exists" if the class of interpolative $T_\cup$-models is elementary. In most of this paper we assume existence of $T^*_\cup$.

Fact 2.1 ([KTW21, Theorem 2.12]). *Suppose each T_i is model-complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed in the class of $T_\cup$-models. Hence, $T^*_\cup$ is precisely the model companion of $T_\cup$, if either of these exists.*

Remark 2.2 ([KTW21, Remark 2.5]). If we change languages in a way that does not change the class of definable sets (with parameters), then the class of interpolative $L_\cup$-structures is not affected. In particular:

(1) Any expansion of an interpolative structure by constants (to any of the languages involved) is interpolative.
(2) Let $L^\diamond_\square$ be an expansion by definitions of $L_\square$ for $\square \in I \cup \{\cap\}$, $L^\diamond_i \cap L^\diamond_j = L^\diamond_\cap$ for distinct i and j in I, and $L^\diamond_\cup = \bigcup_{i\in I} L^\diamond_i$ is the resulting expansion by definitions of $L_\cup$. Then any $L_\cup$-structure $\mathcal{M}_\cup$ has a canonical expansion $\mathcal{M}^\diamond_\cup$ to an $L^\diamond_\cup$-structure, and $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}^\diamond_\cup$ is interpolative.
(3) An interpolative $\mathcal{M}_\cup$-structure remains interpolative after each function symbol f in each of the languages $L_\square$ for $\square \in I \cup \{\cup, \cap\}$ is replaced by a relation symbol R_f, interpreted as the graph of the interpretation of f in $\mathcal{M}_\cup$.
(4) Suppose $\mathcal{M}_\cup$ is an $L_\cup$-structure. Moving to $\mathcal{M}^{\mathrm{eq}}_\cap$ involves the introduction of new sorts and function symbols for quotients by $L_\cap$-definable equivalence relations on M. For all $\square \in I \cup \{\cup, \cap\}$, let $L^{\cap-\mathrm{eq}}_\square$ be the language expanding $L_\square$ produced by adding new symbols for $L_\square$-definable equivalence relations, and let $\mathcal{M}^{\cap-\mathrm{eq}}_\square$ be the natural expansion of $\mathcal{M}_\square$ to $L^{\cap-\mathrm{eq}}_\square$. Then $\mathcal{M}_\cup$ is interpolative if and only

if $\mathcal{M}_\cup^{\cap-\mathrm{eq}}$ is interpolative. This follows from the fact that if $X_\square$ is an $\mathcal{M}_\square^{\cap-\mathrm{eq}}$-definable set in one of the new sorts, corresponding to the quotient of M^x by an $L_\cap$-definable equivalence relation, then the preimage of $X_\square$ under the quotient is $\mathcal{M}_\square$-definable.

By Remark 2.2(2), we are always free to Morleyize each theory T_i, thereby reducing to the case where each T_i has quantifier elimination. Theories with quantifier elimination are model-complete, so Fact 2.1 applies in this situation. However, we will not make a global assumption of quantifier elimination or model-completeness, as it is sometimes desirable to study a theory in its native language.

We recall the following results from [KTW21]. Fact 2.3 can be read as saying that $T_\cup^*$ is the model companion of $T_\cup$ "relative" to $(T_i)_{i\in I}$.

Fact 2.3 ([KTW21, Theorem 2.7])**.** *Assume $T_\cup^*$ exists.*

(1) For any $\mathcal{M}_\cup \models T_\cup$ there is $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup \models T_\cup^*$ such that $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$.
(2) If $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$ are both $T_\cup^*$-models and $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$, then $\mathcal{M}_\cup \preccurlyeq \mathcal{N}_\cup$.

Fact 2.4 is a variation of the Robinson joint consistency theorem.

Fact 2.4 ([KTW21, Corollary 2.3])**.** *Suppose $p_\cap(x)$ is a complete $L_\cap$-type and $p_i(x)$ is a complete L_i-type extending $p_\cap(x)$ for all $i \in I$. Then $\bigcup_{i\in I} p_i(x)$ is a consistent (partial) $L_\cup$-type.*

Note that Fact 2.4 merely tells us that $\bigcup_{i\in I} p_i(x)$ is consistent, not that it is consistent with a background $L_\cup$-theory (e.g., $T_\cup^*$ or the elementary diagram of a model). We now rectify this situation, under the assumption that $T_\cup^*$ exists.

Proposition 2.5. *Assume $T_\cup^*$ exists and $\mathcal{M}_\cup \models T_\cup^*$. Let $p_\cap(x)$ be a complete $L_\cap(M)$-type, and for all $i \in I$, let $p_i(x)$ be a complete $L_i(M)$-type such that $p_\cap(x) \subseteq p_i(x)$. Then $\bigcup_{i\in I} p_i(x)$ is realized in an elementary extension of $\mathcal{M}_\cup$.*

Proof. By Fact 2.4, $\bigcup_{i\in I} p_i(x)$ is consistent. Suppose it is realized by a in a model $\mathcal{N}_\cup$. Since each p_i is a complete $L_i(M)$-type, $\mathrm{Ediag}(\mathcal{M}_i) \subseteq p_i(x)$, so we may assume $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$ and $\mathcal{M}_i \preccurlyeq \mathcal{N}_i$ for all $i \in I$. In particular, $\mathcal{N}_\cup \models T_\cup$, but we may not have $\mathcal{N}_\cup \models T_\cup^*$.

By Fact 2.3(1), there exists $\mathcal{M}'_\cup \models T_\cup^*$ such that $\mathcal{N}_\cup \subseteq \mathcal{M}'_\cup$ and $\mathcal{N}_i \preccurlyeq \mathcal{M}'_i$ for all $i \in I$. In particular, a satisfies $\bigcup_{i\in I} p_i(x)$ in $\mathcal{M}'_\cup$. Also, $\mathcal{M}_i \preccurlyeq \mathcal{M}'_i$ for all $i \in I$, so by Fact 2.3(2), $\mathcal{M}_\cup \preccurlyeq \mathcal{M}'_\cup$. □

Proposition 2.5 is a useful tool for realizing types, but it can only be applied to those $L_\cup$-types which are entailed by a union of L_i-types. We identify complete $L_\cup$-types with this property in Section 3.4.

2.2. **An existence result.** Much of [KTW21] is devoted to developing general conditions ensuring existence of $T_\cup^*$. For the examples in the present paper, we will only need to use one result, Fact 2.6 below. The special case of Fact 2.6 when $T_\cap$ is the theory of an infinite set is due to Winkler [Win75].

Fact 2.6. *Suppose $T_\cap$ is complete, $\aleph_0$-stable, and $\aleph_0$-categorical. Suppose one of the following holds*

(1) T_i^{eq} eliminates $\exists^\infty$ for all $i \in I$, or
(2) $T_\cap$ weakly eliminates imaginaries and each T_i eliminates $\exists^\infty$.

Then $T_\cup^*$ exists.

In [KTW21] we obtain an explicit axiomization for $T^*_\cup$ under the assumptions of Fact 2.6. This axiomization is $\forall\exists$ when each T_i is model complete. We will not use this in the present paper. If T is intepretable in the theory of equality then T $\aleph_0$-categorical and $\aleph_0$-stable, and T^{eq} eliminates $\exists^\infty$. Corollary 2.7 is a useful special case of Fact 2.6.

Corollary 2.7. *If $T_\cap$ is interpretable in the theory of equality and each $(T_i)^{\mathrm{eq}}$ eliminates $\exists^\infty$ then $T^*_\cup$ exists. If each T_i is interpretable in the theory of equality the $T^*_\cup$ exists.*

2.3. **Interpretations.** When dealing with examples, we will need to use some easy facts about interpretations. Let T be an L-theory and T' be an L'-theory. An **interpretation** of T in T', $F{:}\,T' \rightsquigarrow T$, consists of the following data:

1. For every sort s in L, an L'-formula $\varphi_s(x_s)$ and an L'-formula $E_s(x_s, x^*_s)$.
2. For every relation symbol R in L of type $(s_1, \ldots, s_n)$, an L'-formula $\varphi_R(x_{s_1}, \ldots, x_{s_n})$.
3. For every function symbol f in L of type $(s_1, \ldots, s_n) \to s$, an L'-formula $\varphi_f(x_{s_1}, \ldots, x_{s_n}, x_s)$.

We then require that for every model $\mathcal{M}' \models T'$, the formulas above define an L-structure $\mathcal{M} \models T$ in the natural way. See [Hod93, Section 5.3] for details. For every sort s in L, the underlying set M_s of the s sort in $\mathcal{M}$ is the quotient of $\varphi_s(\mathcal{M}')$ by the equivalence relation defined by E_s. We write π_s for the surjective quotient map $\varphi_s(\mathcal{M}') \to M_s$. We sometimes denote $\mathcal{M}$ by $F(\mathcal{M}')$.

An interpretation $F{:}\,T' \rightsquigarrow T$ is an **existential interpretation** if for each sort s in L, the L'-formula $\varphi_s(x_s)$ is T'-equivalent to an existential formula, and all other formulas involved in the interpretation and their negations (i.e., the formulas E_s, $\neg E_s$, φ_R, $\neg\varphi_R$, φ_f, and $\neg\varphi_f$) are also T'-equivalent to existential formulas. See [KTW21, Corollary 2.16] for a proof of Fact 2.8.

A **bi-interpretation** (F, G, η, η') between T and T' consists of an interpretation $F{:}\,T' \rightsquigarrow T$ and an interpretation $G{:}\,T \rightsquigarrow T'$, together with L-formulas and L'-formulas defining for each $\mathcal{M} \models T$ and each $\mathcal{N}' \models T'$ isomorphisms

$$\eta_{\mathcal{M}} : \mathcal{M} \to F(G(\mathcal{M})) \quad \text{and} \quad \eta'_{\mathcal{N}'} : \mathcal{N}' \to G(F(\mathcal{N}')).$$

The bi-interpretation is **existential** if F and G are each existential interpretations, the formulas defining η are T-equivalent to existential formulas, and the formulas defining η' are T'-equivalent to existential formulas.

Fact 2.8. *Suppose T and T' $\forall\exists$-axiomatizable theories which are existentially bi-interpretable. Then T has a model companion if and only if T' has a model companion. Moreover, the model companions are bi-interpretable, if they exist.*

When T^* is the model companion of a $\forall\exists$-axiomatizable theory T, we will sometimes wish to show that T^* is bi-interpretable with an interpolative fusion $T^*_\cup$. We will do this by showing that T is existentially bi-interpretable with a union of model-complete theories $T_\cup$ and then applying Facts 2.1 and 2.8.

2.4. **Flat formulas and $\mathcal{K}$-completeness.** We recall the notion of flat formula from [KTW21].

An **atomic flat formula** has the form $x = y$, $R(x_1, \ldots, x_n)$, or $f(x_1, \ldots, x_n) = y$, where R is an n-ary relation symbol and f is an n-ary function symbol. Here

$x, y, x_1, \dots, x_n$ are single variables, which need not be distinct. A **flat literal** is an atomic flat formula or the negation of an atomic flat formula. A **flat formula** is a conjunction of finitely many flat literals. An **E♭-formula** is a formula of the form $\exists y\, \varphi(x, y)$, where $\vDash \forall x\, \exists^{\leq 1} y\, \varphi(x, y)$ and $\varphi(x, y)$ is flat. Here x and y are finite tuples of variables, which may be empty.

The following result is an easy refinement of Theorem 2.6.1 in [Hod93]. Note Hodges uses the term “unnested” instead of “flat”.

Fact 2.9 ([KTW21, Lemma 2.9 and Corollary 2.10])**.** *Every atomic formula and every negated atomic formula is logically equivalent to an E♭-formula. Every quantifier-free formula is logically equivalent to a finite disjunction of E♭-formulas.*

The **flat diagram** $\mathrm{Fdiag}(\mathcal{A})$ of an L-structure $\mathcal{A}$ is the set of all flat literal $L(A)$-sentences true in $\mathcal{A}$. The flat diagram is logically equivalent to the ordinary Robinson diagram, so we have the following: If $\mathcal{B}$ is an $L(A)$-structure, then $\mathcal{B} \vDash \mathrm{Fdiag}(\mathcal{A})$ if and only if the obvious map $\mathcal{A} \to \mathcal{B}$ is an embedding [Hod93, Lemma 1.4.3].

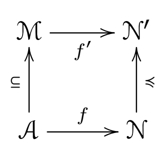
We describe a slight generalization of standard results on model completeness. Let T be an L-theory, and let $\mathcal{K}$ be a class of pairs $(\mathcal{A}, \mathcal{M})$, where $\mathcal{M} \vDash T$ and $\mathcal{A}$ is a substructure of $\mathcal{M}$. We say that T is **$\mathcal{K}$-complete** if for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$, every embedding from $\mathcal{A}$ to another T-model is partial elementary. That is, if $f\colon \mathcal{A} \to \mathcal{N}$ is an embedding and $\mathcal{N} \vDash T$, then $\mathcal{M} \vDash \varphi(a)$ if and only if $\mathcal{N} \vDash \varphi(f(a))$ for any formula $\varphi(x)$ and any $a \in A^x$.

Remark 2.10. The terminology $\mathcal{K}$-complete comes from the following equivalent definition: T is $\mathcal{K}$-complete if and only if for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$,

$$T \cup \mathrm{Fdiag}(\mathcal{A}) \vDash \mathrm{Th}_{L(A)}(\mathcal{M}),$$

i.e., $T \cup \mathrm{Fdiag}(\mathcal{A})$ is a complete $L(A)$-theory. The equivalence follows immediately from the fact that an $L(A)$-structure $\mathcal{N}$ satisfies $\mathrm{Fdiag}(\mathcal{A})$ if and only if the obvious map $\mathcal{A} \to \mathcal{N}$ is an embedding.

We say the class of T-models has the **$\mathcal{K}$-amalgamation property** if whenever $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$, $\mathcal{N} \vDash T$, and $f\colon \mathcal{A} \to \mathcal{N}$ is an embedding, then there is an elementary extension $\mathcal{N} \preccurlyeq \mathcal{N}'$ and an elementary embedding $f'\colon \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$, i.e., the following diagram commutes:



If, in the situation above, we can choose $\mathcal{N}'$ and f' with the further condition that

$$f'(M) \cap N = f'(A) = f(A),$$

then the class of T-models has the **disjoint $\mathcal{K}$-amalgamation property**.

Proposition 2.11. *The theory T is $\mathcal{K}$-complete if and only if the class of T-models has the $\mathcal{K}$-amalgamation property. Further, if T is $\mathcal{K}$-complete, then $\mathcal{A}$ is algebraically closed in $\mathcal{M}$ for all $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$ if and only if the class of T-models has the disjoint $\mathcal{K}$-amalgamation property.*

Proof. We prove the first equivalence. Suppose T is $\mathcal{K}$-complete. The $\mathcal{K}$-amalgamation property follows from [Hod93, Theorem 6.4.1].

Conversely, suppose the class of T-models has the $\mathcal{K}$-amalgamation property. If $\mathcal{M}$ and $\mathcal{N}$ are T-models, $\mathcal{A} \subseteq \mathcal{M}$ is in $\mathcal{K}$, and $f\colon \mathcal{A} \to \mathcal{N}$ is an embedding, then there is an elementary extension $\mathcal{N} \preccurlyeq \mathcal{N}'$ and an elementary embedding $f'\colon \mathcal{M} \to \mathcal{N}'$ such that $f'|_A = f$. For any L-formula $\varphi(x)$ and $a \in A^x$, $\mathcal{M} \models \varphi(a)$ if and only if $\mathcal{N}' \models \varphi(f'(a))$ if and only if $\mathcal{N} \models \varphi(f(a))$. So f is partial elementary. Thus T is $\mathcal{K}$-complete.

Now, assuming T is $\mathcal{K}$-complete, we prove the second equivalence. If every structure in $\mathcal{K}$ is algebraically closed, then the class of T-models has the disjoint $\mathcal{K}$-amalgamation property, by [Hod93, Theorem 6.4.5].

Conversely, suppose the class of T-models has the disjoint $\mathcal{K}$-amalgamation property. Assume towards a contradiction that $(\mathcal{A}, \mathcal{M}) \in \mathcal{K}$ and A is not algebraically closed in $\mathcal{M}$. Then there is some $c \in M \setminus A$ such that $\mathrm{tp}(c/A)$ has exactly k realizations $c_1, \dots, c_k$ in $M \setminus A$. Taking $\mathcal{N} = \mathcal{M}$ and $f = \mathrm{id}_A$ in the disjoint $\mathcal{K}$-amalgamation property, there is an elementary extension $\mathcal{M} \preccurlyeq \mathcal{M}'$ and an elementary embedding $f'\colon \mathcal{M} \to \mathcal{M}'$ which is the identity on A and satisfies $f'(M) \cap M = A$. Then $\mathrm{tp}(c/A)$ has $2k$ distinct realizations $c_1, \dots, c_k, f'(c_1), \dots, f'(c_k)$ in $\mathcal{M}'$, contradiction. □

Suppose T is $\mathcal{K}$-complete. If $\mathcal{K}$ is the class of all pairs $(\mathcal{M}, \mathcal{M})$ such that $\mathcal{M} \models T$, then T is **model-complete**. We say T is **substructure-complete** if $\mathcal{K}$ is the class of all pairs $(\mathcal{A}, \mathcal{M})$ such that $\mathcal{A}$ is a substructure of $\mathcal{M}$. If cl is a closure operator on T-models and $\mathcal{K}$ is the class of all pairs $(\mathcal{A}, \mathcal{M})$ such that $\mathcal{A}$ is a cl-closed substructure of $\mathcal{M}$, i.e., $\mathrm{cl}(A) = A$, then we say T is cl**-complete**. Similarly, we refer to the **(disjoint)** cl**-amalgamation property**.

Model-completeness has a syntactic equivalent: every L-formula is T-equivalent to an existential (hence also a universal) formula [Hod93, Theorem 8.3.1(e)]. Substructure-completeness also has a syntactic equivalent: quantifier elimination. This follows from [Hod93, Theorem 8.4.1] and Proposition 2.11 above.

Many of the theories we consider are acl-complete. Unfortunately, there does not seem to be a natural syntactic equivalent to acl-completeness. For this reason, we introduce a slightly stronger notion, bcl-completeness, which does have a syntactic equivalent.

An L-formula $\varphi(x, y)$ is **bounded in** y with bound k (with respect to T) if

$$T \models \forall x\, \exists^{\leq k} y\, \varphi(x, y).$$

A formula $\exists y\, \psi(x, y)$ is **boundedly existential (b.e.)** (with respect to T) if $\psi(x, y)$ is quantifier-free and bounded in y. We allow y to be the empty tuple of variables, so every quantifier-free formula is b.e. (with bound $k = 1$, by convention). The E♭-formulas introduced above are also b.e. with bound $k = 1$ with respect to the empty theory.

Suppose $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. The **boundedly existential algebraic closure** of A in $\mathcal{M}$, denoted $\mathrm{bcl}(A)$, is the set of all $b \in M$ such that $\mathcal{M} \models \varphi(a, b)$ for some $a \in A^x$ and some L-formula $\varphi(x, y)$ which is b.e. and bounded in y. It follows that $\varphi(a, y)$ is algebraic, so $\mathrm{bcl}(A) \subseteq \mathrm{acl}(A)$. It can also be easily verified that bcl is a closure operator and if $A \subseteq \mathcal{M}$ then $\langle A \rangle \subseteq \mathrm{bcl}(A)$; see Lemma A.3 for details.

Remark 2.12. Every model is acl-closed, every acl-closed set is bcl-closed, and every bcl-closed set is a substructure, therefore:

$$\text{QE} \Leftrightarrow \text{substructure-complete} \Rightarrow \text{bcl-complete} \Rightarrow \text{acl-complete} \Rightarrow \text{model-complete}.$$

Theorem 2.13 clarifies the relationship between acl- and bcl-completeness and provides the promised syntactic equivalent to bcl-completeness. The proof is somewhat involved and may be of independent interest, so we delay it to Appendix A.

Theorem 2.13. *The following are equivalent:*

1. *Every L-formula is T-equivalent to a finite disjunction of b.e. formulas.*
2. *T is* acl*-complete and* $\text{acl} = \text{bcl}$ *in T-models.*
3. *T is* bcl*-complete.*

2.5. **Stationary and extendable independence relations.** In this section, T is a complete L-theory, L' is a language extending L such that L and L' have the same set of sorts, and T' is a complete L'-theory extending T. Let $\mathfrak{M}'$ be a monster model of T' and $\mathfrak{M}$ be the L-reduct of $\mathfrak{M}'$, so $\mathfrak{M}$ is a monster model of T. We isolate some properties of forking independence in stable theories that weakly eliminate imaginaries; these properties will be necessary ingredients for the proofs of preservation of acl- and bcl-completeness in Section 3.

Let $\mathop{\smile\!\!\!\!\!\mid}$ be a ternary relation on small subsets of $\mathfrak{M}$. We consider the following properties that $\mathop{\smile\!\!\!\!\!\mid}$ may satisfy. The first three are specific to T, while the fourth concerns the relationship between T and T'. We let A, B, and C range over arbitrary small subsets of $\mathfrak{M}$.

1. **Invariance:** If σ is an automorphism of $\mathfrak{M}$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\mid}_{\sigma(C)} \sigma(B)$.
2. **Algebraic independence:** If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then $\text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$.
3. **Stationarity over algebraically closed sets:** If $C = \text{acl}(C)$, $\text{tp}_L(A/C) = \text{tp}_L(A^*/C)$, $A \mathop{\smile\!\!\!\!\!\mid}_C B$, and $A^* \mathop{\smile\!\!\!\!\!\mid}_C B$, then $\text{tp}_L(A/BC) = \text{tp}_L(A^*/BC)$.
4. **Full existence over algebraically closed sets in T':** If $C = \text{acl}'(C)$, then there exists A^* with $\text{tp}_{L'}(A^*/C) = \text{tp}_{L'}(A/C)$, and $A^* \mathop{\smile\!\!\!\!\!\mid}_C B$ in $\mathfrak{M}$.

We say $\mathop{\smile\!\!\!\!\!\mid}$ is a **stationary independence relation** in T if it satisfies invariance, algebraic independence, and stationarity over algebraically closed sets. We say $\mathop{\smile\!\!\!\!\!\mid}$ is **extendable** (to T') if it satisfies full existence over algebraically closed sets in T'.

Our definition of a stationary independence relation differs from those used elsewhere, e.g., in [TZ13]. Most natural stationary independence relations satisfy additional axioms (symmetry, monotonicity, etc.). We only require the axioms above.

The main example of a stationary and extendable independence relation that the reader should keep in mind is forking independence $\mathop{\smile\!\!\!\!\!\mid}^f$ in a theory T which is stable with weak elimination of imaginaries. We give a proof of Proposition 2.14 in Appendix B. Note that there are no hypotheses on T'.

Proposition 2.14. *Suppose T is stable with weak elimination of imaginaries. Then $\mathop{\smile\!\!\!\!\!\mid}^f$ is a stationary and independence relation in T which is extendable to T'.*

The next example shows, there are also non-trivial examples of stationary independence relations in unstable theories, which may or may not be extendable.

Example 2.15. Suppose L contains a single binary relation E, and T is the theory of the random graph (the Fraïssé limit of the class of finite graphs). Define:

$$A \mathop{\perp\!\!\!\perp}^{E}_{C} B \iff A \cap B \subseteq C \text{ and } aEb \text{ for all } a \in A \smallsetminus C \text{ and } b \in B \smallsetminus C$$
$$A \mathop{\perp\!\!\!\perp}^{\not E}_{C} B \iff A \cap B \subseteq C \text{ and } \neg aEb \text{ for all } a \in A \smallsetminus C \text{ and } b \in B \smallsetminus C.$$

Both $\mathop{\perp\!\!\!\perp}^{E}$ and $\mathop{\perp\!\!\!\perp}^{\not E}$ are stationary independence relations in T.

Now let $L' = \{E, P\}$, where P is a unary predicate, and let T' be the theory of the Fraïssé limit of the class of finite graphs with a predicate P naming a clique. T' extends T and has quantifier elimination, and $\mathrm{acl}_{L'}(A) = A$ for all sets A.

Then $\mathop{\perp\!\!\!\perp}^{E}$ is extendable to T'. Indeed, for any A, B, and C, let $p(x) = \mathrm{tp}_{L'}(A/C)$, where $x = (x_a)_{a \in A}$ is a tuple of variables enumerating A. The type

$$p(x) \cup \{x_a E b \mid a \in A \smallsetminus C \text{ and } b \in B \smallsetminus C\}$$

is consistent, and for any realization A^* of this type, we have $A^* \mathop{\perp\!\!\!\perp}^{E}_{C} B$ in $\mathbb{M}$.

On the other hand, let a and b be any two elements of $\mathbb{M}'$ satisfying P. Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $P(a^*)$, so $a^* E b$, and $a^* \not\mathop{\perp\!\!\!\perp}^{\not E}_{\varnothing} b$ in $\mathbb{M}$. So $\mathop{\perp\!\!\!\perp}^{\not E}$ is not extendable to T'.

2.6. NSOP_1 **theories.** In this section, we summarize the necessary background on NSOP_1 theories. T is a complete L-theory, $\mathbb{M}$ is a monster model of T, and $\mathcal{M} \prec \mathbb{M}$ is a small submodel with underlying set M.

Let $\leq$ be the tree partial order on $2^{<\omega}$, and let $\nu^\frown\eta$ be the usual concatenation of $\nu, \eta \in 2^{<\omega}$. The formula $\varphi(x; y)$ has $\mathbf{SOP}_1$ (relative to T) if there exist $(b_\eta)_{\eta \in 2^{<\omega}}$ in $\boldsymbol{M}^y$ such that:

(1) For all $\nu, \eta \in 2^{<\omega}$, if $\nu^\frown 0 \leq \eta$, then $\{\varphi(x; b_\eta), \varphi(x; b_{\nu^\frown 1})\}$ is inconsistent.
(2) For all $\sigma \in 2^\omega$, $\{\varphi(x; a_{\sigma|_n}) \mid n \in \omega\}$ is consistent.

The theory T is NSOP_1 if no formula has SOP_1 relative to T. An incomplete theory is NSOP_1 if each of its completions is NSOP_1.

This "negative" definition of NSOP_1 is due to Dzamonja and Shelah [DS04]. We will find it more convenient to work with a "positive" characterization of NSOP_1, due to Chernikov, Ramsey, and Kaplan, in terms of the relation of Kim-independence.

A global type $q \in S_y(\mathbb{M})$ is M**-invariant** if for any formula $\psi(y, z)$ and any $c, c' \in \boldsymbol{M}^z$ with $\mathrm{tp}(c/M) = \mathrm{tp}(c'/M)$, we have $\psi(y, c) \in q$ if and only if $\psi(y, c') \in q$. A sequence $(b_i)_{k \in \omega}$ is a **Morley sequence** for q over M if b_k realizes the restriction of q to $Mb_0 \ldots b_{k-1}$ for all i. Suppose $\varphi(x, b)$ is a formula with $b \in \boldsymbol{M}^y$ and $q \in S_y(\mathbb{M})$ is a global M-invariant type extending $\mathrm{tp}(b/M)$. Then $\varphi(x, b)$ q**-divides over** M if $\{\varphi(x, b_i) \mid i \in \omega\}$ is inconsistent for some (equivalently any) Morley sequence $(b_i)_{i \in \omega}$ for q over M.

A formula $\varphi(x, b)$ **Kim-divides** over M if it q-divides over M for some global M-invariant type q extending $\mathrm{tp}(b/M)$, and $\varphi(x, b)$ **Kim-forks** over M if it implies a disjunction of formulas which Kim-divide over M. We say that A is **Kim-independent** from B over M when no formula in $\mathrm{tp}(A/MB)$ Kim-forks over M. We write $A \mathop{\perp\!\!\!\perp}^{K}_{M} B$ when A is Kim-independent from B over M.

These definitions are made over a model $\mathcal{M}$, rather than over an arbitrary set of parameters C, because a type over C may fail to extend to any global C-invariant type.

We can now state the axiomatic characterization of NSOP_1 and Kim-independence. An earlier version of this criterion appeared in [CR16].

Fact 2.16 ([KR20] Theorem 9.1)**.** *Suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the following for all A, A', B, B' and all models $\mathcal{M}, \mathcal{M}'$:*

1. ***Invariance**: If $A \mathop{\smile\!\!\!\!\!\!|}_M B$ and $MAB \equiv M'A'B'$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{M'} B'$.*
2. ***Existence**: $A \mathop{\smile\!\!\!\!\!\!|}_M M$*
3. ***Monotonicity**: If $A \mathop{\smile\!\!\!\!\!\!|}_M B$ and $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_M B'$.*
4. ***Symmetry**: If $A \mathop{\smile\!\!\!\!\!\!|}_M B$, then $B \mathop{\smile\!\!\!\!\!\!|}_M A$.*
5. ***The independence theorem**: If $A \equiv_M A'$, $A \mathop{\smile\!\!\!\!\!\!|}_M B$, $A' \mathop{\smile\!\!\!\!\!\!|}_M C$, and $B \mathop{\smile\!\!\!\!\!\!|}_M C$, then there exists A'' such that $A'' \equiv_{MB} A$, $A'' \equiv_{MC} A'$, and $A'' \mathop{\smile\!\!\!\!\!\!|}_M BC$.*
6. ***Strong finite character**: If $A \not\mathop{\smile\!\!\!\!\!\!|}_M B$, then there is a formula $\varphi(x, b, m) \in \mathrm{tp}(A/MB)$ such that for any c such that $\boldsymbol{\mathcal{M}} \models \varphi(c, b, m)$, we have $c \not\mathop{\smile\!\!\!\!\!\!|}_M b$.*

Then T is NSOP_1. Suppose $\mathop{\smile\!\!\!\!\!\!|}$ additionally satisfies:

7. ***Witnessing**: If $A \not\mathop{\smile\!\!\!\!\!\!|}_M B$, then there is a formula $\varphi(x, b, m) \in \mathrm{tp}(A/MB)$ which Kim-forks over M.*

Then $\mathop{\smile\!\!\!\!\!\!|}_M = \mathop{\smile\!\!\!\!\!\!|}^K_M$ for all $\mathcal{M}$.

Remark 2.17. If T is NSOP_1, then Kim-independence satisfies all of the properties in Fact 2.16. The only nontrivial properties are symmetry ([KR20] Theorem 5.16) and the independence theorem ([KR20] Theorem 6.5).

Fact 2.18 is a version of Kim's lemma for NSOP_1 theories. Kim's lemma was originally proven for forking in simple theories [Kim98].

Fact 2.18 ([KR20] Theorem 3.15)**.** *Assume T is NSOP_1. If a formula $\varphi(x, b)$ q-divides over M for* some *global M-invariant type q extending* $\mathrm{tp}(b/M)$*, then $\varphi(x, b)$ q-divides for* every *global M-invariant type q extending* $\mathrm{tp}(b/M)$*.*

A consequence of Fact 2.18 is that Kim-forking and Kim-dividing agree in NSOP_1 theories.

Fact 2.19 ([KR20] Proposition 3.19)**.** *Assume T is NSOP_1. A formula $\varphi(x, b)$ Kim-forks over M if and only if it Kim-divides over M.*

Just like forking independence in arbitrary theories, Kim-independence in NSOP_1 theories is blind to algebraic closures.

Fact 2.20 ([KR20] Corollary 5.17)**.** *Assume T is NSOP_1. Then for all sets A and B and all models $\mathcal{M}$, $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ if and only if $\mathrm{acl}(MA) \mathop{\smile\!\!\!\!\!\!|}^K_M \mathrm{acl}(MB)$.*

Every simple theory is NSOP_1, and in a simple theory, Kim-independence $\mathop{\smile\!\!\!\!\!\!|}^K$ agrees with forking independence $\mathop{\smile\!\!\!\!\!\!|}^f$. Fact 2.21 below, which follows from Propositions 8.4 and 8.8 of [KR20], characterizes the simple theories among NSOP_1 theories. We say Kim-independence satisfies **base monotonicity over models** if $a \mathop{\smile\!\!\!\!\!\!|}^K_M Nb$ implies $a \mathop{\smile\!\!\!\!\!\!|}^K_N b$ for all $\mathcal{M} \preceq \mathcal{N}$.

Fact 2.21. *Assume T is NSOP_1. Then the following are equivalent:*

1. *T is simple,*
2. *$\mathop{\smile\!\!\!\!\!\!|}^f_M = \mathop{\smile\!\!\!\!\!\!|}^K_M$ for all models $\mathcal{M}$,*
3. *Kim-independence satisfies base monotonicity over models.*

2.7. **Independence and reducts.** In this section, let $L \subseteq L'$ be languages, T a complete L-theory, T' a complete L'-theory extending T, $\mathbb{M}'$ a monster model of T', and $\mathbb{M}$ its reduct to L. We consider the relationship between notions of independence in $\mathbb{M}'$ and $\mathbb{M}$.

It is not clear from the definition that Kim-independence is preserved under reducts, since the property of being an M-invariant type is not preserved under reducts in general. However, when T' is NSOP$_1$, Fact 2.18 shows that Kim-dividing is always witnessed by q-dividing for a global type q which is finitely satisfiable in M, and this property is preserved under reducts. This gives us Lemma 2.22.

Lemma 2.22. *If T' is* NSOP$_1$*, then:*

(1) T is NSOP$_1$.
(2) Let $\mathcal{M} \prec \mathbb{M}'$, and let $\varphi(x, b)$ be an L-formula. Then $\varphi(x, b)$ Kim-divides over M in $\mathbb{M}$ if and only if it Kim-divides over M in $\mathbb{M}'$.
(3) Kim-independence is preserved by reducts: if $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ in $\mathbb{M}'$, then also $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ in $\mathbb{M}$.

Proof. For (1), the fact that NSOP$_1$ is preserved by reducts is clear from the definition: any formula with SOP$_1$ relative to T also has SOP$_1$ relative to T'.

For (2), fix a global L'-type q' extending $\mathrm{tp}_{L'}(b/M)$, which is finitely satisfiable in M (hence M-invariant). Let $(b_i)_{i\in\omega}$ be a Morley sequence for q' over M. Let q be the restriction of q' to L. Then q is also finitely satisfiable in M (hence M-invariant) and extends $\mathrm{tp}_L(b/M)$, and $(b_i)_{i\in\omega}$ is a Morley sequence for q over M. By Fact 2.18 and (1), $\varphi(x, b)$ Kim-divides over M in $\mathbb{M}$ if and only if $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent if and only if $\varphi(x, b)$ Kim-divides over M in $\mathbb{M}'$.

For (3), suppose $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ in $\mathbb{M}'$. Then no formula in $\mathrm{tp}_{L'}(A/MB)$ Kim-divides over M in $\mathbb{M}'$, so, by (2), no formula in $\mathrm{tp}_L(A/MB)$ Kim-divides over M in $\mathbb{M}$. By Fact 2.19, $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ in $\mathbb{M}$. □

We usually work with a reduct $\mathbb{M}$ which is stable, or at least simple. In this situation, it is useful to consider an independence relation induced on $\mathbb{M}'$ by forking independence $\mathop{\smile\!\!\!\!\!\!|}^f$ in $\mathbb{M}$ (which is defined over arbitrary sets, not just models). Define the relation $\mathop{\smile\!\!\!\!\!\!|}^r$, **independence in the reduct**:

$$A \mathop{\smile\!\!\!\!\!\!|}^r_C B \quad \text{if and only if} \quad \mathrm{acl}'(AC) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}'(C)} \mathrm{acl}'(BC) \text{ in } \mathbb{M}.$$

Note that in the definition of $\mathop{\smile\!\!\!\!\!\!|}^r$, acl' is the algebraic closure in $\mathbb{M}'$. If L is the language of equality and T is the theory of an infinite set then $\mathop{\smile\!\!\!\!\!\!|}^r = \mathop{\smile\!\!\!\!\!\!|}^a$ in $\mathbb{M}'$, where $\mathop{\smile\!\!\!\!\!\!|}^a$ is **algebraic independence**:

$$A \mathop{\smile\!\!\!\!\!\!|}^a_C B \quad \text{if and only if} \quad \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C).$$

Strengthened versions of extension and the independence theorem, adding additional instances of algebraic independence to the conclusion, were established for Kim-independence in NSOP$_1$ theories in [KR18]. Theorem 2.23 and Theorem 2.24 are generalizations of these results, with $\mathop{\smile\!\!\!\!\!\!|}^a$ replaced by the relation $\mathop{\smile\!\!\!\!\!\!|}^r$ induced by the reduct T. The statements are non-trivial because $\mathop{\smile\!\!\!\!\!\!|}^r$ does not satisfy base monotonicity in general.

Theorem 2.23 (Reasonable extension)**.** *Suppose T' is* NSOP$_1$ *and T is simple with stable forking and geometric elimination of imaginaries. For all $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$ and* *for all C, there exists A' such that* $\mathrm{tp}_{L'}(A'/MB) = \mathrm{tp}_{L'}(A/MB)$*,* $A' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} BC$*, and* $A' \mathop{\smile\!\!\!\!\!\!|}^{r}_{MB} C$*.*

Theorem 2.24 (Reasonable independence theorem)**.** *Suppose T' is* NSOP_1 *and T is simple with stable forking and geometric elimination of imaginaries. If* $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} B$*,* $A' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$*,* $B \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$*, and* $\mathrm{tp}_{L'}(A/M) = \mathrm{tp}_{L'}(A'/M)$*, then there exists A'' such that* $\mathrm{tp}_{L'}(A''/MB) = \mathrm{tp}_{L'}(A/MB)$*,* $\mathrm{tp}_{L'}(A''/MC) = \mathrm{tp}_{L'}(A'/MC)$*,* $A'' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} BC$*, and further* $A'' \mathop{\smile\!\!\!\!\!\!|}^{r}_{MC} B$*,* $A'' \mathop{\smile\!\!\!\!\!\!|}^{r}_{MB} C$*, and* $B \mathop{\smile\!\!\!\!\!\!|}^{r}_{MA''} C$*.*

Theorem 2.23 and Theorem 2.24 follow from Example C.1, Theorem C.8, and Theorem C.15 in Appendix C. The hypotheses on T come from Lemma B.2.

When we prove the independence theorem for interpolative fusions in Section 4.2, we need an additional strengthening of the independence theorem for $\mathop{\smile\!\!\!\!\!\!|}^{K}$, which unfortunately we do not know how to prove in general.

Assume T' is NSOP_1 and T is stable. We say that $\mathop{\smile\!\!\!\!\!\!|}^{K}$ in T' satisfies the **T-generic independence theorem** if it satisfies the following strengthening of the independence theorem: Let $\mathcal{M}' \preceq \boldsymbol{\mathcal{M}}'$ be a model with underlying set M, and let A, A', B, and C be acl'-closed sets, each of which contains M. Suppose $\mathrm{tp}_{L'}(A/M) = \mathrm{tp}_{L'}(A'/M)$, $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} B$, $A' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$, and $B \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$ in $\boldsymbol{\mathcal{M}}'$. Then there exists A'' such that $\mathrm{tp}_{L'}(A''/B) = \mathrm{tp}_{L'}(A/B)$, $\mathrm{tp}_{L'}(A''/C) = \mathrm{tp}_{L'}(A'/C)$, and $A'' \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} BC$ in $\boldsymbol{\mathcal{M}}'$, and further, in the stable reduct $\boldsymbol{\mathcal{M}}$:

$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^{f}_{A''B} \mathrm{acl}'(A''C)\mathrm{acl}'(BC)$$
$$\mathrm{acl}'(A''C) \mathop{\smile\!\!\!\!\!\!|}^{f}_{A''C} \mathrm{acl}'(A''B)\mathrm{acl}'(BC)$$
$$\mathrm{acl}'(BC) \mathop{\smile\!\!\!\!\!\!|}^{f}_{BC} \mathrm{acl}'(A''B)\mathrm{acl}'(A''C).$$

Question 2.25. Assume T' is NSOP_1 and T is stable. Does $\mathop{\smile\!\!\!\!\!\!|}^{K}$ in T' always satisfy the T-generic independence theorem? What if T' is simple?

We do not know the answer to Question 2.25 in general. But we will now observe that there are many situations in which we obtain a positive answer, including when T' is stable.

Proposition 2.26. *If T' is stable, then* $\mathop{\smile\!\!\!\!\!\!|}^{K} = \mathop{\smile\!\!\!\!\!\!|}^{f}$ *in T' satisfies the T-generic independence theorem.*

Proof. Let $\mathcal{M}' \preccurlyeq \boldsymbol{\mathcal{M}}'$ be model with underlying set M. By Fact 2.21, $\mathop{\smile\!\!\!\!\!\!|}^{K}_{M} = \mathop{\smile\!\!\!\!\!\!|}^{f}_{M}$ in $\boldsymbol{\mathcal{M}}'$. Let A, A', B, and C be acl'-closed sets, each of which contains M. Suppose $\mathrm{tp}_{L'}(A/M) = \mathrm{tp}_{L'}(A'/M)$, $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} B$, $A' \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} C$, and $B \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} C$ in $\boldsymbol{\mathcal{M}}'$. Applying the independence theorem, we obtain A'' such that $\mathrm{tp}_{L'}(A''/B) = \mathrm{tp}_{L'}(A/B)$, $\mathrm{tp}_{L'}(A''/C) = \mathrm{tp}_{L'}(A'/C)$, and $A'' \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} BC$ in $\boldsymbol{\mathcal{M}}'$.

Then $A'' \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} \mathrm{acl}'(BC)$ in $\boldsymbol{\mathcal{M}}'$. It follows that $\mathrm{tp}_{L'}(A''/\mathrm{acl}'(BC))$ is finitely satisfiable in M, since $\mathcal{M}'$ is a model and T' is stable. To prove that

$$\mathrm{acl}'(BC) \mathop{\smile\!\!\!\!\!\!|}^{f}_{BC} \mathrm{acl}'(A''B)\mathrm{acl}'(A''C) \text{ in } \boldsymbol{\mathcal{M}},$$

it suffices by symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{f}$ to show that $\mathrm{tp}_{L}(\mathrm{acl}'(A''B)\mathrm{acl}'(A''C)/\mathrm{acl}'(BC))$ is finitely satisfiable in BC.

Suppose $\boldsymbol{\mathcal{M}}' \vDash \varphi(d_1, d_2, e)$, where φ is an L-formula, d_1 is a tuple from $\mathrm{acl}'(A''B)$, d_2 is a tuple from $\mathrm{acl}'(A''C)$, and e is a tuple from $\mathrm{acl}'(BC)$. Let $\psi_1(w_1, a, b)$ be an L'-formula isolating $\operatorname{tp}_{L'}(d_1/A''B)$, with a from A'' and b from B, and let $\psi_2(w_2, a', c)$ be an L'-formula isolating $\operatorname{tp}_{L'}(d_2/A''C)$, with a' from A'' and c from C. We may assume that any instances of ψ_1 and ψ_2 are algebraic. Then we have:

$$\mathfrak{M}' \vDash \exists w_1 \exists w_2\, (\psi_1(w_1, a, b) \wedge \psi_2(w_2, a', c) \wedge \varphi(w_1, w_2, e)).$$

Since $\operatorname{tp}_{L'}(A''/\operatorname{acl}'(BC))$ is finitely satisfiable in M, there exist m and m' in M such that:

$$\mathfrak{M}' \vDash \exists w_1 \exists w_2\, (\psi_1(w_1, m, b) \wedge \psi_2(w_2, m', c) \wedge \varphi(w_1, w_2, e)).$$

Let d_1' and d_2' be witnesses to the existential quantifiers. Then $d_1' \in \operatorname{acl}'(MB) = B$, since B is acl'-closed and contains M. Similarly, $d_2' \in C$. This shows that $\varphi(w_1, w_2, e)$ is satisfiable in BC and establishes the claim.

We have proven one of the desired independencies. By basic properties of forking independence in stable theories (symmetry, base monotonicity, and transitivity), $A'' \mathop{\smile\!\!\!\!\!|}^f_M BC$ implies $B \mathop{\smile\!\!\!\!\!|}^f_M A''C$ and $C \mathop{\smile\!\!\!\!\!|}^f_M A''B$ in $\mathfrak{M}'$, and the other two independencies follow by the same argument. □

Note that the proof of Proposition 2.26 only works when M is a model, since it uses the equivalence of non-forking and finite satisfiability over models in stable theories. This raises the following question in pure stability theory.

Question 2.27. Assume T' and T are both stable with elimination of imaginaries. Does $\mathop{\smile\!\!\!\!\!|}^f$ in T' always satisfy the analogue of the T-generic independence theorem where we replace the model M with an arbitrary acl'-closed set?

We say that acl' is **disintegrated relative to** acl if for any sets A and B, we have

$$\operatorname{acl}'(AB) = \operatorname{acl}(\operatorname{acl}'(A)\operatorname{acl}'(B)).$$

Equivalently, for any set A,

$$\operatorname{acl}'(A) = \operatorname{acl}\left(\bigcup_{a \in A} \operatorname{acl}'(a)\right).$$

To simplify terminology, we also say that T' is **relatively disintegrated**.

Proposition 2.28. *Assume T' is* NSOP_1 *and relatively disintegrated, and T is stable. Then $\mathop{\smile\!\!\!\!\!|}^K$ in T' satisfies the T-generic independence theorem.*

Proof. In this case, the additional independencies in $\mathfrak{M}$ are trivial. For example, since A'' and B are acl'-closed sets, we have $\operatorname{acl}'(A''B) = \operatorname{acl}(\operatorname{acl}'(A'')\operatorname{acl}'(B)) = \operatorname{acl}(A''B)$. And it is always true that:

$$\operatorname{acl}(A''B) \mathop{\smile\!\!\!\!\!|}^f_{A''B} \operatorname{acl}'(A''C)\operatorname{acl}'(BC)$$

e.g., by Fact 2.20 applied in the stable theory T. □

We say that T has **disintegrated forking** if $A \not\mathop{\smile\!\!\!\!\!|}^f_C BB'$ implies $A \not\mathop{\smile\!\!\!\!\!|}^f_C B$ or $A \not\mathop{\smile\!\!\!\!\!|}^f_C B'$. Equivalently, if $A \not\mathop{\smile\!\!\!\!\!|}^f_C B$, then $A \not\mathop{\smile\!\!\!\!\!|}^f_C b$ for some singleton $b \in B$. Some authors refer to this as "trivial forking".

The reasonable independence theorem (Theorem 2.24 above) says that in any NSOP_1 theory T', we can witness the independence theorem in such a way that the sets $\operatorname{acl}'(A''B)$, $\operatorname{acl}'(A''C)$, and $\operatorname{acl}'(BC)$ are pairwise independent in the stable reduct T. If T has disintegrated forking, then these pairwise independencies are sufficient to satisfy the T-generic independence theorem.

Proposition 2.29. *Assume T' is* NSOP_1 *and T is stable with elimination of imaginaries and disintegrated forking. Then $\mathop{\smile\!\!\!\!\!\!|}^K$ in T' satisfies the T-generic independence theorem.*

Proof. Let $\mathcal{M}' \preccurlyeq \boldsymbol{\mathcal{M}}'$ be model with underlying set M. Let A, A', B, and C be acl'-closed sets, each of which contains M. Suppose $\mathrm{tp}_{L'}(A/M) = \mathrm{tp}_{L'}(A'/M)$, $A \mathop{\smile\!\!\!\!\!\!|}^f{}_M B$, $A' \mathop{\smile\!\!\!\!\!\!|}^K{}_M C$, and $B \mathop{\smile\!\!\!\!\!\!|}^K{}_M C$ in $\boldsymbol{\mathcal{M}}'$. By Theorem 2.24, we obtain A'' such that $\mathrm{tp}_{L'}(A''/MB) = \mathrm{tp}_{L'}(A/MB)$, $\mathrm{tp}_{L'}(A''/MC) = \mathrm{tp}_{L'}(A'/MC)$, $A'' \mathop{\smile\!\!\!\!\!\!|}^K{}_M BC$ in $\boldsymbol{\mathcal{M}}'$, and additionally in $\boldsymbol{\mathcal{M}}$:

$$\mathrm{acl}'(A''C) \mathop{\smile\!\!\!\!\!\!|}^f_C \mathrm{acl}'(BC)$$
$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^f_B \mathrm{acl}'(BC)$$
$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^f_{A''} \mathrm{acl}'(A''C).$$

By base monotonicity for $\mathop{\smile\!\!\!\!\!\!|}^f$ in $\boldsymbol{\mathcal{M}}$, we have:

$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^f_{A''B} \mathrm{acl}'(BC)$$
$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^f_{A''B} \mathrm{acl}'(A''C)$$

and since T has disintegrated forking,:

$$\mathrm{acl}'(A''B) \mathop{\smile\!\!\!\!\!\!|}^f_{A''B} \mathrm{acl}'(A''C)\mathrm{acl}'(BC).$$

The other two independencies follow by the same argument. □

3. Logical tameness

Throughout this section, we fix the languages $L_\square$ and theories $T_\square$ for $\square \in I \cup \{\cup, \cap\}$, and we assume the interpolative fusion $T^*_\cup$ exists.

We seek to understand when logical tameness properties (completeness, model completeness, quantifier elimination, etc.) of the theories T_i are preserved in passing to the interpolative fusion $T^*_\cup$. We have already seen a close connection between interpolative fusions and model-completeness (Fact 2.1), which we reformulate as a first preservation result in Section 3.1.

In order to understand definable sets and types, we often want something stronger than model-completeness. So Sections 3.2 is devoted to preservation of acl- and bcl-completeness (defined in Section 2.4). This requires stronger hypotheses, which hold, for example, when $T_\cap$ is stable with weak elimination of imaginaries, and with no additional assumptions on the theories T_i. Under the same hypotheses, we also obtain a description of algebraic closusure in $T^*_\cup$. Section 3.3 is about preservation of substructure-completeness (equivalently, quantifier elimination) under even stronger hypothesis. Model-, acl-, bcl-, and substructure-completeness are all instances of the notion of $\mathcal{K}$-completeness discussed in Section 2.4.

In Section 3.4, we Morleyize the theories T_i in order to translate the preservation results of Sections 3.1 through 3.3 to results about relative syntactic complexity of definable sets for general interpolative fusions (without $\mathcal{K}$-completeness assumptions). Finally, in Section 3.5, we deduce a sufficient condition for completeness of the interpolative fusion.

3.1. **Model-completeness.** Flat formulas (defined in Section 2.4) help explain the relevance of $\mathcal{K}$-completeness of $T^*_\cup$ for understanding types, and are useful in expressing syntactic consequences of $\mathcal{K}$-completeness of $T^*_\cup$.

Remark 3.1. Any flat literal $L_\cup$-formula is an L_i-formula for some $i \in I$. Thus:

(1) If $\varphi(x)$ is a flat $L_\cup$-formula, then there is some finite $J \subseteq I$ and a flat L_i-formula $\varphi_i(x)$ for all $i \in J$ such that $\varphi(x)$ is logically equivalent to $\bigwedge_{i \in J} \varphi_i(x)$.
(2) If $A_\cup$ is an $L_\cup$-structure, then $\mathrm{Fdiag}(A_\cup) = \bigcup_{i \in I} \mathrm{Fdiag}(\mathcal{A}_i)$, where $\mathrm{Fdiag}(\mathcal{A}_i)$ is an $L_i(A)$-theory.
(3) By Remark 2.10, if $T^*_\cup$ is $\mathcal{K}$-complete, then for any pair $(\mathcal{A}_\cup, \mathcal{M}_\cup) \in \mathcal{K}$,
$$T^*_\cup \cup \bigcup_{i \in I} \mathrm{Fdiag}(\mathcal{A}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$
As a consequence, when $(\mathcal{A}_\cup, \mathcal{M}_\cup) \in \mathcal{K}$, the complete $L_\cup$-type of A is determined by the quantifier-free L_i-types of A for all $i \in I$.

We now restate Fact 2.1 as preservation of model-completeness. If $T^*_\cup$ is model-complete, then every $L_\cup$-formula is equivalent to an existential $L_\cup$-formula. We give a slight refinement, using the flat formulas defined above.

Theorem 3.2. *Suppose each T_i is model-complete. Then $T^*_\cup$ is model-complete, and every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas of the form*
$$\exists y \bigwedge_{i \in J} \varphi_i(x, y),$$
where $J \subseteq I$ is finite and each $\varphi_i(x, y)$ is a flat L_i-formula.

Proof. Model-completeness follows immediately from Fact 2.1. Since $T^*_\cup$ is model-complete, $\psi(x)$ is $T^*_\cup$-equivalent to an existential $L_\cup$-formula $\exists z\, \varphi(x, z)$. By Fact 2.9, $\varphi(x, z)$ is equivalent to a finite disjunction of $\mathrm{E}\flat$-formulas. Now distribute $\exists z$ over the disjunction and apply Remark 3.1(1). □

3.2. acl**-completeness and** bcl**-completeness.** Our next goal is to find sufficient conditions for preservation of acl and bcl-completeness as introduced in Section 2.4. It turns out that stationary and extendable independence relations, as defined in Section 2.5, play an important role.

We say $T_\cap$ **admits a stationary and extendable independence relation** if whenever $\widehat{T_\cap}$ is a completion of $T_\cap$, and $\widehat{T_i}$ extending $\widehat{T_\cap}$ is a completion of T_i for each i, there is a stationary independence relation $\mathop{\smile\!\!\!\!\!\mid}^{\cap}$ in $\widehat{T_\cap}$ which is extendable to $\widehat{T_i}$ for each i.

In general, this is a hypothesis on the relationship between $T_\cap$ and each T_i, not on $T_\cap$ alone. However, by Proposition 2.14, it is always satisfied by forking independence in $T_\cap$ when $T_\cap$ is stable with weak elimination of imaginaries, with no additional assumptions on the theories T_i. This holds, for instance, when $T_\cap$ is the theory of an infinite set or the theory of algebraically closed fields.

From the proof, we also obtain a characterization of acl in $T^*_\cup$. Let A be a subset of a $T^*_\cup$-model $\mathcal{M}_\cup$. Given $\square \in I \cup \{\cup, \cap\}$, let $\mathrm{acl}_\square(A)$ be the algebraic closure of A in the reduct $\mathcal{M}_\square$. The **combined closure**, $\mathrm{ccl}(A)$, is the smallest set containing A which is acl_i-closed for each $i \in I$. More concretely, $b \in \mathrm{ccl}(A)$ if and only if
$$b \in \mathrm{acl}_{i_n}(\dots(\mathrm{acl}_{i_1}(A))\dots) \text{ for some } i_1, \dots, i_n \in I.$$

We can now state and prove the main result of this section.

Theorem 3.3. *Assume $T_\cap$ admits a stationary and extendable independence relation $\mathop{\perp\!\!\!\perp}^{\cap}$. If each T_i is* acl*-complete, then $T_\cup^*$ is* acl*-complete and* $\mathrm{acl}_\cup = \mathrm{ccl}$.

Proof. Each T_i is model-complete, so $T_\cup^*$ is model-complete by Theorem 3.2. In order to apply Proposition 2.11, we will show that the class of $T_\cup^*$-models has the disjoint ccl-amalgamation property.

So suppose $\mathcal{A}_\cup$ is a ccl-closed substructure of a $T_\cup^*$-model $\mathcal{M}_\cup$, $\mathcal{N}_\cup \models T_\cup^*$, and $f\colon \mathcal{A}_\cup \to \mathcal{N}_\cup$ is an embedding. Let $\boldsymbol{\mathcal{M}}_\cup$ be a monster model of $\mathrm{Th}_{L_\cup}(\mathcal{N}_\cup)$ (this is a completion of $T_\cup^*$), so $\mathcal{N}_\cup \preccurlyeq \boldsymbol{\mathcal{M}}_\cup$. Let $p_\square(x) = \mathrm{tp}_{L_\square}(M/A)$ for each $\square \in I \cup \{\cap\}$, where x is an infinite tuple of variables enumerating M. By acl-completeness of T_i, $f\colon \mathcal{A}_i \to \mathcal{N}_i$ is partial elementary for all $i \in I$, so $f\colon \mathcal{A}_\cap \to \mathcal{N}_\cap$ is also partial elementary, and we can replace the parameters from A in $p_\square(x)$ by their images under f, obtaining a consistent type $p'_\square(x)$ over $A' = f(A) \subseteq N$ for all $\square \in I \cup \{\cap\}$.

Fix $i \in I$. Since A is algebraically closed in $\mathcal{M}_i$ and $f\colon \mathcal{A}_i \to \boldsymbol{\mathcal{M}}_i$ is partial elementary, A' is algebraically closed in $\boldsymbol{\mathcal{M}}_i$. Since $\mathop{\perp\!\!\!\perp}^{\cap}$ satisfies full existence over algebraically closed sets in T_i, there is a realization M'_i of $p'_i(x)$ in $\boldsymbol{\mathcal{M}}_i$ such that $M'_i \mathop{\perp\!\!\!\perp}^{\cap}_{A'} N$ in $\boldsymbol{\mathcal{M}}_\cap$. Let $q_i(x) = \mathrm{tp}_{L_i}(M'_i/N)$.

For all $i, j \in I$, $\mathrm{tp}_{L_\cap}(M'_i/A') = \mathrm{tp}_{L_\cap}(M'_j/A') = p'_\cap(x)$, so since $\mathop{\perp\!\!\!\perp}^{\cap}$ satisfies stationarity over algebraically closed sets, $\mathrm{tp}_{L_\cap}(M'_i/N) = \mathrm{tp}_{L_\cap}(M'_j/N)$. Let $q_\cap(x)$ be this common type, so $q_\cap(x) \subseteq q_i(x)$ for all i. By Proposition 2.5, $\bigcup_{i\in I} q_i(x)$ is realized by a set M' in a model $\mathcal{N}'_\cup$ such that $\mathcal{N}_\cup \preccurlyeq \mathcal{N}'_\cup \prec \boldsymbol{\mathcal{M}}_\cup$.

Let $f'\colon \mathcal{M}_\cup \to \mathcal{N}'_\cup$ be the map induced by the common enumeration of M and M' by the variables x. Then f' is an L_i-embedding for all $i \in I$, so it is an $L_\cup$-embedding. Since $T_\cup^*$ is model-complete, f' is an elementary embedding. If $a \in A$, then a is enumerated by a variable x_a from x, and the formula $x_a = a$ is in $p_\cap(x)$. Then the formula $x_a = f(a)$ is in $p'_\cap(x)$, so $f'(a) = f(a)$. This establishes the ccl-amalgamation property.

$$\begin{array}{ccc} \mathcal{M}_\cup & \xrightarrow{\;f'\;} & \mathcal{N}'_\cup \\ \uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\preccurlyeq} \\ \mathcal{A}_\cup & \xrightarrow{\;f\;} & \mathcal{N}_\cup \end{array}$$

Additionally, we have $M' \mathop{\perp\!\!\!\perp}^{\cap}_{A'} N$ in $\boldsymbol{\mathcal{M}}_\cap$, so since $\mathop{\perp\!\!\!\perp}^{\cap}$ satisfies algebraic independence, $M' \cap N = A'$, and hence $f'(M) \cap N = f(A)$. This establishes the disjoint ccl-amalgamation property.

By Proposition 2.11, $T_\cup^*$ is ccl-complete and every ccl-closed substructure is $\mathrm{acl}_\cup$-closed. It follows that for any set $B \subseteq \mathcal{M} \models T$, $\mathrm{acl}_\cup(B) \subseteq \mathrm{ccl}(B)$.

For the converse, it suffices to show $\mathrm{acl}_\cup(B)$ is acl_i-closed for all $i \in I$. Indeed,

$$\mathrm{acl}_i(\mathrm{acl}_\cup(B)) \subseteq \mathrm{acl}_\cup(\mathrm{acl}_\cup(B)) = \mathrm{acl}_\cup(B).$$

So $\mathrm{acl}_\cup = \mathrm{ccl}$, and hence $T_\cup^*$ is acl-complete. □

Under the same hypotheses, we also obtain preservation of bcl-completeness and a syntactic consequence for $L_\cup$-formulas.

Corollary 3.4. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose each T_i is* bcl*-complete. Then $T_\cup^*$ is* bcl*-complete and every $L_\cup$-formula is $T_\cup^*$-equivalent to a finite disjunction of b.e. formulas of the form*

$$\exists y \bigwedge_{i\in J} \varphi_i(x, y),$$

where $J \subseteq I$ is finite and $\varphi_i(x,y)$ is a flat L_i-formula for all $i \in J$.

Proof. By Theorem 2.13, T_i is acl-complete and $\mathrm{bcl}_i = \mathrm{acl}_i$ for all $i \in I$. We have $\mathrm{bcl}_\cup(A) \subseteq \mathrm{acl}_\cup(A)$ for any subset A of a $T_\cup$-model. But also, for all $i \in I$,

$$\begin{aligned}\mathrm{acl}_i(\mathrm{bcl}_\cup(A)) &= \mathrm{bcl}_i(\mathrm{bcl}_\cup(A))\\ &\subseteq \mathrm{bcl}_\cup(\mathrm{bcl}_\cup(A))\\ &= \mathrm{bcl}_\cup(A).\end{aligned}$$

So $\mathrm{bcl}_\cup(A)$ is acl_i-closed for all $i \in I$, hence

$$\mathrm{ccl}(A) \subseteq \mathrm{bcl}_\cup(A) \subseteq \mathrm{acl}_\cup(A).$$

Theorem 3.3 implies $T^*_\cup$ is acl-complete and $\mathrm{ccl}(A) = \mathrm{acl}_\cup(A)$, so the containments above are equalities:

$$\mathrm{ccl}(A) = \mathrm{bcl}_\cup(A) = \mathrm{acl}_\cup(A).$$

Since $T^*_\cup$ is acl-complete and $\mathrm{acl}_\cup = \mathrm{bcl}_\cup$, $T^*_\cup$ is bcl-complete.

It remains to prove the characterization of $L_\cup$-formulas. By Theorem 2.13, bcl-completeness implies that every $L_\cup$-formula is $T^*_\cup$-equivalent to a finite disjunction of b.e. formulas. Let $\exists y\, \psi(x,y)$ be a b.e. formula appearing in the disjunction. By Fact 2.9, the quantifier-free formula $\psi(x,y)$ is equivalent to a finite disjunction of E$\flat$-formulas $\bigvee_{k=1}^m \exists z_k\, \theta_k(x,y,z_k)$. Distributing the quantifier $\exists y$ over the disjunction, we find that $\exists y \exists z_k\, \theta_k(x,y,z_k)$ is a b.e. formula. Applying Remark 3.1(1) to the flat formula $\theta_k(x,y,z_k)$ yields the result. □

3.3. **Quantifier elimination.** Recall that quantifier elimination is equivalent to substructure-completeness. This follows from [Hod93, Theorem 8.4.1] and Proposition 2.11. In contrast to model-completeness, acl-completeness, and bcl-completeness, we cannot obtain preservation of substructure-completeness without tight control on algebraic closure in the theories T_i.

Theorem 3.5 below is motivated by some comments in the introduction of [MS14] on the failure of quantifier elimination in ACFA. For subsets $A \subseteq B \subseteq \mathcal{M}$, we denote by $\mathrm{Aut}(B/A)$ the group of all partial elementary bijections $f\colon B \to B$ which fix A pointwise.

Theorem 3.5. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose each T_i has quantifier elimination, and*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all $L_\cup$-substructures A of $T^*_\cup$-models and all $i \in I$. Then $T^*_\cup$ has quantifier elimination.

Proof. For all $i \in I$, T_i is substructure-complete, and hence acl-complete. So by Theorem 3.3, $T^*_\cup$ is acl-complete and $\mathrm{acl}_\cup = \mathrm{ccl}$. We will show $T^*_\cup$ is substructure complete.

Suppose $\mathcal{A}_\cup$ is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$, $\mathcal{N}_\cup \models T^*_\cup$, and $f\colon \mathcal{A}_\cup \to \mathcal{N}_\cup$ is an embedding. As each T_i is substructure-complete, f is partial elementary $\mathcal{A}_i \to \mathcal{N}_i$, so f extends to a partial elementary map $g_i\colon \mathrm{acl}_i(A) \to \mathcal{N}_i$. By our hypothesis, $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$, so we have a family of maps $(g_i)_{i\in I}$ defined on $\mathrm{acl}_\cap(A)$ and extending f.

Fix $j \in I$. For all $i \in I$, $(g_i^{-1} \circ g_j) \in \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A)$, so in fact it is an L_i-automorphism of $\mathrm{acl}_\cap(A)$ by our assumption on the automorphism groups. It follows that $g_j = g_i \circ (g_i^{-1} \circ g_j)$ is an L_i-embedding $\mathrm{acl}_\cap(A) \to \mathcal{N}_i$. Since i was arbitrary, g_j is an $L_\cup$-embedding. Now since $\mathrm{acl}_\cap(A)$ is acl_i-closed for all $i \in I$, it is ccl-closed, and hence $\mathrm{acl}_\cup$-closed. So by acl-completeness of $T^*_\cup$, $g_j\colon \mathrm{acl}_\cap(A) \to \mathcal{N}_\cup$ is partial elementary, and hence so is $g_j|_A = f$. □

We prefer hypothesis which can be checked language-by-language, i.e., which refer only to properties of T_i, $T_\cap$, and the relationship between T_i and $T_\cap$, rather than how T_i and T_j relate when $i \neq j$, or how T_i relates to $T_\cup$. The hypotheses of Theorem 3.5 are not strictly language-by-language, because they refer to an arbitrary $L_\cup$-substructure A. However, there are several natural strengthenings of these hypotheses which are language-by-language. One is to simply assume the hypotheses of Theorem 3.5 for all sets A. Simpler language-by-language criteria are given in the following corollaries.

Corollary 3.6. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose each T_i admits quantifier elimination. If either of the following conditions hold for all sets A, then $T^*_\cup$ has quantifier elimination:*

(1) $\mathrm{acl}_i(A) = \langle A\rangle_{L_i}$ *for all* $i \in I$.
(2) $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ *for all* $i \in I$.

Proof. We apply Theorem 3.5, so assume $A = \langle A\rangle_{L_\cup}$.

(1) We have $A \subseteq \mathrm{dcl}_\cap(A) \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \langle A\rangle_{L_i} = A$.
(2) We have $\mathrm{dcl}_\cap(A) \subseteq \mathrm{acl}_\cap(A) \subseteq \mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$.

In either case, $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) = \mathrm{dcl}_\cap(A)$. It follows that the group $\mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_\cap}(\mathrm{dcl}_\cap(A)/A)$ is already trivial, since every partial elementary map which fixes A pointwise also fixes $\mathrm{dcl}_\cap(A)$ pointwise, so the subgroup $\mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$ is also trivial. □

Corollary 3.7. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose each T_i admits quantifier elimination and a universal axiomatization. Then $T^*_\cup$ has quantifier elimination.*

Proof. Every L_i-substructure of a model of T_i is an elementary substructure, and hence acl_i-closed, so we can apply Corollary 3.6(1). □

3.4. **Language-independent consequences.** The results of Sections 3.1 through 3.3 can be lifted to the general case (when we have no $\mathcal{K}$-completeness hypotheses on the T_i) via Morleyization. This allows us to understand $L_\cup$-definable sets and certain complete $L_\cup$-types relative to L_i-definable sets and complete L_i-types.

To set notation: For each i, Morleyization gives a definitional expansion $L_i^\diamond$ of L_i and an extension $T_i^\diamond$ of T_i by axioms defining the new symbols in $L_i^\diamond$. We assume that the new symbols in $L_i^\diamond$ and $L_j^\diamond$ are distinct for $i \neq j$, so that $L_i^\diamond \cap L_j^\diamond = L_\cap$. It follows that each $T_i^\diamond$ has the same set of $L_\cap$ consequences, namely $T_\cap$. We let $L_\cup^\diamond = \bigcup_{i\in I} L_i^\diamond$ and $T_\cup^\diamond = \bigcup_{i\in I} T_i^\diamond$. Then every $T_\cup$-model $\mathcal{M}_\cup$ has a canonical expansion to a $T_\cup^\diamond$-model $\mathcal{M}_\cup^\diamond$, and by Remark 2.2, $\mathcal{M}_\cup$ is interpolative if and only if $\mathcal{M}_\cup^\diamond$ is interpolative.

The first result applies to any interpolative fusion $T^*_\cup$.

Proposition 3.8.

(1) Every $L_\cup$-formula $\psi(x)$ is $T_\cup^*$-equivalent to a finite disjunction of formulas of the form

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y)$$

where $J \subseteq I$ is finite and $\varphi_i(x, y)$ is an L_i-formula for all $i \in J$.

(2) If $\mathcal{M}_\cup$ is a $T_\cup^*$-model, then

$$T_\cup^* \cup \bigcup_{i \in I} \mathrm{Ediag}(\mathcal{M}_i) \models \mathrm{Ediag}(\mathcal{M}_\cup).$$

Proof. For (1), each Morleyized theory $T_i^\diamond$ has quantifier elimination, hence is model-complete, so we can apply Theorem 3.2 to the interpolative fusion $(T_\cup^\diamond)^*$. This says that $\psi(x)$ is $(T_\cup^\diamond)^*$-equivalent to a finite disjunction of formulas of the form $\exists y \bigwedge_{i \in J} \varphi_i(x, y)$, where each $\varphi_i(x, y)$ is a flat $L_i^\diamond$-formula. But since $L_i^\diamond$ is a definitional expansion of L_i, each formula $\varphi_i(x, y)$ can be translated back to an L_i-formula.

Now (2) is just a restatement of Fact 2.3(2). But we will give another proof, to illustrate the Morleyization method, which will be used repeatedly below. By Theorem 3.2, $(T_\cup^\diamond)^*$ is model-complete, so by Remark 3.1(3) we have

$$(T_\cup^*)^\diamond \cup \bigcup_{i \in I} \mathrm{Fdiag}(\mathcal{M}_i^\diamond) \models \mathrm{Ediag}(\mathcal{M}_\cup^\diamond).$$

But since $L_i^\diamond$ is a definitional expansion of L_i, $\mathrm{Fdiag}(\mathcal{M}_i^\diamond)$ is completely determined by $\mathrm{Ediag}(\mathcal{M}_i)$, and the result follows. □

We will now establish a sequence of variants on Proposition 3.8, with stronger hypotheses and stronger conclusions, but with essentially the same proof.

Proposition 3.9. *Assume $T_\cap$ admits a stationary and extendable independence relation. Then:*

(1) In models of $T_\cup^*$, $\mathrm{acl}_\cup = \mathrm{ccl}$.

(2) Every $L_\cup$-formula $\psi(x)$ is $T_\cup^*$-equivalent to a finite disjunction of formulas of the form

$$\exists y \bigwedge_{i \in J} \varphi_i(x, y)$$

where $J \subseteq I$ is finite, $\varphi_i(x, y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i \in J} \varphi_i(x, y)$ is bounded in y (see Appendix A for the definition).

(3) If A is an $\mathrm{acl}_\cup$-closed subset of a $T_\cup^*$-model $M_\cup$, then

$$T_\cup^* \cup \bigcup_{i \in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$

Proof. Each Morleyized theory $T_i^\diamond$ has quantifier elimination, hence is acl-complete and bcl-complete. Further, Morleyization does not affect the stationary and extendable independence relation on $T_\cap$. Then (1) follows from Theorem 3.3, observing that $\mathrm{acl}_\cup$ and ccl are not altered by definitional expansions. (2) and (3) follow just as in Proposition 3.8, using the syntactic result of Corollary 3.4 for (2). □

Proposition 3.10. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose further that*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all $L_\cup$-substructures $\mathcal{A}_\cup$ of $T_\cup^*$-models and all $i \in I$. Then:

(1) *Every formula $\psi(x)$ is $T^*_\cup$-equivalent to a finite disjunction of formulas*

$$\exists y \bigwedge_{i\in J} \varphi_i(x,y)$$

where $J \subseteq I$ is finite, $\varphi_i(x,y)$ is an L_i-formula for all $i \in J$, and $\bigwedge_{i\in J}\varphi_i(x,y)$ is bounded in y with bound 1.

(2) *If $\mathcal{A}_\cup$ is an $L_\cup$-substructure of a $T^*_\cup$-model $\mathcal{M}_\cup$, then*

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$

Proof. Morleyization does not affect our hypotheses on acl_i, $\mathrm{acl}_\cap$, and the stationary and extendable independence relation on $T_\cap$. So by Theorem 3.5, $(T^\diamond_\cup)^*$ has quantifier elimination. This gives us (2) as in the proof of Proposition 3.8.

For (1), $\psi(x)$ is $(T^\diamond_\cup)^*$-equivalent to a quantifier-free $L^\diamond_\cup$-formula $\psi^\diamond(x)$. We cannot translate $\psi^\diamond(x)$ back to a Boolean combination of L_i-formulas, since a single atomic subformula of $\psi^\diamond(x)$ may involve function and relation symbols from distinct languages.

However, by Fact 2.9, $\psi^\diamond(x)$ is equivalent to a finite disjunction of E$\flat$-formulas $\exists y\, \theta^\diamond(x,y)$. Applying Remark 3.1(1) to each flat formula $\theta^\diamond(x,y)$, we obtain a finite disjunction of formulas of the form

$$\exists y \bigwedge_{i\in J} \varphi^\diamond_i(x,y)$$

where $J \subseteq I$ is finite, $\varphi^\diamond_i(x,y)$ is a flat $L^\diamond_i$-formula for all $i \in J$, and $\bigwedge_{i\in J}\varphi_i(x,y)$ is bounded in y with bound 1. Replacing each $L^\diamond_i$-formula $\varphi^\diamond_i$ with an equivalent L_i-formula, we obtain the desired result. □

Remark 3.11. As in Corollary 3.6(1), we can replace the hypotheses of Proposition 3.10 with: $T_\cap$ admits a stationary and extendable independence relation, and for all sets A and all $i \in I$, $\mathrm{acl}_i(A) = \langle A\rangle_{L_i}$. The assumption $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ gives us something stronger, see Remark 3.13 below.

With a slightly stronger hypothesis, we can get true relative quantifier elimination down to L_i-formulas in $T^*_\cup$.

Proposition 3.12. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose further that*

$$\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A) \quad \textit{and} \quad \mathrm{Aut}_{L_\cap}(\mathrm{acl}_\cap(A)/A) = \mathrm{Aut}_{L_i}(\mathrm{acl}_\cap(A)/A)$$

for all sets A and all $i \in I$. Then:

(1) *Every formula is $T^*_\cup$-equivalent to a Boolean combination of L_i-formulas.*

(2) *For any subset A of a $T^*_\cup$-model $\mathcal{M}_\cup$,*

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{Th}_{L_i(A)}(\mathcal{M}_i) \models \mathrm{Th}_{L_\cup(A)}(\mathcal{M}_\cup).$$

Proof. We first move to a relational language by replacing all function symbols by their graphs. Then we proceed just as in the proof of Proposition 3.10, noting that when $L^\diamond_\cup$ is relational, every subset of a $(T^\diamond_\cup)^*$-model is a substructure, and each quantifier-free $L^\diamond_\cup$-formula is already a Boolean combination of $L^\diamond_i$-formulas. □

Remark 3.13. Once again, as in Corollary 3.6(2), we can replace the hypotheses of Proposition 3.12 with: $T_\cap$ admits a stationary and extendable independence relation, and $\mathrm{acl}_i(A) = \mathrm{dcl}_\cap(A)$ for all sets A and all $i \in I$. The assumption $\mathrm{acl}_i(A) =$

$\langle A\rangle_{L_i}$ does not suffice for this, because this condition is lost when passing to a relational language.

3.5. **Completeness.** We view Theorem 3.3 and its avatar Proposition 3.9 as the main results of this section, since they give us useful tools for understanding $L_\cup$-definable sets and complete $L_\cup$-types, while only requiring mild hypotheses (in particular, they apply whenever $T_\cap$ is stable with weak elimination of imaginaries). For example, we obtain from Proposition 3.9 the following criterion for completeness of $T^*_\cup$.

Corollary 3.14. *Assume $T_\cap$ admits a stationary and extendable independence relation. Suppose each T_i is complete and $\varnothing$ is acl_i-closed for all $i \in I$. Then $T^*_\cup$ is complete.*

Proof. Since $\varnothing$ is acl_i-closed for all $i \in I$, it is ccl-closed, and hence $\mathrm{acl}_\cup$-closed by Proposition 3.9(1). So for any model $\mathcal{M}_\cup \models T^*_\cup$, by Proposition 3.9(3),

$$T^*_\cup \cup \bigcup_{i\in I} \mathrm{Th}(\mathcal{M}_i) \models \mathrm{Th}(\mathcal{M}_\cup).$$

Since each T_i is complete, $\bigcup_{i\in I} \mathrm{Th}(\mathcal{M}_i) = T_\cup \subseteq T^*_\cup$, so $T^*_\cup \models \mathrm{Th}(\mathcal{M}_\cup)$. □

In general, when $T_\cap$ admits a stationary and extendable independence relation and each T_i is complete, a completion of $T^*_\cup$ is determined by the $L_\cup$-isomorphism type of $\mathrm{acl}_\cup(\varnothing) = \mathrm{ccl}(\varnothing)$ in any model. For example, this is what happens in ACFA.

We conclude with two counterexamples indicating the sharpness of Theorem 3.3 and Proposition 3.9. In the first example, $T_\cap$ is unstable with elimination of imaginaries, and $\mathop{\smile\!\!\!\!\!\mid}^f$ fails to be stationary. In the second example, $T_\cap$ is stable but fails weak elimination of imaginaries, and $\mathop{\smile\!\!\!\!\!\mid}^f$ is stationary but fails to be extendable. In both examples, $T_\cap$ does not admit any stationary and extendable independence relation, and we do not even get the result of Corollary 3.14.

Example 3.15. Let $L_\cap = \{\leq\}$ where $\leq$ is a binary relation symbol, and let L_i be the expansion of $L_\cap$ by a unary predicate P_i for $i \in \{1,2\}$. Let $T_\cap = \mathrm{DLO}$, and for $i \in \{1,2\}$, let T_i be the theory of a dense linear order equipped with a downwards closed set with an upper bound, but no least upper bound, defined by P_i. Note that T_i is complete and $\varnothing$ is acl_i-closed for $i \in \{1,2\}$. A model $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if $P_1(\mathcal{M}_\cup) \neq P_2(\mathcal{M}_\cup)$, so $T^*_\cup$ exists. But a $T^*_\cup$-model either has $P_1(\mathcal{M}_\cup) \subsetneq P_2(\mathcal{M}_\cup)$ or $P_2(\mathcal{M}_\cup) \subsetneq P_1(\mathcal{M}_\cup)$, so $T^*_\cup$ is not complete.

Example 3.16. Let $L_\cap = \{E\}$ where E is a binary relation symbol, and let L_i be the expansion of $L_\cap$ by a unary predicate P_i for $i \in \{1,2\}$. Let $T_\cap$ be the theory of an equivalence relation with infinitely many infinite classes. For $i \in \{1,2\}$, let T_i be the theory of a $T_\cap$-model with a distinguished equivalence class, defined by P_i. Again, T_i is complete and $\varnothing$ is acl_i-closed for $i \in \{1,2\}$. Every model of $T_\cup$ is interpolative, so $T^*_\cup = T_\cup$. But a $T^*_\cup$-model $\mathcal{M}_\cup$ either has $P_1(\mathcal{M}_\cup) = P_2(\mathcal{M}_\cup)$ or $P_1(\mathcal{M}_\cup) \neq P_2(\mathcal{M}_\cup)$, so $T^*_\cup$ is not complete.

4. Combinatorial tameness

Throughout this section, we fix the languages $L_\square$ and theories $T_\square$ for $\square \in I \cup \{\cup,\cap\}$, and we assume the interpolative fusion $T^*_\cup$ exists.

4.1. **Stability and** NIP**.** Under very strong hypotheses, we can prove that stability and NIP are preserved by interpolative fusions.

Proposition 4.1. *Assume the hypotheses of Proposition 3.12. If each T_i is stable, then $T^*_\cup$ is stable. If each T_i is* NIP*, then $T^*_\cup$ is* NIP*.*

Proof. This follows from Proposition 3.12(1), since any Boolean combination of stable formulas is stable, and any Boolean combination of NIP formulas is NIP. □

We can also use Proposition 3.12(2) to count types.

Proposition 4.2. *Assume the hypotheses of Proposition 3.12, and let κ be an infinite cardinal such that $\kappa^{|I|} = \kappa$. If each T_i is stable in κ, then $T^*_\cup$ is stable in κ.*

Proof. Let A be a subset of a model $\mathcal{M}_\cup \models T^*_\cup$ such that $|A| \leq \kappa$. We would like to understand the size of $S^x_{L_\cup}(A)$, the space of $L_\cup$-types in the finite variable context x over A. By Proposition 3.12(2), a type in $S^x_{L_\cup}(A)$ is completely determined by its restrictions to L_i-types in $S^x_{L_i}(A)$ for all $i \in I$. Since T_i is stable in κ, we have $|S^x_{L_i}(A)| \leq \kappa$, and $|S^x_{L_\cup}(A)| \leq \prod_{i\in I} |S^x_{L_i}(A)| \leq \kappa^{|I|} = \kappa$. So $T^*_\cup$ is stable in κ. □

Since $\kappa^{|I|} = \kappa$ for all infinite κ when I is finite, we obtain the following corollary.

Corollary 4.3. *Assume the hypotheses of Proposition 3.12, and further assume $|I|$ is finite. If each T_i is $\aleph_0$-stable, then $T^*_\cup$ is $\aleph_0$-stable. If each T_i is superstable, then $T^*_\cup$ is superstable.*

We do not expect to obtain preservation of stability or NIP without strong restrictions on acl, as in the hypotheses of Proposition 3.12. The proofs of Propositions 4.1 and 4.2 do not apply to other classification-theoretic properties such as simplicity, NSOP_1, and NTP_2, as these properties are not characterized by counting types, and formulas with these properties are not closed under Boolean combinations in general. Nevertheless, in the subsequent sections we obtain preservation results for simplicity and NSOP_1 under more general hypotheses.

4.2. NSOP_1**.** Toward proving that $T^*_\cup$ is NSOP_1, we define an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ on subsets of any monster model $\boldsymbol{\mathcal{M}}_\cup \models T^*_\cup$. Then we seek to apply Fact 2.16 to show that $T^*_\cup$ is NSOP_1 and $\mathop{\smile\!\!\!\!\!\mid}$ is Kim-independence.

Assume each T_i is NSOP_1, and let $\boldsymbol{\mathcal{M}}_\cup \models T^*_\cup$ be a monster model (the choice of $\boldsymbol{\mathcal{M}}_\cup$ amounts to the choice of a completion of $T^*_\cup$). For all $A, B \subseteq \boldsymbol{M}$ and $\mathcal{M}_\cup \preceq \boldsymbol{\mathcal{M}}_\cup$, we define:

$$A \mathop{\smile\!\!\!\!\!\mid}_M B \quad \text{if and only if} \quad \mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!\mid}^K_M \mathrm{acl}_\cup(MB) \text{ in } \boldsymbol{\mathcal{M}}_i \text{ for all } i \in I.$$

This definition is motivated by the following considerations. If $\mathrm{Th}(\boldsymbol{\mathcal{M}}_\cup)$ is NSOP_1, then $A \mathop{\smile\!\!\!\!\!\mid}^K_M B$ implies $\mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!\mid}^K_M \mathrm{acl}_\cup(MB)$ in $\boldsymbol{\mathcal{M}}_\cup$ by Fact 2.20. Then by Lemma 2.22, we have $\mathrm{acl}_\cup(MA) \mathop{\smile\!\!\!\!\!\mid}^K_M \mathrm{acl}_\cup(MB)$ in $\boldsymbol{\mathcal{M}}_i$ for all i. Conversely, it is reasonable to hope that Kim-forking between $\mathrm{acl}_\cup(MA)$ and $\mathrm{acl}_\cup(MB)$ in some $\boldsymbol{\mathcal{M}}_i$ is the only source of Kim-forking between A and B in $\boldsymbol{\mathcal{M}}_\cup$.

In fact, with the exception of the independence theorem over models, all of the properties required by Fact 2.16 follow easily for $\mathop{\smile\!\!\!\!\!\mid}$.

Proposition 4.4. *Assume each T_i is* NSOP_1*. Then $\mathop{\smile\!\!\!\!\!\mid}$ satisfies invariance, existence, monotonicity, symmetry, and strong finite character. If* $\mathrm{Th}(\boldsymbol{\mathcal{M}}_\cup)$ *is* NSOP_1*, then $\mathop{\smile\!\!\!\!\!\mid}$ also satisfies witnessing.*

Proof. Invariance, existence, monotonicity, symmetry: Clear from the definition, using the corresponding properties of Kim-independence in each $\mathfrak{M}_i$.

Strong finite character: Suppose $A \not\perp_M B$. Then for some $i \in I$, we have $\mathrm{acl}_\cup(MA) \not\perp^K_M \mathrm{acl}_\cup(MB)$ in $\mathfrak{M}_i$. So there is some $a' \in \mathrm{acl}_\cup(MA)$ and $b' \in \mathrm{acl}_\cup(MB)$ such that $a' \not\perp^K_M b'$ in $\mathfrak{M}_i$. Let $\varphi(x', b', m)$ be an L_i-formula in $\mathrm{tp}_{L_i}(a'/Mb')$ which Kim-forks over M in $\mathfrak{M}_i$, let $\psi(x', a, m)$ be an $L_\cup$-formula isolating the algebraic type $\mathrm{tp}_{L_\cup}(a'/MA)$, and let $\theta(y', b, m)$ be an $L_\cup$-formula isolating the algebraic type $\mathrm{tp}_{L_\cup}(b'/MB)$. Note that by replacing ψ with $\psi(x', a, m) \wedge (\exists^{\leq k} x'\, \psi(x', a, m))$ for some k, we may assume $\psi(x', c, m)$ has only finitely many realizations for any c.

We claim that the following formula $\chi(x, b, m)$ witnesses strong finite character:

$$\exists x' \,\exists y' \,[\varphi(x', y', m) \wedge \psi(x', x, m) \wedge \theta(y', b, m)].$$

Certainly we have $\chi(x, b, m) \in \mathrm{tp}_{L_\cup}(A/MB)$. Suppose we are given c such that $\mathfrak{M}_\cup \models \chi(c, b, m)$. Then picking witnesses c' and b'' for the existential quantifiers, we have that $c' \in \mathrm{acl}_\cup(Mc)$ (since $\mathfrak{M}_\cup \models \psi(c', c, m)$) and $b'' \in \mathrm{acl}_\cup(Mb)$ (since $\mathfrak{M}_\cup \models \theta(b'', b, m)$). Further, $b'' \equiv_{MB} b'$, so $\varphi(x', b'', m)$ Kim-forks over M in $\mathfrak{M}_i$. Since $\mathfrak{M}_\cup \models \varphi(c', b'', m)$, we have $c' \not\perp^K_M b''$ in $\mathfrak{M}_i$, so $c \not\perp_M b$.

Witnessing: For this property, we assume $\mathrm{Th}(\mathfrak{M}_\cup)$ is NSOP_1. Suppose again $A \not\perp_M B$. We use the same notation as in the proof of strong finite character, and we seek to show that $\chi(x, b, m)$ Kim-forks over M in $\mathfrak{M}_\cup$.

If not, then by compactness we can find a complete $L_\cup$-type $p(x)$ over Mb which contains $\chi(x, b, m)$ but does not Kim-fork over M. Let e realize this type. Then we have $e \perp^K_M b$ in $\mathfrak{M}_\cup$, so by Fact 2.20, $\mathrm{acl}_\cup(Me) \perp^K_M \mathrm{acl}_\cup(Mb)$ in $\mathfrak{M}_\cup$. But since $\mathfrak{M}_\cup \models \chi(e, b, m)$, there is some $e' \in \mathrm{acl}_\cup(Me)$ and some $b'' \in \mathrm{acl}_\cup(Mb)$ such that $\mathfrak{M}_\cup \models \varphi(e', b'', m)$. This is a contradiction, since by Lemma 2.22 and the fact that $\mathrm{tp}_{L_\cup}(b''/M) = \mathrm{tp}_{L_\cup}(b'/M)$, $\varphi(x', b'', m)$ Kim-forks over M in $\mathfrak{M}_\cup$. □

It remains to show that $\perp$ satisfies the independence theorem, which could also be called "independent 3-amalgamation over models". In Theorem 3.3 above, we used stationarity of an independence relation in $T_\cap$ to establish disjoint 2-amalgamation over algebraically closed sets. For the independence theorem, we need to appeal to a strengthening of stationarity, namely 3-uniqueness, which holds over models in stable theories.

Assume T is stable. Let $B, A_1, A_2, A_3 \subseteq \boldsymbol{M}$ be sets. We say A_1, A_2, A_3 is an **independent triple over** B if $A_2 \perp^f_B A_1$ and $A_3 \perp^f_B A_1A_2$. By basic properties of forking independence in stable theories (symmetry, base monotonicity, and transitivity), it follows that whenever $\{i, j, k\} = \{1, 2, 3\}$:

$$A_i \mathop{\perp^f}_{B} A_jA_k \quad \text{and} \quad A_i \mathop{\perp^f}_{BA_j} A_k.$$

Now assume the stable theory T has elimination of imaginaries and B is acl-closed. Suppose A_1, A_2, A_3 is an independent triple over B. For each $i \in \{1, 2, 3\}$, let a_i be a tuple enumerating A_i. The three types $\mathrm{tp}(a_i/B)$ for $i \in \{1, 2, 3\}$ uniquely determine $\mathrm{tp}(a_1a_2a_3/B)$ in the following sense. If A'_1, A'_2, A'_3 is another independent triple over B, with each A'_i enumerated by a tuple a'_i, and if $\mathrm{tp}(a_i/B) = \mathrm{tp}(a'_i/B)$ for all $i \in \{1, 2, 3\}$, then by stationarity of $\perp^f$ over B, $\mathrm{tp}(a_1a_2a_3/B) = \mathrm{tp}(a'_1a'_2a'_3/B)$.

The analogous statement may not hold if we consider the types of the sets $\mathrm{acl}(A_iA_j)$. Fix again an independent triple A_1, A_2, A_3 over and acl-closed set

B, and assume that for each $i \in \{1,2,3\}$, A_i is algebraically closed and $B \subseteq A_i$. For all $i \in \{1,2,3\}$, let a_i be a tuple enumerating A_i, and for all $1 \le i < j \le 3$, let a_{ij} be a tuple extending a_i and a_j and enumerating $\mathrm{acl}(A_iA_j)$. Then we say T has **3-uniqueness over** B if for any other independent triple A_1', A_2', A_3' over B, with each A_i enumerated by a tuple a_i' and each $\mathrm{acl}(A_iA_j)$ enumerated by a tuple a_{ij}' extending a_i' and a_j', if $\mathrm{tp}(a_i/B) = \mathrm{tp}(a_i'/B)$ for all $i \in \{1,2,3\}$ and $\mathrm{tp}(a_{ij}/B) = \mathrm{tp}(a_{ij}'/B)$ for all $1 \le i < j \le 3$, then $\mathrm{tp}(a_{12}a_{13}a_{23}/B) = \mathrm{tp}(a_{12}'a_{13}'a_{23}'/B)$.

Fact 4.5 ([DPKM06, Proposition 1.6(2)])**.** *Every stable theory with elimination of imaginaries has* 3*-uniqueness over models.*

In its general form, 3-uniqueness was introduced by Hrushovski in [Hru12]. The reference [DPKM06] cited for Fact 4.5 actually shows that stable theories satisfy a stronger property, called n-complete amalgamation, over models. In the special case of 3-uniqueness, the result follows easily from [Hru12, Lemma 4.2] together with the fact that in a stable theory, any type which does not fork over a model M is finitely satisfiable in M.

We are now ready to proceed with the main theorem. See Section 2.7 for the definition of the $T_\cap$-generic independence theorem.

Theorem 4.6. *Assume $T_\cap$ is stable, each T_i is* NSOP_1*, and $\mathop{\smile\!\!\!\!\!\mid}^K$ satisfies the $T_\cap$-generic independence theorem in each T_i. Then $T_\cup^*$ is* NSOP_1 *and $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^K$.*

Proof. To make our notation more compact, in this proof we write $[X]_\square$ for $\mathrm{acl}_\square(X)$ when $\square \in I \cup \{\cap, \cup\}$. By Remark 2.2, we may assume that $T_\cap$ eliminates imaginaries.

Fix a monster model $\boldsymbol{\mathcal{M}}_\cup \models T_\cup^*$ (equivalently, a completion of $T_\cup^*$). We would like to show that $\mathrm{Th}(\boldsymbol{\mathcal{M}}_\cup)$ is NSOP_1. By Proposition 4.4, $\mathop{\smile\!\!\!\!\!\mid}$ satisfies invariance, existence, monotonicity, symmetry, and strong finite character. If we show that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the independence theorem, then by Fact 2.16, $\mathrm{Th}(\boldsymbol{\mathcal{M}}_\cup)$ is NSOP_1. Proposition 4.4 then tells us that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies witnessing, so $\mathop{\smile\!\!\!\!\!\mid}_M = \mathop{\smile\!\!\!\!\!\mid}^K{}_M$ for all models $\mathcal{M}_\cup \preccurlyeq \boldsymbol{\mathcal{M}}_\cup$. Toward the independence theorem, suppose we are given $A, A', B, C \subseteq \boldsymbol{M}$ and $\mathcal{M}_\cup \preccurlyeq \boldsymbol{\mathcal{M}}_\cup$ such that:

$$\mathrm{tp}_{L_\cup}(A/M) = \mathrm{tp}_{L_\cup}(A'/M), \quad A \mathop{\smile\!\!\!\!\!\mid}_M B, \quad A' \mathop{\smile\!\!\!\!\!\mid}_M C, \quad \text{and} \quad B \mathop{\smile\!\!\!\!\!\mid}_M C.$$

By adding elements to A, A', B, and C, we may assume $A = [MA]_\cup$, $A' = [MA']_\cup$, $B = [MB]_\cup$, and $C = [MC]_\cup$. Then by definition of $\mathop{\smile\!\!\!\!\!\mid}$, we have, for all $i \in I$:

$$\mathrm{tp}_{L_i}(A/M) = \mathrm{tp}_{L_i}(A'/M), \quad A \mathop{\smile\!\!\!\!\!\mid}^K_M B, \quad A' \mathop{\smile\!\!\!\!\!\mid}^K_M C, \quad \text{and} \quad B \mathop{\smile\!\!\!\!\!\mid}^K_M C \quad \text{in } \boldsymbol{\mathcal{M}}_i.$$

Let us fix some notation. Since $\mathrm{tp}_{L_i}(A/M) = \mathrm{tp}_{L_i}(A'/M)$, there is a partial elementary bijection $A \to A'$ which fixes M pointwise. Let x_A be a tuple of variables simultaneously enumerating A and A' according to this bijection. Let x_B and x_C be tuples enumerating B and C, respectively. For all $\square \in I \cup \{\cap, \cup\}$, we define:

$$p_\square^{AB}(x_{AB}) = \mathrm{tp}_{L_\square}([AB]_\cap/M)$$
$$p_\square^{AC}(x_{AC}) = \mathrm{tp}_{L_\square}([A_i'C]_\cap/M)$$
$$p_\square^{BC}(x_{BC}) = \mathrm{tp}_{L_\square}([BC]_\cap/M)$$

where x_{AB} is a tuple of variables extending x_A and x_B and enumerating $[AB]_\cap$, and similarly for x_{AC} and x_{BC}. We additionally define:

$$q_\square^{AB}(x_{AB}, y_{AB}) = \mathrm{tp}_{L_\square}([AB]_\cup/M)$$
$$q_\square^{AC}(x_{AC}, y_{AC}) = \mathrm{tp}_{L_\square}([A'C]_\cup/M)$$
$$q_\square^{BC}(x_{BC}, y_{BC}) = \mathrm{tp}_{L_\square}([BC]_\cup/M)$$

where y_{AB} is a tuple of variables enumerating $[AB]_\cup \setminus [AB]_\cap$, and similarly for y_{AC} and y_{BC}.

Now $T_\cap$ is stable with elimination of imaginaries, so by Proposition 3.9(3), $\bigcup_{i\in I} q_i^{AB}$ axiomatizes $q_\cup^{AB}$ relative to $T_\cup^*$. Similarly, $\bigcup_{i\in I} q_i^{AC}$ axiomatizes $q_\cup^{AC}$ and $\bigcup_{i\in I} q_i^{BC}$ axiomatizes $q_\cup^{BC}$.

For the moment, fix $i \in I$. Since $\mathop{\smile\!\!\!\!\!\mid}^K$ in T_i satisfies the $T_\cap$-generic independence theorem, there exists A_i'' such that $\mathrm{tp}_{L_i}(A_i''/MB) = \mathrm{tp}_{L_i}(A/MB)$, $\mathrm{tp}_{L_i}(A_i''/MC) = \mathrm{tp}_{L_i}(A'/MC)$, $A_i'' \mathop{\smile\!\!\!\!\!\mid}^K_M BC$ in $\mathfrak{M}_i$, and further, in $\mathfrak{M}_\cap$:

$$[A_i''B]_i \mathop{\smile\!\!\!\!\!\mid}^f_{A_i''B} [A_i''C]_i[BC]_i \tag{1}$$

$$[A_i''C]_i \mathop{\smile\!\!\!\!\!\mid}^f_{A_i''C} [A_i''B]_i[BC]_i \tag{2}$$

$$[BC]_i \mathop{\smile\!\!\!\!\!\mid}^f_{BC} [A_i''B]_i[A_i''C]_i. \tag{3}$$

Since Kim-independence is preserved under reducts (Lemma 2.22) and agrees with forking independence in a stable theory (Fact 2.21), we have $B \mathop{\smile\!\!\!\!\!\mid}^f_M C$ and $A_i'' \mathop{\smile\!\!\!\!\!\mid}^f_M BC$ in $\mathfrak{M}_\cap$, so A_i'', B, C is an independent triple over M in $\mathfrak{M}_\cap$.

Let $E_i = A_i''BC$. By base monotonicity from (1), (2), and (3), $[A_i''B]_i$, $[A_i''C]_i$, $[BC]_i$ is an independent triple over E_i in $\mathfrak{M}_\cap$. In particular:

$$[BC]_i \mathop{\smile\!\!\!\!\!\mid}^f_{E_i} [A_i''B]_i[A_i''C]_i. \tag{4}$$

Our goal is to extend the sets $[A_i''B]_i$, $[A_i''C]_i$, $[BC]_i$ to realizations of the types q_i^{AB}, q_i^{AC}, and q_i^{BC}, in such a way that these realizations also form an independent triple over E_i in $\mathfrak{M}_\cap$

By Fact 2.20, $A_i'' \mathop{\smile\!\!\!\!\!\mid}^K_M BC$ implies $A_i'' \mathop{\smile\!\!\!\!\!\mid}^K_M [BC]_i$ in $\mathfrak{M}_i$. By reasonable extension (Theorem 2.23), we can find a realization A_i''' of $\mathrm{tp}(A_i''/[BC]_i)$ such that:

$$A_i''' \mathop{\smile\!\!\!\!\!\mid}^K_M [BC]_\cup \quad \text{and} \quad A_i''' \mathop{\smile\!\!\!\!\!\mid}^r_{[BC]_i} [BC]_\cup \quad \text{in } \mathfrak{M}_i.$$

Let σ be an automorphism of $\mathfrak{M}_i$ which fixes $[BC]_i$ pointwise and moves A_i''' to A_i'', and let $D_i^{BC} = \sigma([BC]_\cup)$. Then D_i^{BC} realizes $q_i^{BC} = \mathrm{tp}_{L_i}([BC]_\cup/M)$, and we have in $\mathfrak{M}_i$:

$$A_i'' \mathop{\smile\!\!\!\!\!\mid}^K_M D_i^{BC} \tag{5}$$

$$A_i'' \mathop{\smile\!\!\!\!\!\mid}^r_{[BC]_i} D_i^{BC}. \tag{6}$$

By definition of $\mathop{\smile\!\!\!\!|}^{r}$, since $[A''_iB]_i$ and $[A''_iC]_i$ are both subsets of $[A''_i[BC]_i]_i$, we have in $\mathcal{M}_\cap$:

$$D_i^{BC} \mathop{\smile\!\!\!\!|}^{f}_{[BC]_i} [A''_iB]_i[A''_iC]_i \quad \text{by symmetry and monotonicity, from (6)} \tag{7}$$

$$D_i^{BC} \mathop{\smile\!\!\!\!|}^{f}_{E_i[BC]_i} [A''_iB]_i[A''_iC]_i \quad \text{by base monotonicity, from (7)} \tag{8}$$

$$D_i^{BC} \mathop{\smile\!\!\!\!|}^{f}_{E_i} [A''_iB]_i[A''_iC]_i \quad \text{by transitivity, from (4) and (8).} \tag{9}$$

Thus $[A''_iB]_i$, $[A''_iC]_i$, D_i^{BC} is an independent triple over E_i in $\mathcal{M}_\cap$. In particular:

$$[A''_iB]_i \mathop{\smile\!\!\!\!|}^{f}_{E_i} [A''_iC]_iD_i^{BC}. \tag{10}$$

Since $\mathrm{tp}_{L_i}(A''_iB/M) = \mathrm{tp}_{L_i}(AB/M)$, we can extend $[A''_iB]_i$ to a realization D_i^{AB} of $q_i^{AB} = \mathrm{tp}_{L_i}([AB]_\cup/M)$. Further, since $\mathop{\smile\!\!\!\!|}^{f}$ in $\mathcal{M}_\cap$ is extendable to $\mathcal{M}_i$ (Proposition 2.14), we may assume that in $\mathcal{M}_\cap$:

$$D_i^{AB} \mathop{\smile\!\!\!\!|}^{f}_{[A''_iB]_i} [A''_iC]_iD_i^{BC}. \tag{11}$$

Repeating the argument above, we have in $\mathcal{M}_\cap$:

$$D_i^{AB} \mathop{\smile\!\!\!\!|}^{f}_{E_i[A''_iB]_i} [A''_iC]_iD_i^{BC} \quad \text{by base monotonicity, from (11)} \tag{12}$$

$$D_i^{AB} \mathop{\smile\!\!\!\!|}^{f}_{E_i} [A''_iC]_iD_i^{BC} \quad \text{by transitivity, from (10) and (12).} \tag{13}$$

Thus $[A''_iC]_i$, D_i^{AB}, D_i^{BC} is an independent triple over E_i in $\mathcal{M}_\cap$. In particular:

$$[A''_iC]_i \mathop{\smile\!\!\!\!|}^{f}_{E_i} D_i^{AB}D_i^{BC}. \tag{14}$$

Similarly, since $\mathrm{tp}_{L_i}(A''_iC/M) = \mathrm{tp}_{L_i}(A'C/M)$, we can extend $[A''_iC]_i$ to a realization D_i^{AC} of $q_i^{AC} = \mathrm{tp}_{L_i}([A'C]_\cup/M)$ such that in $\mathcal{M}_\cap$:

$$D_i^{AC} \mathop{\smile\!\!\!\!|}^{f}_{[A''_iC]_i} D_i^{AB}D_i^{BC}. \tag{15}$$

Repeating the argument one more time, we have in $\mathcal{M}_\cap$:

$$D_i^{AC} \mathop{\smile\!\!\!\!|}^{f}_{E_i[A''_iC]_i} D_i^{AB}D_i^{BC} \quad \text{by base monotonicity, from (15)} \tag{16}$$

$$D_i^{AC} \mathop{\smile\!\!\!\!|}^{f}_{E_i} D_i^{AB}D_i^{BC} \quad \text{by transitivity, from (14) and (16).} \tag{17}$$

Thus D_i^{AB}, D_i^{AC}, D_i^{BC} is an independent triple over E_i in $\mathcal{M}_\cap$.

With all the pieces in place, we set

$$q_i^{ABC}(x_{AB}, y_{AB}, x_{AC}, y_{AC}, x_{BC}, y_{BC}) = \mathrm{tp}_{L_i}(D_i^{AB}, D_i^{AC}, D_i^{BC}/M).$$

We now claim that for all $i, j \in I$, the restrictions of q_i^{ABC} and q_j^{ABC} to $L_\cap$ are equal:

$$\mathrm{tp}_{L_\cap}(D_i^{AB}, D_i^{AC}, D_i^{BC}/M) = \mathrm{tp}_{L_\cap}(D_j^{AB}, D_j^{AC}, D_j^{BC}/M).$$

As noted above, A''_i, B, C and A''_j, B, C are independent triples over M in $\mathfrak{M}_\cap$. We also have:

$$\begin{aligned} \mathrm{tp}_{L_\cap}([A''_i B]_\cap/M) &= \mathrm{tp}_{L_\cap}([A''_j B]_\cap/M) = p_\cap^{AB} \\ \mathrm{tp}_{L_\cap}([A''_i C]_\cap/M) &= \mathrm{tp}_{L_\cap}([A''_j C]_\cap/M) = p_\cap^{AC} \\ \mathrm{tp}_{L_\cap}([BC]_\cap/M) &= p_\cap^{BC}. \end{aligned}$$

So by 3-uniqueness over M (Fact 4.5),

$$\mathrm{tp}_{L_\cap}([A''_i B]_\cap, [A''_i C]_\cap, [BC]_\cap/M) = \mathrm{tp}_{L_\cap}([A''_j B]_\cap, [A''_j C]_\cap, [BC]_\cap/M).$$

It follows that there exists a partial $L_\cap$-elementary bijection $\tau\colon [E_i]_\cap \to [E_j]_\cap$ extending the identity on $[BC]_\cap$ and the elementary bijections $[A''_i B]_\cap \to [A''_j B]_\cap$ and $[A''_i C]_\cap \to [A''_j C]_\cap$ given by the enumerations of these sets by the variables x_{AB} and x_{AC}.

Extending τ to an automorphism of $\mathfrak{M}_\cap$, we may identify $[E_i]_\cap$ with $[E_j]_\cap$ and call this set just $[E]_\cap$. This also has the effect of identifying A''_i with A''_j and E_i with E_j, and similarly we call these sets just A'' and E, respectively.

By (1) and (11) above, in $\mathfrak{M}_\cap$:

$$[A''B]_i \mathop{\smile\!\!\!\!\!\!\;|}^f_{A''B} E \quad \text{and} \quad D_i^{AB} \mathop{\smile\!\!\!\!\!\!\;|}^f_{[A''B]_i} E$$

so by transitivity and closing under $\mathrm{acl}_\cap$, and applying the same argument to D_j^{AB}:

$$D_i^{AB} \mathop{\smile\!\!\!\!\!\!\;|}^f_{[A''B]_\cap} [E]_\cap \quad \text{and} \quad D_j^{AB} \mathop{\smile\!\!\!\!\!\!\;|}^f_{[A''B]_\cap} [E]_\cap.$$

We have $\mathrm{tp}_{L_\cap}(D_i^{AB}/[A''B]_\cap) = \mathrm{tp}_{L_\cap}(D_j^{AB}/[A''B]_\cap)$, since both agree with $q_\cap^{AB}$, so by stationarity, $\mathrm{tp}_{L_\cap}(D_i^{AB}/[E]_\cap) = \mathrm{tp}_{L_\cap}(D_j^{AB}/[E]_\cap)$.

The same argument, using (2), (3), (7), and (15), shows $\mathrm{tp}_{L_\cap}(D_i^{AC}/[E]_\cap) = \mathrm{tp}_{L_\cap}(D_j^{AC}/[E]_\cap)$ and $\mathrm{tp}_{L_\cap}(D_i^{BC}/[E]_\cap) = \mathrm{tp}_{L_\cap}(D_j^{BC}/[E]_\cap)$.

Since forking independence over a set agrees with forking independence over the algebraic closure of that set, D_i^{AB}, D_i^{AC}, D_i^{BC} and D_j^{AB}, D_j^{AC}, D_j^{BC} are both independent triples over $[E]_\cap$. So by stationarity:

$$\mathrm{tp}_{L_\cap}(D_i^{AB}, D_i^{AC}, D_i^{BC}/[E]_\cap) = \mathrm{tp}_{L_\cap}(D_j^{AB}, D_j^{AC}, D_j^{BC}/[E]_\cap).$$

In particular, the restrictions of these types to M agree, which establishes the claim. It follows that that there is a complete $L_\cap$-type $q_\cap^{ABC}$ over M such that $q_\cap^{ABC} \subseteq q_i^{ABC}$ for all $i \in I$.

By Proposition 2.5, $\bigcup_{i\in I} q_i^{ABC}$ is consistent and realized in $\mathfrak{M}_\cup$ by sets (D_{AB}, D_{AC}, D_{BC}). Let A^*, B^*, and C^* be the subsets enumerated by x_A, x_B, and x_C, respectively.

Since D_{BC} satisfies q_i^{BC} for all $i \in I$, D_{BC} satisfies $q_\cup^{BC} = \mathrm{tp}_{L_\cup}(\mathrm{acl}_\cup(BC)/M)$. So after applying an automorphism of $\mathfrak{M}_\cup$ which fixes M pointwise, we may assume $D_{BC} = \mathrm{acl}_\cup(BC)$, and in particular $B^* = B$ and $C^* = C$. Similarly, D_{AB} and D_{AC} satisfy q_i^{AB} and q_i^{AC} for all $i \in I$, so they satisfy $q_\cup^{AB}$ and $q_\cup^{AC}$, and hence $\mathrm{tp}_{L_\cup}(A^*B/M) = \mathrm{tp}_{L_\cup}(AB/M)$ and $\mathrm{tp}_{L_\cup}(A^*C/M) = \mathrm{tp}_{L_\cup}(A'C/M)$.

For all $i \in I$, $\mathrm{tp}_{L_i}(A^* D_{BC}/M) = \mathrm{tp}_{L_i}(A''_i D_i^{BC}/M) \subseteq q_i^{ABC}$, so $A^* \mathop{\smile\!\!\!\!\!\!\;|}^K_M \mathrm{acl}_\cup(BC)$ in $\mathfrak{M}_i$. Since $\mathrm{acl}_\cup(MA^*) = A^*$, we have $A^* \mathop{\smile\!\!\!\!\!\!\;|}_M BC$, as desired. □

We will now draw some immediate corollaries, using the sufficient conditions for the $T_\cap$-generic independence theorem derived in Section 2.7.

Corollary 4.7. *Assume $T_\cap$ is stable and for each i, T_i is stable or T_i is* NSOP$_1$ *and relatively disintegrated. Then $T^*_\cup$ is* NSOP$_1$ *and $\mathop{\smile\kern-0.9em\mid\kern0.5em} = \mathop{\smile\kern-0.9em\mid\kern0.5em}^K$.*

Corollary 4.8. *Assume $T_\cap$ is stable with disintegrated forking. If each T_i is* NSOP$_1$*, then $T^*_\cup$ is* NSOP$_1$ *and $\mathop{\smile\kern-0.9em\mid\kern0.5em} = \mathop{\smile\kern-0.9em\mid\kern0.5em}^K$.*

In many of the examples in Section 6, $T_\cap$ is interpretable in the theory of an infinite set. If T has disintegrated forking and finite U-rank, then any theory interpretable in T has disintegrated forking [Goo91, §5]. So any theory interpretable in the theory of an infinite set has disintegrated forking, and Corollary 4.9 follows from Corollary 4.8.

Corollary 4.9. *Assume $T_\cap$ is interpretable in the theory of an infinite set. If each T_i is* NSOP$_1$*, then $T^*_\cup$ is* NSOP$_1$ *and $\mathop{\smile\kern-0.9em\mid\kern0.5em} = \mathop{\smile\kern-0.9em\mid\kern0.5em}^K$.*

Finally, we observe that Corollary 4.7 implies that an unstable NIP theory cannot be decomposed as a fusion of stable theories.

Corollary 4.10. *Assume each T_i is stable. If $T^*_\cup$ is* NIP*, then $T^*_\cup$ is stable.*

Proof. By Corollary 4.7, $T^*_\cup$ is NSOP$_1$, and hence does not have the strict order property. Every NIP theory without the strict order property is stable. □

4.3. **Simplicity.** Having obtained sufficient conditions for the preservation of NSOP$_1$, we can now use Fact 2.21 to improve this to preservation of simplicity, under stronger hypotheses.

We do this in two ways. In Theorem 4.11, we assume that each T_i is simple and relatively disintegrated (see Section 2.7 for the definition). In Theorem 4.13, we relax the hypotheses on a single T_{i^*}, only requiring the $T_\cap$-generic independence theorem for $\mathop{\smile\kern-0.9em\mid\kern0.5em}^f$ in T_{i^*}; but in this case we have to assume that all the other T_i $(i \neq i^*)$ fail to add new algebraicity or forking to $T_\cap$. These two theorems generalize Propositions 6.3.13 and 6.3.15 in [Wag00], which concern the special case when $T_\cap$ is the theory of an infinite set. The second is inspired by a theorem of Tsuboi from [Tsu01], see Fact 4.12 below.

Theorem 4.11. *Assume $T_\cap$ is stable. If each T_i is simple and relatively disintegrated, then $T^*_\cup$ is simple.*

Proof. By Corollary 4.7, $T^*_\cup$ is NSOP$_1$, and $\mathop{\smile\kern-0.9em\mid\kern0.5em} = \mathop{\smile\kern-0.9em\mid\kern0.5em}^K$. By Fact 2.21, it suffices to show that $\mathop{\smile\kern-0.9em\mid\kern0.5em}^K$ satisfies base monotonicity over models.

So fix $M \prec N \prec \mathbb{M}_\cup$, $M \subseteq A$, and $N \subseteq B$. Assume that $A = \mathrm{acl}_\cup(A)$, $B = \mathrm{acl}_\cup(B)$, and $A \mathop{\smile\kern-0.9em\mid\kern0.5em}^K_M B$. It suffices to show that $A \mathop{\smile\kern-0.9em\mid\kern0.5em}^K_N B$. Since each T_i is simple, $\mathop{\smile\kern-0.9em\mid\kern0.5em}^f$ satisfies base monotonicity in $\mathbb{M}_i$. So we have:

$$\begin{aligned} A \mathop{\smile\kern-0.9em\mid\kern0.5em}^K_M B \text{ in } \mathbb{M}_\cup &\Rightarrow A \mathop{\smile\kern-0.9em\mid\kern0.5em}^f_M B \text{ in } \mathbb{M}_i \text{ for all } i \in I \\ &\Rightarrow A \mathop{\smile\kern-0.9em\mid\kern0.5em}^f_N B \text{ in } \mathbb{M}_i \text{ for all } i \in I \\ &\Rightarrow \mathrm{acl}_i(NA) \mathop{\smile\kern-0.9em\mid\kern0.5em}^f_N B \text{ in } \mathbb{M}_i \text{ for all } i \in I \end{aligned}$$

If we can improve this to

$$\mathrm{acl}_\cup(NA) \mathop{\smile\kern-0.9em\mid\kern0.5em}^f_N B \text{ in } \mathbb{M}_i \text{ for all } i \in I,$$

then it follows from the characterization of $\mathop{\smile\kern-0.9em\mid\kern0.5em}^K$ that $A \mathop{\smile\kern-0.9em\mid\kern0.5em}^K_N B$ in $\mathbb{M}_\cup$, as desired.

Since each acl_i is disintegrated relative to $\mathrm{acl}_\cap$, we have

$$\mathrm{acl}_i(NA) = \mathrm{acl}_\cap(\mathrm{acl}_i(N)\mathrm{acl}_i(A)) = \mathrm{acl}_\cap(NA),$$

since N and A are $\mathrm{acl}_\cup$-closed. Thus $\mathrm{acl}_\cap(NA)$ is acl_i-closed for all $i \in I$, and hence $\mathrm{acl}_\cup$-closed, since $\mathrm{acl}_\cup = \mathrm{ccl}$ by Proposition 3.9. So

$$\mathrm{acl}_\cup(NA) = \mathrm{acl}_\cap(NA) = \mathrm{acl}_i(NA),$$

and we have already proven what we wanted. □

The next result is inspired by the following theorem of Tsuboi.

Fact 4.12 ([Tsu01], [Wag00] Proposition 6.3.15)**.** *Suppose that $I = \{1, 2\}$, $L_1 \cap L_2 = \varnothing$, and T_1 and T_2 are simple and eliminate $\exists^\infty$. Then $T^*_\cup$ exists. If acl_1 is trivial and T_1 has U-rank one, then $T^*_\cup$ is simple.*

A theory T has trivial algebraic closure and U-rank one if and only if algebraic closure and forking in T agrees with algebraic closure and forking in the theory of an infinite set. So Theorem 4.13 generalizes Fact 4.12.

Theorem 4.13. *Assume $T_\cap$ is stable and each T_i is simple. Fix $i^* \in I$ and assume that:*

(1) $\mathop{\smile\!\!\!\!\!\!|}^f$ in T_{i^*} satisfies the $T_\cap$-generic independence theorem.
(2) For all $i \neq i^*$, $\mathrm{acl}_i = \mathrm{acl}_\cap$.
(3) For all $i \neq i^*$ and all sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ in $\mathfrak{M}_i$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ in $\mathfrak{M}_\cap$.

Then $T^*_\cup$ is simple.

Proof. For all $i \neq i^*$, T_i is relatively disintegrated, and hence $\mathop{\smile\!\!\!\!\!\!|}^f$ in T_i satisfies the $T_\cap$-generic independence theorem by Proposition 2.28. So by Theorem 4.6, $T^*_\cup$ is NSOP$_1$, and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^K$. By Fact 2.21, it suffices to show that $\mathop{\smile\!\!\!\!\!\!|}^K$ satisfies base monotonicity over models.

We begin just as in the proof of Theorem 4.11, fixing $M \prec N \prec \mathfrak{M}_\cup$, $M \subseteq A$, and $N \subseteq B$ such that $A = \mathrm{acl}_\cup(A)$, $B = \mathrm{acl}_\cup(B)$, and $A \mathop{\smile\!\!\!\!\!\!|}^K_M B$. Then we can show that for all $i \in I$,

$$\mathrm{acl}_i(NA) \mathop{\smile\!\!\!\!\!\!|}^f_N B \text{ in } \mathfrak{M}_i,$$

and we are done if we can improve this to

$$\mathrm{acl}_\cup(NA) \mathop{\smile\!\!\!\!\!\!|}^f_N B \text{ in } \mathfrak{M}_i.$$

Unlike the relatively disintegrated case, we may not have $\mathrm{acl}_i(NA) = \mathrm{acl}_\cup(NA)$ for all i. But by our hypothesis and Proposition 3.9, $\mathrm{acl}_\cup = \mathrm{acl}_{i^*}$, so

$$\mathrm{acl}_\cup(NA) \mathop{\smile\!\!\!\!\!\!|}^f_N B \text{ in } \mathfrak{M}_{i^*}.$$

Taking the reduct (Lemma 2.22), we also have

$$\mathrm{acl}_\cup(NA) \mathop{\smile\!\!\!\!\!\!|}^f_N B \text{ in } \mathfrak{M}_\cap.$$

And since for all $i \neq i^*$, forking in $\mathfrak{M}_i$ agrees with forking in $\mathfrak{M}_\cap$, we have

$$\mathrm{acl}_\cup(NA) \mathop{\smile\!\!\!\!\!\!|}^f_N B \text{ in } \mathfrak{M}_i \text{ for all } i \neq i^*.$$

This completes the proof. □

5. $\aleph_0$-CATEGORICITY

We do not assume that $T^*_\cup$ exists in this section.

5.1. **Existence and Preservation.** Applying the preservation results from Section 3, we show if each T_i is $\aleph_0$-categorical and certain extra hypotheses hold, then $T^*_\cup$ exists and is $\aleph_0$-categorical. This section closely follows work of Pillay and Tsuboi [PT97]. Our principle innovation is to note that their assumptions are satisfied by a number of examples, see Section 6.

Proposition 5.1. *Assume $T_\cap$ admits a stationary and extendable independence relation. Assume also that all languages have only finitely many sorts. Suppose that each T_i is $\aleph_0$-categorical and that there is some $i^* \in I$ such that $\mathrm{acl}_i(A) = \mathrm{acl}_\cap(A)$ for all $i \neq i^*$. Then $T^*_\cup$ exists.*

Proof. A $T_\cup$-model $\mathcal{M}_\cup$ has the **joint consistency property** if for every finite $B \subseteq M$ such that $B = \mathrm{acl}_{i^*}(B)$ and every family $(p_i(x))_{i\in J}$ such that J is a finite subset of I, if $p_i(x)$ is a complete L_i-type over B for all $i \in J$, and the p_i have a common restriction $p_\cap(x)$ to $L_\cap$, then $\bigcup_{i\in I} p_i(x)$ is realized in $\mathcal{M}_\cup$.

Note that the joint consistency property is elementary. Indeed, by $\aleph_0$-categoricity, there is an L_{i^*}-formula $\psi(y)$ expressing the property that the set B enumerated by a tuple b is acl_{i^*}-closed. Since B is finite, every complete L_i-type $p_i(x)$ over B is isolated by a single formula. And the property that the L_i-formula $\varphi_i(x, b)$ isolates a complete L_i-type over B whose restriction to $L_\cap$ is isolated by the $L_\cap$-formula $\varphi_\cap(x, b)$ is definable by a formula $\theta_{\varphi_i,\varphi_\cap}(b)$. So the class of $T_\cup$-models with the joint consistency property is axiomatized by $T_\cup$ together with sentences of the form

$$\forall y \left[\left(\psi(y) \wedge \bigwedge_{i\in J} \theta_{\varphi_i,\varphi_\cap}(y)\right) \to \exists x \bigwedge_{i\in J} \varphi_i(x, y)\right].$$

It remains to show that a structure $\mathcal{M}_\cup$ is interpolative if and only if it has the joint consistency property. So suppose $\mathcal{M}_\cup$ is interpolative, let B and $(p_i(x))_{i\in J}$ be as in the definition of the joint consistency property, and suppose for contradiction that $\bigcup_{i\in J} p_i(x)$ is not realized in $\mathcal{M}_\cup$. Note that since B is acl_{i^*}-closed, it is also acl_i-closed for all $i \neq i^*$, since $\mathrm{acl}_i(B) = \mathrm{acl}_\cap(B) \subseteq \mathrm{acl}_{i^*}(B) = B$.

Each $p_i(x)$ is isolated by a single L_i-formula $\varphi_i(x, b)$, and

$$\mathcal{M}_\cup \models \neg\exists x \bigwedge_{i\in J} \varphi_i(x, b).$$

It follows that the φ_i are separated by a family of $L_\cap$-formulas $(\psi^i(x, c_i))_{i\in J}$. Let $C = B \cup \{c_i \mid i \in J\}$. By full existence for $\mathop{\perp\!\!\!\!\perp}$ in T_i, since B is acl_i-closed, $p_i(x)$ has an extension to a type $q_i(x)$ over C such that for any realization a_i of $q_i(x)$, $a \mathop{\perp\!\!\!\!\perp}_B C$. By stationarity, the types $q_i(x)$ have a common restriction $q_\cap$ to $L_\cap$. Now for all $i \in J$, since $\varphi_i(x, b) \in p_i(x)$, $\psi^i(x, c_i) \in q_i(x)$, and hence $\psi^i(x, c_i) \in q_\cap(x)$. This is a contradiction, since $\{\psi^i(x, c_i) \mid i \in J\}$ is inconsistent.

Conversely, suppose $\mathcal{M}_\cup$ has the joint consistency property. Let $(\varphi_i(x, a_i))_{i\in J}$ be a family of formulas which are not separated. Let $B = \mathrm{acl}_{i^*}((a_i)_{i\in J})$. Since T_{i^*} is $\aleph_0$-categorical and J is finite, B is finite. For each $i \in J$, there is an $L_\cap$-formula $\psi^i(x, b)$ such that $\mathcal{M}_\cup \models \psi^i(a, b)$ if and only if $\mathrm{tp}_{L_\cap}(a/B)$ is consistent with $\varphi_i(x, a_i)$ (we may take $\psi^i(x, b)$ to be the disjunction of formulas isolating each of the finitely many such types). Since the formulas $\psi^i(x, b)$ do not separate the formulas $\varphi_i(x, a_i)$, there must be some element $a \in M^x$ satisfying $\bigwedge_{i\in J} \psi^i(x, b)$. Then $p_\cap(x) = \mathrm{tp}_{L_\cap}(a/B)$ is consistent with each $\varphi_i(x, a_i)$, so $p_\cap(x) \cup \{\varphi_i(x, a_i)\}$ can be extended to a complete L_i-type $p_i(x)$ over B. By the joint consistency property, there is some element in M^x realizing $\bigcup_{i \in J} p_i(x)$, and in particular satisfying $\bigwedge_{i \in J} \varphi_i(x, a_i)$. □

Theorem 5.2 follows by a type-counting argument as in Proposition 4.2.

Theorem 5.2. *Assume the hypotheses of Proposition 5.1, and let $T^*_\cup$ be the interpolative fusion. Assume additionally that I is finite. Then every completion of $T^*_\cup$ is $\aleph_0$-categorical.*

Proof. Let $\widehat{T}$ be a completion of $T^*_\cup$. It suffices to show that for any finite tuple of variables x, there are only finitely many $L_\cup$-types over the empty set in the variables x relative to $\widehat{T}$. Since $\mathrm{acl}_\cup = \mathrm{acl}_{i^*}$ is uniformly locally finite, there is an upper bound m on the size of $\mathrm{acl}_\cup(a)$ for any tuple $a \in M^x$ when $M \models \widehat{T}$.

By Proposition 3.9, $\mathrm{tp}_{L_\cup}(\mathrm{acl}_\cup(a))$ is determined by $\bigcup_{i \in I} \mathrm{tp}_{L_i}(\mathrm{acl}_\cup(a))$. So the number of possible $L_\cup$-types of a is bounded above by the product over all i of the number of L_i-types of m-tuples relative to T_i. This is finite, since I is finite and each T_i is $\aleph_0$-categorical. □

Corollary 5.3 follows from Corollary 3.14.

Corollary 5.3. *Assume the hypotheses of Proposition 5.1. Suppose that I is finite and $\varnothing$ is acl_i-closed for all $i \in I$. Then $T^*_\cup$ is complete and $\aleph_0$-categorical.*

The following result of Pillay and Tsuboi is a special case of Theorem 5.2.

Corollary 5.4 ([PT97, Corollary 5])**.** *Assume $T_\cap$ is stable with weak elimination of imaginaries. Let $I = \{1, 2\}$, suppose T_1 and T_2 are $\aleph_0$-categorical single-sorted theories, and suppose $\mathrm{acl}_1(A) = \mathrm{acl}_\cap(A)$ for all $A \subseteq \boldsymbol{M}_1$. Then $T_\cup$ admits an $\aleph_0$-categorical completion.*

5.2. **A counterexample.** We now construct an example where the interpolative fusion of two NSOP_1 theories with trivial acl over a unstable reduct $T_\cap$ exists but is not NSOP_1. This demonstrates that the stability of $T_\cap$ in Theorem 4.6 is necessary.

Let $\mathcal{K}$ and $\mathcal{K}'$ be Fraïssé classes in the languages $L \subseteq L'$, respectively. We say $\mathcal{K}'$ is a **Fraïssé expansion** of $\mathcal{K}$ if:

(1) $\mathcal{K} = \{\mathcal{A}|_L \mid \mathcal{A} \in \mathcal{K}'\}$
(2) For every $\mathcal{A} \in \mathcal{K}$, every one-point extension $\mathcal{A} \subseteq \mathcal{B}$ with $\mathcal{B} \in \mathcal{K}$, and every expansion $\mathcal{A}' \in \mathcal{K}'$ with $\mathcal{A}'|_L = \mathcal{A}$, there exists $\mathcal{B}' \in \mathcal{K}'$ such that $\mathcal{A}'$ is an L'-substructure of $\mathcal{B}'$ and $\mathcal{B}'|_L = \mathcal{B}$.

See, for example, [Kru19, Theorem 2.7] for a proof of Fact 5.5.

Fact 5.5. *Let $\mathcal{K}'$ and $\mathcal{K}$ be as above. Then $\mathcal{K}'$ is a Fraïssé expansion of $\mathcal{K}$ if and only if the Fraïssé limit of $\mathcal{K}'$ is an expansion of the Fraïssé limit of $\mathcal{K}$.*

Recall that a 3-hypergraph is a set V with a symmetric ternary relation R on V such that $R(a, b, c)$ implies that a, b, c are distinct. Finite 3-hypergraphs form a Fraïssé class, the Fraïssé limit of which is known as the random 3-hypergraph. It is well-known that the theory of random 3-hypergraph is simple but unstable. We leave the proof of the following Lemma to the reader.

Lemma 5.6. *Let $L = \{R\}$ with R a ternary relation symbol, and view the Fraïssé class $\mathcal{K}$ of finite 3-hypergraphs as a class of L-structures. Set $L_1 = \{R, E_1\}$, and*

$L_2 = \{R, E_2\}$, where E_1 and E_2 are binary relation symbols. Let $\mathcal{K}_1$ be the class of all finite L_1-structures such that R is a 3-hypergraph relation, E_1 is a graph relation (symmetric and anti-reflexive), and

$$\forall xyz\,((E_1(x,y) \wedge E_1(y,z) \wedge E_1(z,x)) \to R(x,y,z)).$$

Let $\mathcal{K}_2$ be the class of all finite L_2-structures such that R is a 3-hypergraph relation, E_2 is a graph relation, and

$$\forall xyz\,((E_2(x,y) \wedge E_2(y,z) \wedge E_2(z,x)) \to \neg R(x,y,z)).$$

Then each $\mathcal{K}_i$ is a Fraïssé class with disjoint amalgamation and is a Fraïssé expansion of $\mathcal{K}$.

Recall that the class of finite triangle-free graphs is a Fraïssé class, the Fraïssé limit of which is known as the Henson graph. The theory of the Henson graph is SOP_3 and NSOP_4; see for example [Con17]. In particular, it is SOP_1. We now construct the promised example.

Proposition 5.7. *There are simple $\aleph_0$-categorical theories $T_\cap$, T_1, T_2, each with trivial* acl*, such that $T^*_\cup$ exists, is complete and $\aleph_0$-categorical, and has* SOP_3*. In particular, the theory of the Henson graph is interpretable in $T^*_\cup$.*

Proof. Let $L_\cap = \{R\}$, $L_1 = \{R, E_1\}$, and $L_2 = \{R, E_2\}$, where R is a ternary relation symbol and E_1 and E_2 are binary relation symbols. Let $\mathcal{K}$, $\mathcal{K}_1$, and $\mathcal{K}_2$ be as in Lemma 5.6, let $T_\cap$ be the complete theory of the Fraïssé limit of $\mathcal{K}$ (the random 3-hypergraph), and let T_i be the complete theory of the Fraïssé limit of $\mathcal{K}_i$ for $i = 1$ and 2. It follows from Lemma 5.6 that T_1 and T_2 have a common set of $L_\cap$-consequences, namely $T_\cap$.

To show that $T^*_\cup$ exists and is complete and $\aleph_0$-categorical, we will apply Proposition 5.1 and Corollary 5.3. The theories $T_\cap$, T_1, and T_2 are Fraïssé limits in finite relational languages and hence are $\aleph_0$-categorical with quantifier elimination. Since $\mathcal{K}$, $\mathcal{K}_1$, and $\mathcal{K}_2$ have disjoint amalgamation, $\mathrm{acl}_1(A) = \mathrm{acl}_2(A) = \mathrm{acl}_\cap(A) = A$ for all sets A. So it is enough to construct a stationary and extendable independence relation on $T_\cap$.

If A, B, C are subsets of a $T_\cap$-model $(V; R)$ define $A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_C B$ if and only if ABC is a free amalgam of AC and BC over C, i.e. $AC \cap BC = C$, and if $a, b, c \in ABC$ and $R(a,b,c)$ then either $a, b, c \in AC$ or $a, b, c \in BC$. This is a stationary independence relation on $T_\cap$ (by quantifier elimination), and extendibility to T_1 and T_2 follows from the fact that $\mathcal{K}_1$ and $\mathcal{K}_2$ admit free amalgamation. So $T^*_\cup$ exists by Proposition 5.1 and is complete and $\aleph_0$-categorical by Corollary 5.3.

Let $(V; R, E_1, E_2)$ be a countable model of $T^*_\cup$. Define a binary relation E on V by $E(x,y)$ if and only if $E_1(x,y) \wedge E_2(x,y)$. It is immediate that $(V; E)$ is an $\aleph_0$-categorical graph. It is triangle-free because if a, b, c is an E-triangle, then it is both an E_1-triangle and an E_2-triangle, and hence $R(a,b,c)$ and $\neg R(a,b,c)$, contradiction. We show that $(V; E)$ is isomorphic to the Henson graph by verifying that it satisfies the usual extension axioms. Suppose that A, B are disjoint finite subsets of V and that there are no E-edges between elements of A. We need to find $c \in V$ such that c is connected to every element of A and not connected to any element of B.

We first show that we can add a new element satisfying these properties. Suppose that $c \notin V$, and let $W = V \cup \{c\}$. We extend E_1 and E_2 to W by setting $E_1(c,a)$ and $E_2(c,a)$ (and symmetrically, $E_1(a,c)$ and $E_2(a,c)$) for all $a \in A$. We do not add any other instances of E_1 or E_2. We extend R to W by setting $R(c,a,a')$ (and its symmetrical instances) when $a,a' \in A$ and c, a, a' is an E_1-triangle. We do not add any other instances of R.

For $i \in \{1,2,\cap\}$, let p_i be the complete quantifier-free L_i-type of c over V. The reduct of W to L_i satisfies the universal theory of T_i, so it embeds in a model of T_i. Thus p_i is consistent with T_i, and by quantifier elimination it axiomatizes a complete L_i-type over V. Now both p_1 and p_2 extend $p_\cap$, so by Proposition 2.5, $p_1 \cup p_2$ is realized by an element c' in an elementary extension of V. Since $T^*_\cup$ is $\aleph_0$-categorical and AB is finite, $\mathrm{tp}_{L_\cup}(c'/AB)$ is realized in V. By construction, we have $E(c',a)$ for all $a \in A$ and $\neg E(c',b)$ for all $b \in B$. This completes the proof. □

We end this section with a question:

Question 5.8. Must an interpolative fusion of simple theories be NSOP?

6. Examples

We describe a number of motivating examples which illustrate applications and sharpness of our theorems; see [KTW21] for other examples. The first two examples are among the original motivating examples of unstable simple theories: the random (hyper-)graph and ACFA. The remaining examples are mainly various kinds of generic constructions which preserve NSOP_1.

6.1. **Random hypergraphs and relations.** Fix $n \geq 2$. Let $L = \{E\}$ where E is an n-ary relation symbol. An n-hypergraph is an L-structure $(V;E)$ such that E is symmetric and $E(a_1,\dots,a_n)$ implies $a_i \neq a_j$ for all $i \neq j$. The **random n-hypergraph** is the Fraïssé limit of the class of finite n-hypergraphs. (Strictly speaking, the definition given here is for the generic n-hypergraphs in the sense of Fraïssé. However, the theory of finite n-hypergraphs satisfies a $0-1$ law and the associated almost sure theory is the theory of the random n-hypergraph.) Let T be the L-theory of infinite n-hypergraphs. It is well-known that the L-theory of the random n-hypergraph is the model companion T^* of T, so the random n-hypergraph is also generic in the sense of Robinson.

Proposition 6.1. *There are $T_\cap, T_1, T_2$ such that the theory of the random hypergraph is bi-interpretable $T^*_\cup$, T_1 and T_2 are both interpretable in the theory of equality, acl_2 agrees with $\mathrm{acl}_\cap$, and forking in T_2 agrees with forking in $T_\cap$.*

The proof of Proposition 6.1, in fact, shows that T is existentially bi-interpretable with $T_\cup$. As T_1 and T_2 are interpretable in the theory of equality existence of T^* follows from Corollary 2.7 and Fact 2.6.

By combining Theorem 4.13 and Proposition 6.1 we recover the well known fact that the random n-hypergraph is simple for any n. This shows that the conclusion of Theorem 4.13 is sharp as the random n-hypergraph is unstable. By combining Corollary 5.3 and Proposition 6.1 we recover $\aleph_0$-categoricity the random n-hypergraph.

Proof. Given a set V let Δ_V be the set of $(v_1,\dots,v_n) \in V^n$ such that $v_i \neq v_j$ for all $i \neq j$ and $\sim_V$ be the equivalence relation on Δ_V given by $(v_1,\dots,v_n) \sim_V (v'_1,\dots,v'_n)$ if and only if $\{v_1,\dots,v_n\} = \{v'_1,\dots,v'_n\}$.

Let $L_\cap$ be the two-sorted empty language. Let $T_\cap$ be such that $(V,S) \models T_\cap$ when V,S are both infinite. Let L_1 be the expansion of $L_\cap$ by an $(n+1)$-ary relation symbol π on $V^n \times S$. Let T_1 be the L_1-theory such that $(V, S; \pi) \models T_1$ if π is the graph of a surjection $\Delta_V \to S$ such that $\pi(a) = \pi(b)$ if and only if $a \sim b$. Let L_2 be the expansion of $L_\cap$ by a unary predicate P of sort S. Let T_2 be such that $(V, S; P) \models T_2$ if $(V, S) \models T_\cap$ and P is infinite and co-infinite. It is easy to see that T_1 and T_2 are interpretable in the theory of equality, acl_2 agrees with $\mathrm{acl}_\cap$, and forking in T_2 agrees with forking in $T_\cap$.

We observe T is existentially bi-interpretable with $T_\cup$ and then apply Fact 2.8. Suppose that $(V, S; \pi, P) \models T_\cup$. Let $E \subseteq V^m$ be the preimage of P under π. Then $(V; E)$ is an infinite n-hypergraph. Suppose that $(V; E)$ is an infinite n-hypergraph. Let π_V be the quotient map $\Delta_V \to \Delta_V / \sim_V$ and P_V be the image of E under π_V. Then $(V, \Delta_V / \sim_V; \pi_V, P_V)$ is a $T_\cup$-model. These observations may be formalized to construct an existential bi-interpretation between T and $T_\cup$. □

We now describe a second realization of the theory of random n-hypergraph as an interpolative fusion. Our second realization shows that the conditions for preservation of stability and NIP provided in Proposition 4.1 are sharp. Both T_1 and T_2 are bi-interpretable with the theory of an infinite set, acl_1 and acl_2 agree both agree with $\mathrm{acl}_\cap$. However, $\mathrm{acl}_\cap$ does not agree with $\mathrm{dcl}_\cap$.

Proposition 6.2. *There are $T_\cap$, T_1, and T_2 such that T_1 and T_2 are both bi-interpretable with the theory of equality,* acl_1 *and* acl_2 *both agree with* $\mathrm{acl}_\cap$*, and the theory of the random n-hypergraph is bi-interpretable with $T^*_\cup$.*

Proof. We let Δ_V and $\sim_V$ be as in the proof of Proposition 6.1. Let $L_\cap$ be the two sorted language containing a single ternay relation D and let $T_\cap$ be the $L_\cap$-theory such that $(V, S; D) \models T_\cap$ if D is a subset of $\Delta_V \times D$ satisfying

(1) if $a, a' \in \Delta_V, s \in S$ and $a \sim_V a'$ then $D(a, s)$ implies $D(a', s)$,
(2) for every $a \in \Delta_V$ there are exactly two $s \in S$ such that $D(a, s)$.

For each $i \in \{1, 2\}$ let L_i be the expansion of $L_\cap$ by a binary relation g_i and T_i be the L_i-theory such that $(V, S; D, g_i) \models T_i$ if $(V, S; D) \models T_\cap$ and g_i is the graph of a function $g_i : \Delta_V \to S$ such that $D(a, g_i(a))$ for all Δ_V and if $a, a' \in \Delta_V$ and $a \sim_V a'$ then $g_1(a) = g_2(a')$.

Let V be a set. We let Q be the quotient of Δ_V by $\sim_V$ and identify the $\sim_V$-class of $(v_1, \ldots, v_n)$ with $\{v_1, \ldots, v_n\}$. Let $c_1, c_2 \in V$ be distinct, $S = (\{c_1\} \times Q) \cup (\{c_2\} \times Q)$, π be the projection $S \to Q$, and D be the set of $(v_1, \ldots, v_n, a)$ in $\Delta_V \times S$ such that $\pi(a) = \{v_1, \ldots, v_n\}$. Let $g_1 : \Delta_V \to S$ be given by declaring $g_1(v_1, \ldots, v_n) = (c_1, \{v_1, \ldots, v_n\})$. Then $(V, S; D, g_1) \models T_1$. It follows that T_1 and T_2 are both bi-interpretable with the theory of equality and that acl_1 and acl_2 both agree with $\mathrm{acl}_\cap$.

By Fact 2.8 it suffices to show that $T_\cup$ is existentially bi-interpretable with T. Suppose $(V; E)$ is an n-hypergraph. Fix distinct $c_1, c_2 \in V$ and let S and D be defined as in the proceeding paragraph. Given $(v_1, \ldots, v_n) \in \Delta_V$ we declare

$$\begin{aligned} g_1(v_1, \ldots, v_n) &= (c_1, \{v_1, \ldots, v_n\}), \\ g_2(v_1, \ldots, v_n) &= (c_1, \{v_1, \ldots, v_n\}) \quad \text{if} \quad E(v_1, \ldots, v_n), \text{ and} \\ g_2(v_1, \ldots, v_n) &= (c_2, \{v_1, \ldots, v_n\}) \quad \text{if} \quad \neg E(v_1, \ldots, v_n). \end{aligned}$$

Suppose $(V, S; D, g_1, g_2)$ is a model of $T_\cup$. Given $v_1, \ldots, v_n \in V$ we declare $E(v_1, \ldots, v_n)$ if and only if $(v_1, \ldots, v_n) \in \Delta_V$ and $g_1(v_1, \ldots, v_n) = g_2(v_1, \ldots, v_n)$. Then $(V; E)$ is an n-hypergraph. These observations are easily formalized to obtain an existential bi-interpretation between $T_\cup$ and T. □

The asymmetric version of the argument in Proposition 6.1 gives a construction of the theory of a generic n-ary relation. Fix $n \geq 2$ and suppose that L contains a single n-ary relation symbol R. The collection of finite L-structures form a Fraïssé class, the Fraïssé limit of this class is the **generic n-ary relation**. The theory T^*_R of the generic n-ary relation is the model companion of the empty L-theory. When $n = 2$, T^*_R is the theory of the generic directed graph. Simplicity of the generic n-ary relation follows from Theorem 4.13 and Proposition 6.3. We recover $\aleph_0$-categoricity of the generic n-ary relation by combining Corollary 5.3 and Proposition 6.3.

Proposition 6.3. *There are $T_\cap$, T_1, and T_2 such that T^*_R is bi-interpretable $T^*_\cup$, T_1 and T_2 are both interpretable in the theory of equality,* acl_2 *agrees with* $\mathrm{acl}_\cap$, *and forking in T_2 agrees with forking in $T_\cap$.*

Proof. Let $L_\cap$ be the two-sorted empty language. Let $T_\cap$ be such that $(V,S) \vDash T_\cap$ when V,S are both infinite. Let L_1 be the expansion of $T_\cap$ by an n-ary function $\pi : V^n \to S$. Let T_1 be such that $(V,S;\pi) \vDash T_1$ when π is a bijection $V^n \to S$. Let L_2 be the expansion of $L_\cap$ by a unary relation of sort S. Let T_2 be such that $(V,S;P) \vDash T_2$ if P is infinite and co-infinite. Observe that T_1 and T_2 are both interpretable in the theory of equality, acl_2 agrees with $\mathrm{acl}_\cap$, and forking in T_2 agrees with forking in $T_\cap$.

By Fact 2.8 it suffices to show that $T_\varnothing$ is existentially bi-interpretable wuth $T_\cup$. Suppose that $(V,S;\pi,P) \vDash T_\cup$. Let E be the pre-image of P under π. Then $(V;E)$ is an n-ary relation. Suppose $(V;E)$ is an n-ary relation. Let ι be the identity map $V^n \to V^n$. Then $(V,V^n;\iota,E) \vDash T_\cup$. These observations may be formalized to construct an existential bi-interpretation between T and $T_\cup$. $\square$

We can also realize the theory of a generic relation as a relatively disintegrated fusion. Proposition 6.4 follows by a straightforward asymmetric version of the proof of Proposition 6.2. We leave the details to the reader.

Proposition 6.4. *There are $T_\cap, T_1, T_2$ such that T^*_R is bi-interpretable with $T^*_\cup$, T_1 and T_2 are both bi-interpretable with the theory of equality, and* acl_1 *and* acl_2 *both agree with* $\mathrm{acl}_\cap$.

6.2. **Generic automorphisms.** Arguably the most important unstable simple theory is the theory ACFA of existentially closed difference fields. It turns out that ACFA is bi-interpretable with a fusion of two theories, each of which is bi-interpretable with ACF. This is a special case of a more general fact about theories of generic automorphisms. We work with the theory of a structure equipped with a generic family of (non-commuting) automorphisms. It is no more difficult to handle this via our approach than the expansion by a single generic automorphism.

Let J be an index set. Suppose that T is a model complete L-theory with infinite models. Let $L_{J-\mathrm{Aut}}$ be the extension of L by a family $(\sigma_j)_{j\in J}$ of unary function symbols and $T_{J-\mathrm{Aut}}$ be an $L_{J-\mathrm{Aut}}$-theory such that $(\mathcal{M};(\sigma_j)_{j\in J}) \vDash T_{J-\mathrm{Aut}}$ if and only if $\mathcal{M} \vDash T$ and each σ_j is an L-automorphism of $\mathcal{M}$. We let $T^*_{J-\mathrm{Aut}}$ be the model companion of $T_{J-\mathrm{Aut}}$ if it exists. We drop the J when $|J| = 1$. It is conjectured that T^*_{Aut} does not exist when T is unstable. The case of Fact 6.5 when $|J| = 1$ is due to Chatzidakas and Pillay.

Fact 6.5. *If T is stable and $T^*_{J-\mathrm{Aut}}$ exists then $T^*_{J-\mathrm{Aut}}$ is simple.*

This is sharp in the sense that $T^*_{J-\mathrm{Aut}}$ is almost never stable. Suppose that $|J| = 1$, L is the empty language, and T is the theory of an infinite set. Suppose $(M;\sigma) \models T^*_{\mathrm{Aut}}$. It is easy to see that $\phi(x,y) := [\sigma(x) = y]$ is unstable. Fact 6.5 follows from Theorem 4.11 and Proposition 6.6.

Proposition 6.6. *Let $I = J \cup \{0\}$. Suppose that $T^*_{J-\mathrm{Aut}}$ exists. Then there are $T_\cap$ and $(T_i)_{i\in I}$ such that $T^*_{J-\mathrm{Aut}}$ is bi-interpretable with $T^*_\cup$, $T_\cap$ is mutually interpretable with T, each T_i is bi-interpretable with T and disintegrated relative to $T_\cap$.*

As usual, we only sketch the proof.

Proof. We let $T_\cap$ be the theory of two disjoint T-models with no additional structure, i.e. $(\mathcal{M},\mathcal{N}) \models T_\cap$ when $\mathcal{M},\mathcal{N} \models T$. So $T_\cap$ is mutually existentially interpretable, but not bi-intepretable, with T. For each $i \in I$ let T_i be the theory of $(\mathcal{M},\mathcal{N};\tau_i)$ where $(\mathcal{M},\mathcal{N}) \models T_\cap$ and τ_i is an L-isomorphism $\mathcal{M} \to \mathcal{N}$. If $(\mathcal{M},\mathcal{N};(\sigma_i)_{i\in I})$ is a $T_\cup$-model then $(\mathcal{M};(\tau_j^{-1} \circ \tau_0)_{j\in J}) \models T_{J-\mathrm{Aut}}$ and if $(\mathcal{M};(\sigma_i)_{i\in J})$ is a $T_{J-\mathrm{Aut}}$-model then $(\mathcal{M},\mathcal{M};\mathrm{id},(\sigma_i)_{i\in J}) \models T_\cup$. It easily follows that $T_\cup$ is existentially bi-interpretable with $T_{J-\mathrm{Aut}}$. The first claim now follows by Fact 2.8.

Fix $i \in I$. It remains to show that T_i is disintegrated relative to $T_\cap$. Suppose that $(\mathcal{M},\mathcal{N};\sigma_i) \models T_i$ and $A \subseteq M, B \subseteq N$. Observe that $\mathrm{acl}_\cap(A,B) = (\mathrm{acl}_T(A),\mathrm{acl}_T(B))$ and that $\mathrm{acl}_i(A,B) = (\mathrm{acl}_T(A\tau_i^{-1}(B)),\mathrm{acl}_T(\tau_i(A)B))$. So for all $A,A' \subseteq M$ and $B,B' \subseteq N$ we have $\mathrm{acl}_i(AA',BB') = \mathrm{acl}_\cap(\mathrm{acl}_i(A,B)\mathrm{acl}_i(A',B'))$. ☐

6.3. **Generic selections.** We describe the generic selection of a definable equivalence relation, and then discuss several specific cases: generic tournaments, functions, and Skolemizations. Let E be an equivalence relation on a set X. A **selection** of E is a set that contains exactly one element from each E-class and a **quotient function** for E is a (not necessarily surjective) function $f : X \to Y$ such that for all $a,a' \in X$ we have $f(a) = f(a')$ if and only if $E(a,a')$.

In this section T is a one-sorted, complete, and model complete L-theory, and $\phi(x)$ and $\psi(x,y)$ are L-formulas such that $\psi(x,y)$ defines an equivalence relation $E_{\mathcal{M},\psi}$ on $\phi(M^k)$ for every $\mathcal{M} \models T$, here $|x| = k$. (Everything easily generalizes to the case when T is multi-sorted and not model complete.) Let L_{Sel} be the expansion of L by an n-ary predicate P and T_{Sel} be the L_{Sel}-theory such that $(\mathcal{M};P) \models T_{\mathrm{Sel}}$ if and only if $\mathcal{M} \models T$ and P is a selection of $E_{\mathcal{M},\psi}$. We say that T eliminates $\psi(x,y)$ if there is a formula $\varphi(x,z), |z| = l$ such that for any $\mathcal{M} \models T$, $\varphi(M^k,M^l)$ is the graph of a quotient function $\phi(M^k) \to M^l$ for $E_{\mathcal{M},\psi}$.

Theorem 6.7. *Suppose one of the following holds:*

1. *T^{eq} eliminates $\exists^\infty$, or*
2. *T eliminates $\exists^\infty$ and T eliminates $\psi(x,y)$.*

Then T_{Sel} has a model companion T^*_{Sel} and there are $T_\cap$, T_1, T_2 such that

1. *T^*_{Sel} is bi-interpretable with $T^*_\cup$,*
2. *T_1 is bi-interpretable with T,*
3. *T_2 is interpretable in the theory of equality, and*
4. *if each $E_{\mathcal{M},\psi}$-class is finite then acl_2 agrees with $\mathrm{acl}_\cap$ and forking in T_2 agrees with forking in $T_\cap$.*

Proof. For each $n \geq 1$ let e_n be the number of $E_{\mathcal{M},\psi}$-classes with exactly n-elements if there are finitely many such $E_{\mathcal{M},\psi}$-classes and declare $e_n = \infty$ otherwise. Let e_∞

be the number of infinite $E_{\mathcal{M},\psi}$-classes if there are finitely many infinite ψ-classes and declare $e_\infty = \infty$ otherwise.

Let $L_\cap$ be the three sorted language with a binary relation E on the second sort and a binary relation ρ. Let $T_\cap$ be the $L_\cap$-theory such that $(M, N, Q; E, \rho) \vDash T_\cap$ if M, N are infinite, E is an equivalence relation on N such that for all $n \geq 1$

(1) if $e_n < \infty$ then there are e_n E-classes with exactly n elements,
(2) if $e_n = \infty$ then there are infinitely many E-classes with n elements,
(3) if $e_\infty < \infty$ then there are e_∞ infinite E-classes, and
(4) if $e_\infty = \infty$ then there are infinitely many infinite E-classes,

and ρ is the graph of a quotient function $N \to Q$ for E. Note that in either case (1) or (2) we have $e_n = 0$ for sufficiently large n. It easily follows that $T_\cap$ is interpretable in the theory of equality. The quotient function ρ is included to ensure weak elimination of imaginaries.

Let L_1 be the expansion of $L_\cap$ by L, where all relations and functions are on the first sort, together with an $(k+1)$-ary relation symbol π. Let T_1 be the L_1-theory such that $(\mathcal{M}, N, Q; E, \rho, \pi) \vDash T_1$ when $(M, N; E, \rho) \vDash T_\cap$, $\mathcal{M} \vDash T$, and π is the graph of an isomorphism $(\phi(M^k); E_{\mathcal{M},\psi}) \to (N; E)$. It is easy to see that T_1 is bi-interpretable with T.

Let L_2 be the expansion of $L_\cap$ by a unary predicate P of the second sort. Let T_2 be the L_2-theory such that $(M, N, Q; E, \rho, P) \vDash T_2$ if $(M, N, Q; E, \rho) \vDash T_\cap$ and P contains exactly one element from each E-class. Note that T_2 is interpretable in the theory of equality. It is easy to see that if every $E_{\mathcal{M},\psi}$-class is finite then forking and algebraic closure in T_2 agree with forking and algebraic closure in $T_\cap$.

Suppose $(\mathcal{M}, N, Q; E, \rho, \pi, P) \vDash T_\cup$. Let S be the preimage of P under π. Then S is a selection for $E_{\mathcal{M},\psi}$, hence $(\mathcal{M}; S) \vDash T_{\mathrm{Sel}}$. Suppose that $(\mathcal{M}; S) \vDash T_{\mathrm{Sel}}$. Let Q be $\phi(M^k)/E_{\mathcal{M},\psi}$, ρ be the quotient $\phi(M^k) \to Q$, N be a copy of $\phi(M^k)$, π be a bijection $\phi(M^k) \to N$, E be the pushforward of $E_{\mathcal{M},\psi}$ by π, and P be the image of S under π. Then $(\mathcal{M}, N, Q; E, \rho, \pi, P) \vDash T_\cup$. These observations may be formalized to construct an existential bi-interpretation between T_{Sel} and $T_\cup$.

As $T_\cap$ is interpretable in the theory of equality we see that $T_\cap$ is $\aleph_0$-stable and $\aleph_0$-categorical. It is also easy to see that $T_\cap$ weakly eliminates imaginaries. Note that in either case (1) or (2) T_1 eliminates $\exists^\infty$. Furthemore T_2^{eq} eliminates $\exists^\infty$ as T_2 is interpretable in the theory of equality. So $T_\cup^*$ exists by Fact 2.6. Finally, an application of Fact 2.8 shows that T_{Sel}^* exists and is bi-interpretable with $T_\cup^*$. □

The reader may now apply the results of [KTW21] to obtain an explicit ∀∃-axiomization of T_{Sel}^*. Corollary 6.8 follows by Corollary 4.9 as $T_\cap$ is interpetable in the theory of equality.

Corollary 6.8. *If T is* NSOP$_1$ *and T^{eq} eliminates $\exists^\infty$, then T_{Sel}^* is* NSOP$_1$.

We now discuss some specific cases.

6.3.1. *The generic tournament.* A tournament is a set V together with a binary relation E such that $(a, a) \notin E$ for all $a \in V$ and for all distinct $a, b \in V$ we either have $(a, b) \in E$ or $(b, a) \in E$, but not both. Finite tournaments form a Fraïssé class, we refer to the Fraïssé limit of this class is the **generic tournament**. Let T_{Tour} be the theory of tournaments. The theory of the generic tournament is the model companion T_{Tour}^* of T_{Tour}.

A tournament is a selection. Fix a set V, let Δ be the set of $(a,b) \in V^2$ such that $a \neq b$ and $\sim$ be the equivalence relation on Δ where $(u,v) \sim (u',v')$ if and only if $\{u,v\} = \{u',v'\}$. Note that a tournament E on V is a selection of $\sim$. So Proposition 6.9 is a special case of Theorem 6.7.

Proposition 6.9. *There are $T_\cap, T_1, T_2$ such that the theory of the generic tournament is bi-interpretable $T^*_\cup$, T_1 and T_2 are both interpretable in the theory of equality,* acl_2 *agrees with* $\mathrm{acl}_\cap$*, and forking in T_2 agrees with forking in $T_\cap$.*

Combining Theorem 4.13 and Proposition 6.9 we recover simplicity of the generic tournament. This is sharp as the generic tournament is unstable. We recover $\aleph_0$-categoricity of the generic tournament from Corollary 5.3 and Proposition 6.9.

6.3.2. *The generic n-ary function.* Fix $n \geq 1$, let L_f be the language containing a single n-ary function symbol f, and T_f be the empty L_f-theory. The model companion T^*_f of T_f is the theory of a generic n-ary function.

An n-ary function is a selection. Let M be a set. Let $\sim$ be the equivalence relation on $M^n \times M$ where $(a,b) \sim (a',b')$ if and only if $a = a'$. A selection for $\sim$ is the graph of a function $M^n \to M$. So Proposition 6.10 is a special case of Theorem 6.7.

Proposition 6.10. *Then there are $T_\cap$, T_1, T_2 such that T^*_f is bi-interpretable with $T^*_\cup$ and T_1 and T_2 are interpretable in the theory of equality.*

Apply Theorem 4.6 we recover the fact that the theory of a generic n-ary function is NSOP_1. This is sharp as the theory of a generic n-ary function is TP_2 [KR18, Proposition 3.14]. We describe another presentation of the generic n-ary function as a fusion of stable theories in Section 6.4.

6.3.3. *Generic Skolem functions.* Suppose that T is a complete, model complete, L-theory with infinite models. Let $\theta(x,y)$ be an L-formula and $\Gamma(x) = \exists y \theta(x,y)$. Given $\mathcal{M} \models T$ a Skolem function for $\theta(x,y)$ is a function $f : \Gamma(M^{|x|}) \to M^{|y|}$ such that $\mathcal{M} \models \theta(a, f(a))$ for all $a \in M^{|x|}$. Let $L_{\theta\text{-Skol}}$ be the expansion of L by an $(|x|+|y|)$-ary predicate f and $T_{\theta\text{-Skol}}$ be the $L_{\theta\text{-Skol}}$-theory such that $(\mathcal{M}; f) \models T_{\theta\text{-Skol}}$ when f is the graph of a Skolem function for $\theta(x,y)$. The first claim of Fact 6.11 is due to Winkler [Win75]. The second claim is due to Kruckman and Ramsey [KR18].

Fact 6.11. *Suppose T eliminates $\exists^\infty$. Then $T_{\theta\text{-Skol}}$ has a model companion $T^*_{\theta\text{-Skol}}$. If T is* NSOP_1 *then $T^*_{\theta\text{-Skol}}$ is* NSOP_1.

Let $|x'| = |x|$, $|y'| = |y|$ and $\psi(x,y,x',y')$ be $\theta(x,y) \wedge \theta(x',y') \wedge (x = x')$. So if $\mathcal{M} \models T$ then ψ defines an equivalence relation on $\theta(M^{|x|}, M^{|y|})$ and any selection for this equivalence relation is the graph of a Skolem function for $\theta(x,y)$. So Fact 6.11 follows from Theorem 6.7.

Fact 6.12 is due to Nübling [Nüb04].

Fact 6.12. *Suppose that $\theta(x,y)$ is bounded in y. If T is simple then $T^*_{\theta\text{-Skol}}$ is simple. If T is $\aleph_0$-categorical then any completion of $T^*_{\theta\text{-Skol}}$ is $\aleph_0$-categorical.*

The first claim of Fact 6.12 follows from Theorem 4.13 and Theorem 6.7. The second claim folows from Theorem 5.2 and Theorem 6.7. The second claim of Fact 6.12 is reasonably sharp, see [Nüb04, 2.3.1].

One can prove Fact 6.13 by iterating Fact 6.11. The first claim is also due to Winkler [Win75] and the second claim is due to Kruckman and Ramsey [KR18].

Fact 6.13. *Suppose that T eliminates $\exists^\infty$. Then there is a language $L \subseteq L_{\mathrm{Skol}}$ and a model complete L_{Skol}-theorem $T \subseteq T^*_{\mathrm{Skol}}$ which has definable Skolem functions. If T is* NSOP_1 *then T^*_{Skol} is* NSOP_1.

One constructs T^*_{Skol} as the union of a countable chain of theories, each of which defines Skolem functions for the previous theory and then applies the fact that a union of a chain of NSOP_1 theories is NSOP_1.

6.4. **The model companion of the empty theory.** Let L be a first order language and $\varnothing_L$ be the empty L-theory. Winkler [Win75] showed that $\varnothing_L$ has a model companion $\varnothing^*_L$. Fact 6.14 is due to Kruckman and Ramsey [KR18] and independently Jeřábek [Jeř19].

Fact 6.14. *$\varnothing^*_L$ is* NSOP_1.

Fact 6.14 is sharp as $\varnothing^*_L$ is TP_2 and hence not simple when L contains a function symbol of airity at least two [KR18, Proposition 3.14]. Proposition 6.15 allows us to realize $\varnothing^*_L$ as a fusion of stable theories. In particular, this allows us to recover Fact 6.4 using Theorem 4.6 and Proposition 2.26.

Proposition 6.15. *There is $T_\cap$, and $(T_i)_{i\in I}$ such that each T_i is interpretable in the theory of equality and $\varnothing^*_L$ is bi-interpretable with $T^*_\cup$.*

So existence of $\varnothing^*_L$ follows by Corollary 2.7. Fact 6.14 follows from Proposition 6.15 and Corollary 4.9.

Proof. Let $\mathcal{C}$ be the set of constant symbols in L, $\mathcal{R}$ be the set of relation symbols, and $\mathcal{F}$ be the set of function symbols. Let T be the L-theory such that $\mathcal{M} \models T$ if

(1) every n-ary $R \in \mathcal{R}$ defines an infinite and co-infinite subset of M^n, and
(2) whenever $f \in \mathcal{F}$ is n-ary then $f : M^n \to M$ is surjective and $f^{-1}(a)$ is infinite for all $a \in M$.

It is easy to see that $\varnothing^*_L$ is a model companion of T. We construct $(T_i)_{i\in I}$, show that $T_\cup$ is existentially bi-interpretable with T, and apply Fact 2.8.

Let $I = L \cup \{1\}$. Let $L_\cap$ be the countably-sorted empty language and $T_\cap$ be the $L_\cap$-theory such that $(M, (N_k)_{k\geq 1}) \models T_\cap$ if M and each N_k is infinite.

For each $c \in \mathcal{C}$ let L_c be the expansion of $L_\cap$ by a constant symbol c and T_c be the L_c-theory such that $(M, (N_k)_{k\geq 1}; c) \models T_c$ when c defines an element of M. So each T_c is interpretable in the theory of equality.

Let L_1 be the expansion of $L_\cap$ by function symbols $(\pi_k)_{k\geq 1}$ such that each π_k is k-ary. Let T_1 be the L_1-theory such that $(M, (N_k)_{k\geq 1}; (\pi_k)_{k\geq 1}) \models T_1$ if each π_k is a bijection $M^k \to N_k$. It is easy to see that T_1 is interpretable in the theory of equality.

For each n-ary $R \in \mathcal{R}$ let L_R be the expansion of $L_\cap$ by a unary predicate P_R and let T_R be the L_R-theory such that $(M, (N_k)_{k\geq 1}; P_R) \models T_R$ if and only if P_R defines an infinite and co-infinite subset of N_n. For each n-ary $f \in \mathcal{F}$ let L_f be the expansion of $T_\cap$ by a unary function symbol g_f and T_f be the L_f-theory such that $(M, (N_k)_{k\geq 1}; g_f) \models T_f$ when g_f is a surjection $N_k \to M$ and $g_f^{-1}(a)$ is infinite for every $a \in M$. It is easy to see that each T_R and T_f is interpretable in theory of equality.

Suppose that $\mathcal{M} \models T$. We describe a $T_\cup$-model which is existentially interpretable in $\mathcal{M}$. For each $n \geq 1$ let N_n be M^n and π_n be the identity $M^n \to N_n$. Interpret constant symbols in the obvious way. For each n-ary $R \in \mathcal{R}$ let P_R be $\{\pi_n(a) : \mathcal{M} \models$

$R(a)\}$. For each n-ary $f \in \mathcal{F}$ let $g_f : N_n \to M$ be $f \circ \pi_n^{-1}$. Then $(M, (N_k)_{k\geq 1}; \dots)$ is a model of $T_\cup$.

Now suppose that $\mathcal{M}_\cup = (M, (N_k)_{k\geq 1}; \dots) \models T_\cup$. We describe a T-model with domain M which is existentially interpretable in $\mathcal{M}_\cup$. Again interpret constant symbols in the obvious way. Let each n-ary $R \in \mathcal{R}$ define $\{a \in M^n : \mathcal{M}_\cup \models P_R(\pi_n(a))\}$. Let each n-ary $f \in \mathcal{F}$ be $g_f \circ \pi_n$. This is easy seen to produce a T-model. These observations are easily formalized to construct an existential bi-interpretation between T and $T_\cup$. □

Fact 6.16. *Suppose that L does not contain any function symbol of airity at least two. Then $\varnothing^*_L$ is simple.*

We sketch a proof of Fact 6.16. We will use some elementary facts about the theory of a generic unary function which we leave to the reader. Suppose that L does not contain any function symbol of airity at least two. Let $\mathcal{C}, \mathcal{R}$, and $\mathcal{F}$ be as above. Let $I = L$. Let $L_\cap$ be the one-sorted empty language and $T_\cap$ be the theory of an infinite set. For each $c \in \mathcal{C}$ let T_c be the theory of an infinite set with a distinguished element. For each n-ary $R \in \mathcal{R}$ let T_R be the theory of a generic n-ary predicate and for each $f \in \mathcal{F}$ let T_f be the theory of a generic unary function. So $\varnothing^*_L$ is $T^*_\cup$. Each T_c and T_f is stable and each T_R is simple. We observe that each T_i is disintegrated, an application of Theorem 4.11 shows that $\varnothing^*_L$ is simple. Each T_c and T_R has trivial algebraic closure. In each T_f the algebraic closure of a set is the substructure generated by that set, hence T_f is unary.

6.4.1. *The generic expansion.* Suppose that T is a one-sorted model complete L'-theory and L is a language containing L'. We consider T to be an L-theory. If T eliminates $\exists^\infty$ then T, considered as an L-theory, has a model companion T^*_L [Win75]. If L' is the empty language and T is the theory of equality then T^*_L is $\varnothing^*_L$. If $L \setminus L'$ contains a single unary predicate then T^*_L is the expansion of T by a generic unary predicate. Fact 6.17 is due to Kruckman and Ramsey [KR18].

Fact 6.17. *Suppose that T eliminates $\exists^\infty$. If T is* NSOP$_1$ *then T^*_L is* NSOP$_1$.

Suppose that T eliminates $\exists^\infty$. Let $I = \{1, 2\}$, $L_\cap$ be the one-sorted empty language, $T_\cap$ be the theory of an infinite set, L_1 be L' , T_1 be T, L_2 be $L \setminus L'$, and T_2 be $\varnothing^*_{L\setminus L'}$. Then T^*_L agrees with $T^*_\cup$. So Fact 6.17 follows from Corollary 4.9.

One can also adapt the proof of Proposition 6.15 to show that if T is stable then T^*_L is bi-interpretable with the fusion of a family of stable theories.

6.5. **The generic variation.** Suppose that L is a relational language and T is a complete, one-sorted L-theory with infinite models. The generic variation T^*_{Var} of T was defined by Baudisch [Bau02]. This theory exists when T eliminates $\exists^\infty$. This example is essentially a natural generalization of one of the most basic examples of a theory that is NSOP$_1$ and not simple, see 6.5.1.

Let L_{Var} be the language containing an $(n+1)$-ary relation symbol R_{Var} for each n-ary relation symbol R in L. Suppose $\mathcal{N}$ is an L_{Var}-structure. For each $a \in N$ we put an L-structure $\mathcal{N}[a]$ on N by declaring $\mathcal{N}[a] \models R(b)$ if and only if $\mathcal{N} \models R_{\mathrm{Var}}(a, b)$ for any relation symbol R from L. It is easy to describe an L_{Var}-theory T_{Var} such that $\mathcal{N} \models T_{\mathrm{Var}}$ if and only if $\mathcal{N}[a] \models T$ for every $a \in N$. Fact 6.18 is proven in [Bau02].

Fact 6.18. *If T eliminates $\exists^\infty$ then T_{Var} has a model companion T^*_{Var}.*

If T is the theory of the generic n-ary relation then T^*_{Var} is the theory of the generic $(n+1)$-ary relation, so one may produce the theory of a generic n-ary relation from the (stable) theory of an infinite and co-infinite unary relation by iterating the construction $n-1$ times.

Proposition 6.19. *Suppose that T eliminates $\exists^\infty$. Then there are $T_\cap$, T_1, T_2 such that T^*_{Var} is b-interpretable with $T^*_\cup$, T_1 is bi-interpretable with the theory of equality, and T_2 is mutually interpretable with T.*

Proof. Let $L_\cap$ be the two sorted language with a single unary function symbol π_1. Let $T_\cap$ be the $L_\cap$ theory such that $(M,N;\pi_1) \models T_\cap$ if M,N are both infinite and $\pi_1 : N \to M$ is a surjection such that $\pi_1^{-1}(b)$ is infinite for all $b \in N$. Let L_1 be the expansion of $L_\cap$ by a unary function π_2. Let T_1 be the L_1-theory such that $(M,N;\pi_1,\pi_2) \models T_1$ if the map $N \to M^2$ given by $a \mapsto (\pi_1(a),\pi_2(a))$ is a bijection. It is easy to see that T_2 is bi-interpretable with the theory of equality.

Let L_2 be the expansion of $L_\cap$ by an $(n+1)$-ary relation R_2 for each n-ary relation $R \in L$ such that the first variable of R_2 is of sort M and the last n variables are of sort N. Suppose $\mathcal{M} = (M,N;\pi_1,(R_2)_{R\in L_\cap})$ is an L_2-structure. We let M_a be the set of $b \in N$ such that $\pi_1(b) = a$ for all $a \in M$. For each $a \in M$ we equip M_a with an L-structure $\mathcal{M}(a)$ by letting the interpertation of each n-ary $R \in L$ be $\{b \in (M_a)^n : \mathcal{N} \models R_2(a,b)\}$. Then T_2 is the L_2-theory such that $\mathcal{M} \models T_2$ when $\mathcal{M}(a) \models T$ for all $a \in Y$ and

$$T_2 \models \forall x, y_1, \ldots, y_n \, (R_2(x,y_1,\ldots,y_n) \to [\pi_1(y_1) = x \wedge \ldots \wedge \pi_1(y_n) = x])$$

for every n-ary $R \in L$. Note that T_2 interprets T. We show that T interprets T_2. Suppose $\mathcal{M} \models T$. Let $\rho : M^2 \to M$ be the projection onto the first coordinate and for each n-ary relation symbol $R \in L$ declare $R_2(a,b)$ if $a \in M$, $b \in M^n$, and $\mathcal{M} \models R(a)$. Then $(M,M^2;\rho,(R_2)_{R\in L}) \models T_2$.

We show that $T_\cup$ and T_{Var} are existentially bi-interpretable and apply Fact 2.8. Suppose that $\mathcal{N} = (M,N;\pi_1,\pi_2,(R_2)_{R\in L})$ is a $T_\cup$-model. For each n-ary $R \in L$ and $a \in M$, $b_1,\ldots,b_n \in N$ we declare $R_{\mathrm{Var}}(a,b_1,\ldots,b_n)$ if and only if $\mathcal{N} \models R_2(a,c_1,\ldots,c_n)$ where each c_k is the unique element of N such that $\pi_1(c_k) = a$ and $\pi_2(c_k) = b_k$. Then $(M;(R_{\mathrm{Var}})_{R\in L}) \models T_{\mathrm{Var}}$.

Now suppose that $\mathcal{M} = (M;(R_{\mathrm{Var}})_{R\in L}) \models T_{\mathrm{Var}}$. Let π_1,π_2 be the projections $M^2 \to M$ onto the first and second coordinates, respectively. For each n-ary $R \in L$ and $a \in M$, $b_1,\ldots,b_n \in M^2$ we declare $R_2(a,b_1,\ldots,b_n)$ if and only if $\pi_1(b_1) = \ldots = \pi_1(b_n) = a$ and $\mathcal{M} \models R_{\mathrm{Var}}(a,\pi_2(b_1),\ldots,\pi_2(b_n))$. These observations may be formalized to construct an existential bi-interpretation between $T_\cup$ and T_{Var}. □

We also observe that Fact 6.18 follows from Corollary 2.7. Suppose that T eliminates $\exists^\infty$. It easily follows that T_1 eliminates $\exists^\infty$. Note that T_1 eliminates $\exists^\infty$ as T_1 is interpretable in the theory of equality. Finally, $T_\cap$ is interpretable in the theory of equality.

Dobrowolski [Dob18] showed that T^*_{Var} is NTP_1 when T is NTP_1. It is conjectured that NSOP_1 and NTP_1 are equivalent. We obtain Corollary 6.20 by combining Corollary 4.9 and Proposition 6.19.

Corollary 6.20. *If T is NSOP_1 and eliminates $\exists^\infty$ then T^*_{Var} is NSOP_1.*

Corollary 6.20 generalizes the result of Baudisch that if T is stable and eliminates $\exists^\infty$ then T^*_{Var} does not have the strict order property [Bau02, Theorem 4.4]. Corollary 6.20 is sharp. Baudisch shows that if there is an L-formula $\varphi(x,y)$, $\mathcal{M} \models T$, and a sequence $(a_i)_{i\in\omega}$ from $\mathcal{M}$ such that the $\varphi(M^{|x|}, a_i)$ are pairwise disjoint and infinite then T^*_{Var} is TP_2 [Bau02, Lemma 4.2]. The same argument shows that if T has U-rank at least two then T^*_{Var} is TP_2.

6.5.1. T^*_{Feq}. Let L_{Feq} be the two sorted theory containing a single ternay relation D. Let T_{Feq} be the L_{Feq}-theory such that $(M, N; D) \models T_{\mathrm{Feq}}$ if M, N are infinite, $D(x, y, z)$ is a ternary relation where x is of sort M and y, z are of sort N, and $R(a, x, y)$ is an equivalence relation for any $a \in M$. Then T_{Feq} has a model companion T^*_{Feq}. Chernikov and Ramsey [CR16] show that T^*_{Feq} is NSOP_1.

Proposition 6.21 follows by a slight modification of Proposition 6.19 and the observation that the theory of an equivalence relation with infinitely many infinite classes is interpretable in the theory of equality. We leave the details to the reader.

Proposition 6.21. *There are $T_\cap, T_1, T_2$ such that T^*_{Feq} is bi-interpretable with $T^*_\cup$ and T_1 and T_2 are both interpretable in the theory of equality.*

Combining Theorem 4.6 and Proposition 6.21 we recover the fact that T^*_{Feq} is NSOP_1. This is sharp as T^*_{Feq} is easily seen to be TP_2.

6.6. **Generic subspaces.** Fix a finite field $\mathbb{F}$. Let T_{Vec} be the theory of $\mathbb{F}$-vector spaces, L be a language extending the language of $\mathbb{F}$-vector spaces, T be a complete, model complete, L-theory extending T_{Vec}, and T_{Sub} be the theory of a T-model equipped with an $\mathbb{F}$-vector subspace. Fact 6.22 is a special case of work of d'Elbée [d'E21a].

Fact 6.22. *Suppose that T eliminates $\exists^\infty$. Then the model companion T^*_{Sub} of T_{Sub} exists. If T is stable then T^*_{Sub} is* NSOP_1.

We do not know if Fact 6.22 remains true when "stable" is replaced by "NSOP_1". Fix a prime p. A special case of Fact 6.22 is that the theory of an algebraically closed field of characteristic p equipped with an additive subgroup has an NSOP_1 model companion. This is sharp as this theory is TP_2, see [d'E21a].

The proof of Proposition 6.23 is easy and left to the reader.

Proposition 6.23. *Let $I = \{1, 2\}$, $T_1 = T$, and T_2 be the theory of $(\mathcal{V}; P)$ where $\mathcal{V}$ is an infinite dimensional $\mathbb{F}$-vector space and P is a unary predicate defining an infinite dimensional and infinite codimensional subspace of $\mathcal{V}$. Then $T^*_\cup$ exists and equals T^*_{Sub}.*

Note that that a T_2-model $(\mathcal{V}; P)$ is determined up to isomorphism by the dimensions of P and $\mathcal{V}$. It follows that $(\mathcal{V}; P)$ is $\aleph_0$-categorical and $\aleph_0$-stable, so in particular T_2 eliminates $\exists^\infty$. So the first claim of Fact 6.22 is a special case of Fact 2.6 and the second claim is a special case of Corollary 4.7.

We describe a related construction of d'Elbée. Fix an algebraically closed field K and an abelian variety A defined over K. Suppose that the ring of endomorphisms of A is $\mathbb{Z}$. Let T be the theory of the expansion of K by a subgroup of the K-points $A(K)$ of A. Then d'Elbée [d'E21b] shows that T has a model companion T^* which is NSOP_1 and TP_2. One can show that T^* is bi-interpretable with a fusion of two stable theories, hence Corollary 4.7 implies that T^* is NSOP_1. We leave the details of this to the reader.

6.7. **Fusion over the theory of equality.** Let $L_\cap$ be the language of equality and $T_\cap$ be the theory of an infinite set. Fact 6.24 is essentially due to Winkler [Win75]. The first part can be recovered as a special case of Corollary 2.7.

Fact 6.24. *If each T_i eliminates $\exists^\infty$ then $T^*_\cup$ exists. Conversely, if T_1 is complete and does not eliminate $\exists^\infty$ then there is T_2 such that $T^*_\cup$ does not exist.*

Fact 6.25 is proven by Ramsey and the first author in [KR18] and independently by Jeřábek in [Jeř19]. It can be recovered as a special case of Corollary 4.9.

Fact 6.25. *If each T_i is* NSOP$_1$ *and $T^*_\cup$ exists, then $T^*_\cup$ is* NSOP$_1$*.*

Tsuboi conjectured in [Tsu01] that if $L_1 \cap L_2 = \varnothing$ and T_1, T_2 are both simple and eliminate $\exists^\infty$ then $T_\cup$ has a simple completion. The following corollary of Fact 6.24 and Fact 6.25 is the NSOP$_1$ analogue of Tsuboi's conjecture.

Corollary 6.26. *If $L_i \cap L_j = \varnothing$ for distinct $i, j \in I$ and each T_i is* NSOP$_1$ *and eliminates $\exists^\infty$ then $T_\cup$ has an* NSOP$_1$ *completion.*

We now describe some special cases from the literature where more than NSOP$_1$ is preserved. Suppose that $I = \{1, 2\}$, T_1 is an arbitrary theory which eliminates $\exists^\infty$, $L_2 = \{P\}$ for a unary predicate P, and T_2 is the theory of an infinite and co-infinite unary predicate. Then $T^*_\cup$ is the expansion of T_1 by a generic predicate defined by Chatzidakas and Pillay [CP98]. Fact 6.27 is proven in [CP98]. Note that Fact 6.27 is a special case of Theorem 4.13.

Fact 6.27. *If T_1 is simple and eliminates $\exists^\infty$, then the expansion of T_1 by a generic unary predicate is simple.*

Proposition 6.28 describes a family of examples which shows that fusions over the theory of equality are typically TP$_2$.

Proposition 6.28. *Suppose $I = \{1, 2\}$, T_1 expands the theory of some infinite group, T_2 has U-rank at least two, and $T^*_\cup$ exists. Then $T^*_\cup$ has* TP$_2$*.*

So for example the fusion of a vector space and an equivalence relation with infinitely many infinite classes over the theory of equality is TP$_2$. This once again demonstrates the sharpness of the NSOP$_1$ conclusion in Corollary 4.9.

Proof. Let $\mathcal{M}_\cup \models T^*_\cup$ be $\aleph_0$-saturated. Morleyizing if necessary, we can arrange that T_i admits quantifier elimination. By Fact 2.1, $T^*_\cup$ is model complete and $\mathcal{M}_\cup$ is an existentially closed model of $T^*_\cup$. As T_2 has U-rank at least two there is an L_2-formula $\varphi(x, y)$ and a sequence $(b_k)_{k\in\omega}$ of tuples such that $\varphi(M^{|x|}, b_k)$ is infinite for all $k \in \omega$ and $\{\varphi(x, b_k) \mid k \in \omega\}$ is n-inconsistent for some n. Let $(a_l)_{l\in\omega}$ be a sequence of distinct elements of M. Using $\cdot$ to denote the group operation in T_1, we define $\psi(x, y, z) = \varphi(x \cdot z^{-1}, y)$ so that $\psi(M^{|x|}, b_k, c_l) = \varphi(M^{|x|}, b_k) \cdot c_l$ for all k, l. Note that $\{\psi(x, b_k, c_l) \mid k \in \omega\}$ is n-inconsistent for all l. Fix $m \in \omega$ and $f : m \to \omega$. An easy application of existential closedness produces $a \in M^{|x|}$ such that we have $\mathcal{M}_\cup \models \psi(a, b_{f(l)}, c_l)$ for all $l \le m$. So $\psi(x, y, z)$ together with the array $(b_k, c_l) \mid k, l \in \omega)$ together witness that $T^*_\cup$ has TP$_2$. ☐

Simon and Shelah [SS12] show that if T is NIP and algebraic closure in T is trivial then the expansion of T by a generic dense linear order is NIP. Proposition 6.29 generalizes this. Proposition 6.29 follows from Fact 6.24 and Proposition 4.1.

Proposition 6.29. *Suppose that each T_i is* NIP *and that each* acl_i *is trivial. Then $T^*_\cup$ exists and is* NIP*.*

6.8. **Fusions of blowups.** Example 3.16 is a member of a more general family of examples which we now discuss. These examples witness sharpness of Proposition 3.9. Suppose L is a relational language and T is a complete L-theory with infinite models. We define the **blow up** of T. Let L_{Blow} be the language containing a binary relation symbol E and an n-ary relation symbol R_{Blow} for each n-ary relation symbol $R \in L$. Suppose $\mathcal{M} \models T$. A blow up of $\mathcal{M}$ is an L_{Blow}-structure $\mathcal{N}$ such that there is a surjection $\rho : N \to M$ satisfying the following:

(1) For all $a, b \in N$ we have $\rho(a) = \rho(b)$ if and only if $\mathcal{N} \models E(a, b)$,
(2) $\rho^{-1}(a)$ is infinite for all $a \in N$,
(3) for every n-ary relation symbol $R \in L$ and $a_1, \dots, a_n \in N$ we have $\mathcal{N} \models R_{\text{Blow}}(a_1, \dots, a_n)$ if and only if $\mathcal{M} \models R(\rho(a_1), \dots, \rho(a_n))$.

Let T_{Blow} be the L_{Blow}-theory of blowups of models of T. It is easy to see that T_{Blow} is complete and algebraic closure in T_{Blow} is trivial. Given $i \in \{1, 2\}$ let L_i be a relational language, T_i be a complete L_i-theory with infinite models, suppose that T_i^{eq} eliminates $\exists^\infty$ and $L_1 \cap L_2 = \emptyset$. So $T_\cup^*$ exists by Fact 2.6. Let $S_\cap$ be the $\{E\}$-theory of an equivalence relation with infinitely many infinite classes and given $i \in \{1, 2\}$ let S_i be $(T_i)_{\text{Blow}}$. It is easy to see that each S_i^{eq} eliminates $\exists^\infty$ so $S_\cup^*$ exists by Fact 2.6. It is also easy to see that $S_\cup^*$ is the blow up of $T_\cup^*$. It follows that algebraic closure in $S_\cup^*$ is trivial and $S_\cup^*$ interprets $T_\cup^*$.

We show that $S_\cup^*$ is in general not acl-closed. So if $S_\cup^*$ is acl-complete then every $S_\cup^*$-formula is equivalent to a boolean combination of S_1 and S_2 formulas. As stable formulas are closed under boolean combinations it would follow that if S_1 and S_2 are stable then $S_\cup^*$ is stable. It is easy to produce examples of stable T_1, T_2 such that $T_\cup^*$ has TP_2, see Proposition 6.28. For example if T_1 is theory of an equivalence relation with infinitely many infinite classes and T_2 is the theory of infinite vector spaces over a fixed finite field $\mathbb{F}$, then $T_\cup^*$ has TP_2, so $S_\cup^*$ is not acl-closed.

6.9. **Multiple valuations.** Let $T_\cap$ be the theory of algebraically closed fields and for each $i \in I$ let T_i be the theory of an algebraically closed field with a nontrivial valuation v_i. The $T_\cup^*$ exists and is the theory of an algebraically closed field with an independent family $(v_i)_{i \in I}$ of valuations. It is well known that the theory of an algebraically closed valued field is NIP. Fact 6.30 is due to Johnson [Joh16].

Fact 6.30. *The theory $T_\cup^*$ has* NTP_2 *and* IP.

This witnesses sharpness of Proposition 4.1. Let K be an algebraically closed field and $(v_i)_{i \in I}$ be an independent family of non-trivial valuations on K. Each $(K; v_i)$ is NIP and algebraic closure in each $(K; v_i)$ agrees with algebraic closure in K, see for example [vdD89]. However, algebraic and definable closure do not agree in K.

We discuss a seemingly similar structure. Let P be the set of primes. Let $L_\cap$ be the language of abelian groups and $T_\cap$ be the theory of $(\mathbb{Z}; +)$. For each $p \in P$ let L_p be the expansion of $L_\cap$ by a binary relation $\preccurlyeq_p$ and let T_p be the theory of $(\mathbb{Z}; +, \preccurlyeq_p)$ where $k \preccurlyeq_p k'$ if and only if the p-adic valuation of k is less than or equal to the p-adic valuation of k'. Alouf and d'Elbée [Ad19] prove Fact 6.31.

Fact 6.31. *The model companion $T_\cup^*$ of $T_\cup$ exists and agrees with the theory of $(\mathbb{Z}; +, (\preccurlyeq_p)_{p \in P})$. Furthermore every $L_\cup$-formula is $T_\cup^*$-equivalent to a boolean combination of formulas from the various L_p.*

It is also shown in [Ad19] that each T_p is NIP and furthermore that $T^*_\cup$ is NIP. Note that that the second claim follows from the first claim and Fact 6.31 as a boolean combination of NIP formulas is NIP.

This is a natural example where the conditions of Proposition 4.1 are satisfied. Algebraic closure in each $(\mathbb{Z};+,\preccurlyeq_p)$ agrees with algebraic closure in $(\mathbb{Z};+)$ [Ad19] and it is well known that algebraic and definable closure agree in $(\mathbb{Z};+)$.

6.10. **Cyclic multiplicative orders on $\mathbb{F}$.** We now describe the original motivating example of a fusion. Let G be an abelian group. A **cyclic group order** on G is a ternary relation C such that for all $a,b,c,d \in G$

(1) $C(a,b,c)$ implies $C(b,c,a)$,
(2) $C(a,b,c)$ implies $\neg C(c,b,a)$,
(3) $C(a,b,c)$ and $C(a,c,d)$ implies $C(a,b,d)$,
(4) if a,b,c are distinct then either $C(a,b,c)$ or $C(c,b,a)$,
(5) $C(a,b,c)$ implies $C(da,db,dc)$.

$(1)-(4)$ are the usual axioms of a cyclic order on a set. Let $(F;+,\times)$ be a field. A **cyclic multiplicative order** on $(F;+,\times)$ is a ternary relation C on F such that

(6) $C(a,b,c)$ implies $a,b,c \in F^\times$ for all $a,b,c \in F$, and
(7) the restriction of C to $F^\times$ is a cyclic group order on $(F^\times;\times)$.

We are primarily interested in the case when F is an algebraic closure of a finite field. In this case any cyclic multiplicative order on $(F;+,\times)$ is the pullback of the usual cyclic group order on $(\mathbb{R}/\mathbb{Z};+)$ via an injective character $(F^\times;\times) \to (\mathbb{R}/\mathbb{Z};+)$.

Let $L_\cap$ be the language containing a binary function symbol $\times$ and $T_\cap$ be the $L_\cap$ theory of $(F;\times)$, where $(F;+,\times)$ is an algebraically closed field. (Note that $T_\cap$ is bi-interpretable with the theory of the multiplicative group of an algebraically closed field.) Let L_1 be the expansion of $L_\cap$ by a binary function symbol $+$ and T_1 be ACF. Let L_2 be the expansion of $L_\cap$ by a ternary function symbol C and T_2 be the L_2-theory such that $(F;\times,C) \models T_2$ if and only if $(F;\times) \models T_\cap$, $C(a,b,c)$ implies $a,b,c \in F^\times$ for all $a,b,c \in F$, and the restriction of C to $F^\times$ is a cyclic group order on $(F^\times;\times)$. Fact 6.32 is essentially proven in [Tra17].

Fact 6.32. *The fusion $T^*_\cup$ exists and agrees the theory of an algebraic closure of a finite field equipped with a cyclic multiplicative group order.*

It is easy to see that $T_\cap$ is stable and has weak elimination of imaginaries. Fact 6.33 follows by Corollary 3.4. Fact 6.33 is also proven in [Tra17].

Fact 6.33. *Every $L_\cup$-formula $\psi(x)$ is $T^*_\cup$-equivalent to a a finite disjunction of b.e. formulas of the form $\exists y[\varphi_1(x,y) \wedge \varphi_2(x,y)]$ where φ_i is an L_i-formula for $i \in \{1,2\}$.*

Note that T is SOP. An argument similar to the proof of Proposition 6.28 shows that T has TP_2. So T does not satisfy any known classification theoretic property. However, $T_\cap$ is strongly minimal, T_1 and T_2 are both NIP, and we have a reasonable description of T-definable sets in terms of T_1- and T_2-definable sets. So we conjecture that T satisfies some as-yet-undiscovered classification theoretic property. More generally, we make the following vague conjecture. Recall that a set system $(X;\mathcal{S})$ consists of a set X and a family $\mathcal{S}$ of subsets of X.

Vague Conjecture. *There is a classification-theoretic property* ★ *such that:*

(1) *there is a property P of set systems such that T is* ★ *if and only if* $(M^x; \{\delta(M^x, b) : b \in M^y\})$ *has P for every formula $\delta(x,y)$ and $\mathcal{M} \models T$,*
(2) *both* NSOP_1 *and* NTP_2 *imply* ★*,*
(3) NSOP_1 *is equivalent to the conjunction of* ★ *and* NSOP*,*
(4) ★ *is preserved by fusions under reasonably general conditions, and*
(5) *any fusion of* NIP *theories over a stable base is* ★*.*

We describe two other examples of theories which are SOP and TP_2, and should have ★. One example is the model companion of the theory of a real closed ordered field $(R; +, \times, <)$ equipped with a subgroup of $(R^\times; \times)$, this theory was shown to exist in [Blo21]. The second example requires slightly more explanation. Let $\mathbb{F}$ be a field. Granger studied the theory of an $\mathbb{F}$-vector space equipped with a generic symmetric bilinear form [Gra]. This theory always has TP_2. When $\mathbb{F}$ is algebraically closed this theory is NSOP_1 [CR16], we expect that it is ★ when $\mathbb{F}$ is real closed.

7. Dense linear orders are indecomposable

We have given many examples of theories which "decompose" as fusions. We finish with an example of a theory which does not. There should be many such examples, but it appears difficult to establish this. Let T be a complete L-theory. We say that T is **interpolatively decomposable** if there are $(T_i)_{i\in I}$ such that $T^*_\cup$ exists, T is bi-interpretable with $T^*_\cup$, and T is not interpretable in any T_i. For example an unstable theory which is bi-interpretable with a fusion of a family of stable theories is interpolatively decomposable. We say that T is **interpolatively indecomposable** if it is not interpolatively decomposable. We expect that ACF and RCF are interpolatively indecomposable, but we do not have a proof at present. We do not even have a proof that the theory of $(\mathbb{N}, <)$ is interpolatively indecomposable.

Theorem 7.1. *The theory* DLO *of dense linear orders without endpoints is interpolatively indecomposable.*

We will use a deep result on NIP structures. Suppose that T is $\aleph_0$-categorical, unstable, and NIP. Simon [Sim18] shows that T interprets an infinite linear order. While this is not explicitly stated in that paper, the proof shows that T interprets a dense linear order. (It is also not difficult to show directly that an infinite $\aleph_0$-categorical linear order interprets a dense linear order.) So we have Fact 7.2.

Fact 7.2. *If T is $\aleph_0$-categorical,* NIP*, and unstable, then T interprets* DLO*.*

We now prove Theorem 7.1.

Proof. Suppose that DLO is bi-interpretable with $T^*_\cup$. So each T_i is interpretable in DLO. By Corollary 4.10 some T_j is unstable. Note that T_j is $\aleph_0$-categorical and NIP as DLO is $\aleph_0$-categorical and NIP. Apply Fact 7.2. □

The most basic example of an unstable NSOP theory (the random graph) decomposes as a fusion of two stable theories and the most basic example of an unstable NIP theory (DLO) is indecomposable.

Finally we let ACF_p be the theory of algebraically closed fields of characteristic p (possibly $p = 0$) and sketch a heuristic argument that ACF_p is interpolatively indecomposable. Suppose that ACF_k is bi-interpretable with $T^*_\cup$. So each T_i is interpretable in ACF_p. It is a conjecture of Zil'ber that any structure interpretable in an algebraically closed field is either one-based or interprets an infinite field. Any infinite field interpretable in an algebraically closed field K is definably isomorphic to K [Bou89]. We expect that a fusion of a family of one-based theories cannot interpret an infinite field, but we do not have a proof.

Appendix A. bcl-completeness

Recall the definitions of b.e. formula and bcl from Section 2.4.

Remark A.1. The class of b.e. formulas is closed (up to T-equivalence) under conjunction: if $\exists y_1\, \psi_1(x, y_1)$ and $\exists y_2\, \psi_2(x, y_2)$ are b.e. with bounds k_1 and k_2 respectively, then

$$(\exists y_1\, \psi_1(x, y_1)) \wedge (\exists y_2\, \psi_2(x, y_2))$$

is T-equivalent to

$$\exists y_1 y_2\, (\psi_1(x, y_1) \wedge \psi_2(x, y_2)),$$

which is b.e. with bound $k_1 \cdot k_2$.

Remark A.2. A formula $\varphi(x, y, z)$ is bounded in yz if and only if it is bounded in z and $\exists z\, \varphi(x, y, z)$ is bounded in y. As a consequence, $b \in \mathrm{bcl}(A)$ if and only if b satisfies a formula $\exists z\, \varphi(a, y, z)$, such that $a \in A^x$ and $\varphi(x, y, z)$ is quantifier-free and bounded in yz.

Lemma A.3. *If $A \subseteq \mathcal{M}$ then $\langle A \rangle \subseteq \mathrm{bcl}(A)$. Furthermore,* bcl *is a closure operator.*

Proof. Fix $A \subseteq \mathcal{M}$. Suppose $b \in \langle A \rangle$. Then $t(a) = b$ for a term $t(x)$ and a tuple a from A. Then the formula $t(x) = y$ is b.e. (taking z to be the empty tuple of variables) and bounded in y (with bound 1), so it witnesses $b \in \mathrm{bcl}(A)$.

It follows that $A \subseteq \mathrm{bcl}(A)$, and it is clear that $A \subseteq B$ implies $\mathrm{bcl}(A) \subseteq \mathrm{bcl}(B)$. It remains to show bcl is idempotent.

Suppose $b \in \mathrm{bcl}(\mathrm{bcl}(A))$. Then by Remark A.2, $\mathcal{M} \models \exists z\, \varphi(a, b, z)$ for some quantifier-free formula $\varphi(x, y, z)$ which is bounded in yz and some tuple $a = (a_1, \ldots, a_n)$ from $\mathrm{bcl}(A)$. For each $1 \leq j \leq n$, since a_j is in $\mathrm{bcl}(A)$, $\mathcal{M} \models \exists z_j\, \psi_j(d_j, a_j, z_j)$ for some quantifier-free formula $\psi_j(w_j, x_j, z_j)$ which is bounded in $x_j z_j$, and some tuple d_j from A.

Then the quantifier-free formula

$$\left(\bigwedge_{j=1}^{n} \psi_j(w_j, x_j, z_j) \right) \wedge \varphi(x_1, \ldots, x_n, y, z)$$

is bounded in $x_1 \ldots x_n y z_1 \ldots z_n z$ (by the product of the bounds for φ and the ψ_j), and

$$\mathcal{M} \models \exists x_1 \ldots x_n z_1 \ldots z_n z \left(\bigwedge_{j=1}^{n} \psi_j(d_j, x_j, z_j) \right) \wedge \varphi(x_1, \ldots, x_n, b, z),$$

so $b \in \mathrm{bcl}(A)$ by Remark A.2. □

Theorem A.4. *The following are equivalent:*

1. *Every L-formula is T-equivalent to a finite disjunction of b.e. formulas.*
2. *T is* acl*-complete and* $\mathrm{acl} = \mathrm{bcl}$ *in T-models.*
3. *T is* bcl*-complete.*

Proof. We assume (1) and prove (2). We first show acl and bcl agree. Suppose $A \subseteq \mathcal{M} \models T$ and $b \in \mathrm{acl}(A)$, witnessed by an algebraic formula $\varphi(a, y)$ with parameters a

from A. Suppose there are exactly k tuples in M^y satisfying $\varphi(a,y)$. Let $\varphi'(x,y)$ be the formula

$$\varphi(x,y) \wedge \exists^{\leq k} y'\, \varphi(x,y'),$$

and note $\varphi'(x,y)$ is bounded in y. By assumption, $\varphi'(x,y)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\psi(x,y)$ such that $T \vDash \psi(x,y) \to \varphi'(x,y)$ and $\mathcal{M} \vDash \psi(a,b)$. Since $\varphi'(x,y)$ is bounded in y, so is $\psi(x,y)$, and hence $b \in \mathrm{bcl}(A)$.

We continue to assume (1) and show T is acl-complete. Suppose $\mathcal{A}$ is an algebraically closed substructure of $\mathcal{M} \vDash T$ and $f{:}\mathcal{A} \to \mathcal{N} \vDash T$ is an embedding. We show that for any formula $\varphi(x)$, if $\mathcal{M} \vDash \varphi(a)$, where $a \in A^x$, then $\mathcal{N} \vDash \varphi(f(a))$. By our assumption, $\varphi(x)$ is equivalent to a finite disjunction of boundedly existential formulas, so there is some boundedly existential formula $\exists y\, \psi(x,y)$ such that

$$T \vDash (\exists y\, \psi(x,y)) \to \varphi(x) \quad \text{and} \quad \mathcal{M} \vDash \exists y\, \psi(a,y).$$

Let $b \in M^y$ be a witness for the existential quantifier. Then each component of the tuple b is in $\mathrm{acl}(a) \subseteq A$, since A is algebraically closed. And ψ is quantifier-free, so $\mathcal{N} \vDash \psi(f(a), f(b))$, and hence $\mathcal{N} \vDash \varphi(f(a))$.

It is clear that (2) implies (3).

We now assume (3) and prove (1). For any finite tuple of variables x, let Δ_x be the set of boundedly existential formulas with free variables from x.

Claim: For all models $\mathcal{M}$ and $\mathcal{N}$ of T and all tuples $a \in M^x$ and $a' \in N^x$, if $\mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a')$, then $\mathrm{tp}(a) = \mathrm{tp}(a')$.

Proof of claim: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are models of T, $a \in M^x$, $a' \in N^x$, and $\mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a')$. Let y be a tuple of variables enumerating the elements of $\mathrm{bcl}(a)$ which are not in a. Let $p(x,y) = \mathrm{qftp}(\mathrm{bcl}(a))$, and let $q(x) = \mathrm{tp}(a')$. We claim that $T \cup p(x,y) \cup q(x)$ is consistent.

Let $b = (b_1, \ldots, b_n)$ be a finite tuple from $\mathrm{bcl}(a)$ which is disjoint from a, and let $\psi(x,y')$ be a quantifier-free formula such that $\mathcal{M} \vDash \psi(a,b)$ (where $y' = (y_1, \ldots, y_n)$ is the finite subtuple of y enumerating b).

For each $1 \leq j \leq n$, the fact that $b_j \in \mathrm{bcl}(a)$ is witnessed by $\mathcal{M} \vDash \exists z_j\, \varphi_j(a, b_j, z_j)$, where $\varphi_j(x, y_j, z_j)$ is quantifier-free and bounded in $y_j z_j$ (by Remark A.2). Letting $z = (z_1, \ldots, z_n)$, the conjunction $\bigwedge_{j=1}^n \varphi_j(x, y_j, z_j)$ is a quantifier-free formula $\varphi(x, y', z)$ which is bounded in $y'z$. It follows that $\varphi(x,y',z) \wedge \psi(x,y')$ is also bounded in $y'z$, and $\mathcal{M} \vDash \exists z\, (\varphi(a,b,z) \wedge \psi(a,b))$. Then

$$\exists y'z\, (\varphi(x,y',z) \wedge \psi(x,y')) \in \mathrm{tp}_{\Delta_x}(a) \subseteq \mathrm{tp}_{\Delta_x}(a'),$$

so $\mathcal{N} \vDash \exists y'z\, (\varphi(a',y',z) \wedge \psi(a',y'))$. Letting $b' \in N_{y'}$ be a witness for the first block of existential quantifiers, $\mathcal{N} \vDash \psi(a',b')$, so $T \cup \{\psi(x,y')\} \cup q(x)$ is consistent.

By compactness, $T \cup p(x,y) \cup q(x)$ is consistent, so there exists a model $\mathcal{N}' \vDash T$, a tuple $a'' \in (N')^x$ realizing $q(x)$, and an embedding $f{:}\,\mathrm{bcl}(a) \to \mathcal{N}'$ such that $f(a) = a''$. By bcl-completeness, we have $\mathrm{tp}(a) = \mathrm{tp}(a'') = \mathrm{tp}(a')$, as was to be shown.

Having established the claim, we conclude with a standard compactness argument. Let $\varphi(x)$ be an L-formula. Suppose $\mathcal{M} \vDash T$ and $\mathcal{M} \vDash \varphi(a)$. Let $p_a(x) = \mathrm{tp}_{\Delta_x}(a)$. By the claim, $T \cup p_a(x) \cup \{\neg\varphi(x)\}$ is inconsistent. Since $p_a(x)$ is closed under finite conjunctions (up to equivalence) by Remark A.1, there is a formula $\psi_a(x) \in p_a(x)$ such that $T \vDash \psi_a(x) \to \varphi(x)$.

Now

$$T \cup \{\varphi(x)\} \cup \{\neg\psi_a(x) \mid \mathcal{M} \models T \text{ and } \mathcal{M} \models \varphi(a)\}$$

is inconsistent, so there are finitely many $a_1, \ldots, a_n$ such that $T \models \varphi \to (\bigvee_{i=1}^n \psi_{a_i}(x))$. Since also $T \models (\bigvee_{i=1}^n \psi_{a_i}(x)) \to \varphi(x)$, we have shown that $\varphi(x)$ is T-equivalent to $\bigvee_{i=1}^n \psi_{a_i}(x)$. □

It may be surprising that acl-completeness does not already imply every formula is equivalent to a finite disjunction of b.e. formulas, i.e., acl-completeness is not equivalent to bcl-completeness. We give a counterexample.

Example A.5. Let L be the language with a single unary function symbol f. We denote by $E(x, y)$ the equivalence relation defined by $f(x) = f(y)$. We say an element of an L-structure is **special** if it is in the image of f. Let T be the theory asserting the following:

(1) Models of T are nonempty.
(2) There are no cycles, i.e., for all $n \geq 1$, $\forall x\, f^n(x) \neq x$.
(3) Each E-class is infinite and contains exactly one special element.

Every T-model can be decomposed into a disjoint union of **connected components**, each of which is a chain of E-classes, $(C_n)_{n\in\mathbb{Z}}$, such that each class C_n contains a unique special element a_n, and $f(b) = a_n$ for all $b \in C_{n-1}$.

Let A be a subset of a T-model. Then $\text{acl}(A)$ consists of A, together with the $\mathbb{Z}$-indexed chain of special elements in each connected component which meets A. But $\text{bcl}(A)$ is just the substructure generated by A: it only contains the special elements from E-classes further along in the chain than some element of A. Indeed, if a_n is the unique special element in class C_n, $a_n \notin A$, and no element of A is in any class C_m with $m < n$ in the same connected component, then a_n does not satisfy any bounded and b.e. formula with parameters from A.

It is not hard to show that T is acl-complete (and hence complete, since $\text{acl}(\varnothing) = \varnothing$), but not bcl-complete. For an explicit example of a formula which is not equivalent to a finite disjunction of b.e. formulas, consider the formula

$$\exists y\, f(y) = x$$

defining the special elements.

Appendix B. Extendability of forking independence

In this appendix, we adopt the notation and terminology of Section 2.5. Our goal is to understand when forking independence $\mathop{\smile\!\!\!\!\!\!\;|}^f$ in a theory T is extendable to an arbitrary expansion T'. In particular, we show that when T is stable with weak elimination of imaginaries, $\mathop{\smile\!\!\!\!\!\!\;|}^f$ is always stationary and extendable to T'.

We recall a few variations on the notion of elimination of imaginaries (see [CF04]).

(1) T has **elimination of imaginaries** if every $a \in \mathbf{M}^{\text{eq}}$ is interdefinable with some $b \in \mathbf{M}$, i.e., $a \in \text{dcl}^{\text{eq}}(b)$ and $b \in \text{dcl}^{\text{eq}}(a)$.
(2) T has **weak elimination of imaginaries** if for every $a \in \mathbf{M}^{\text{eq}}$ there is some $b \in \mathbf{M}$ such that $a \in \text{dcl}^{\text{eq}}(b)$ and $b \in \text{acl}^{\text{eq}}(a)$.
(3) T has **geometric elimination of imaginaries** if every $a \in \mathbf{M}^{\text{eq}}$ is interalgebraic with some $b \in \mathbf{M}$, i.e., $a \in \text{acl}^{\text{eq}}(b)$ and $b \in \text{acl}^{\text{eq}}(a)$.

Let $\delta(x,y)$ be a formula. An **instance** of δ is a formula $\delta(x,b)$ with $b \in \boldsymbol{M}^y$, and a **δ-formula** is a Boolean combination of instances of δ. A **global δ-type** is a maximal consistent set of δ-formulas with parameters from $\boldsymbol{M}$. We denote by $S_\delta(\boldsymbol{M})$ the Stone space of global δ-types.

The following lemma is a well-known fact about the existence of weak canonical bases for δ-types when $\delta(x,y)$ is stable.

Lemma B.1. *Suppose T has geometric elimination of imaginaries, and $\delta(x,y)$ is a stable formula. For any $q(x) \in S_\delta(\boldsymbol{M})$, there exists a tuple d such that:*

(1) $q(x)$ has finite orbit under automorphisms of $\mathfrak{M}$ fixing d.
(2) d has finite orbit under automorphisms of $\mathfrak{M}$ fixing $q(x)$.
(3) $q(x)$ does not divide over d.

If T has weak elimination of imaginaries, we can arrange that d is fixed by automorphisms of $\mathfrak{M}$ fixing $q(x)$. And if T has elimination of imaginaries, we can further arrange that $q(x)$ is fixed by automorphisms of $\mathfrak{M}$ fixing d.

Proof. Let $e \in \mathfrak{M}^{\mathrm{eq}}$ be the canonical base for $q(x)$. Then $q(x)$ is fixed by all automorphisms fixing e, e is fixed by all automorphisms fixing $q(x)$, and $q(x)$ does not divide over e. By geometric elimination of imaginaries, e is interalgebraic with a real tuple d, and (1), (2), and (3) follow immediately. The cases when T has elimination of imaginaries or weak elimination of imaginaries are similar. □

T has **stable forking** if whenever a complete type $p(x)$ over B forks over $A \subseteq B$, then there is a stable formula $\delta(x,y)$ such that $\delta(x,b) \in p(x)$ and $\delta(x,b)$ forks over A. Every theory with stable forking is simple; the converse is the Stable Forking Conjecture, which remains open (see [KP00]).

The following lemma is essentially the same idea as [PT97, Lemma 3], which itself makes use of key ideas from [HP94, Lemmas 5.5 and 5.8].

Lemma B.2. *Suppose T has stable forking and geometric elimination of imaginaries. Then $\mathop{\smile\!\!\!\!\!\mid}^{f}$ in T is extendable to T', i.e., has full existence over algebraically closed sets in T'.*

Proof. Suppose towards a contradiction that there exist sets A, B, and C in $\mathfrak{M}'$ such that $C = \mathrm{acl}_{L'}(C)$, and for any A^* with $\mathrm{tp}_{L'}(A^*/C) = \mathrm{tp}_{L'}(A/C)$, $A^* \not\mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B$ in $\mathfrak{M}$. We may assume $C \subseteq B$. Let $p(x) = \mathrm{tp}_{L'}(A/C)$. Since T has stable forking, the fact that $\mathrm{tp}_L(A^*/B)$ forks over C is always witnessed by a stable L-formula. So the partial type

$$p(x) \cup \{\neg\delta(x,b) \mid \delta(x,y) \in L \text{ is stable, and } \delta(x,b) \text{ forks over } C \text{ in } \mathfrak{M}\}$$

is not satisfiable in $\mathfrak{M}'$. By saturation and compactness, we may assume that A is finite and x is a finite tuple of variables. And as stable formulas and forking formulas are closed under disjunctions, there is an $L'(C)$-formula $\varphi(x) \in p(x)$, a stable L-formula $\delta(x,y)$, and $b \in \boldsymbol{M}^y$ such that $\delta(x,b)$ forks over C, and

$$\mathfrak{M}' \models \forall x\,(\varphi(x) \to \delta(x,b)).$$

Since forking and dividing agree in simple theories [Cas11, Prop. 5.17], $\delta(x,b)$ divides over C.

Let $[\varphi]$ be the set of all δ-types in $S_\delta(\boldsymbol{M})$ which are consistent with $\varphi(x)$. This is a closed set in $S_\delta(\boldsymbol{M})$: it consists of all global δ-types $r(x)$ such that $\chi(x) \in r(x)$

whenever $\chi(x)$ is a δ-formula and $\varphi(\mathcal{M}') \subseteq \chi(\mathcal{M}')$. In particular, if $r(x) \in [\varphi]$, then $\delta(x,b) \in r(x)$. Since δ is stable, $[\varphi]$ contains finitely many points of maximal Cantor-Bendixson rank. Let $q(x)$ be such a point.

Let d be the weak canonical base for $q(x)$ obtained in Lemma B.1. Since $[\varphi]$ is fixed setwise by any L'-automorphism fixing C, $q(x)$ has finitely many conjugates under such automorphisms. It follows that d too has finitely many conjugates, so $d \in C$, as C is algebraically closed in $\mathcal{M}'$. But then $q(x)$ does not divide over C, contradicting the fact that $\delta(x,b) \in q(x)$. □

Remark B.3. The following counterexample shows the assumptions of geometric elimination of imaginaries in T and $C = \mathrm{acl}(C)$ in $\mathcal{M}'$ (not just in $\mathcal{M}$) are necessary in Lemma B.2. Let T be the theory of an equivalence relation with infinitely many infinite classes. Let T' be the expansion of this theory by a single unary predicate P naming one of the classes. Let a and b be two elements of the class named by P in $\mathcal{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathcal{M}$ and $\mathcal{M}'$). For any a^* such that $\mathrm{tp}_{L'}(a^*/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, we have a^*Eb, and xEb forks over $\varnothing$ in $\mathcal{M}$. To fix this, we move to $\mathcal{M}^{\mathrm{eq}}$, so we have another sort containing names for all the E-classes. Note that $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathcal{M}$ still doesn't contain any of these names. But $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$ in $\mathcal{M}'$ contains the name for the class named by P, since it is fixed by L'-automorphisms. And we recover the lemma, since xEb does not fork over the name for the E-class of b.

Remark B.4. The conclusion of Lemma B.2 also fails when there are unstable forking formulas in T. Let T be be the theory of $(\mathbb{Q},<)$ and T' be the expansion of T by a unary predicate P defining an open interval (p,p'), where $p < p'$ are irrational real numbers. Let $b_1 < a < b_2$ be elements of $\mathcal{M}'$ such that $a \in P$ and $b_1, b_2 \notin P$. Let $C = \varnothing$ (which is algebraically closed in $\mathcal{M}'$). Then for any realization a^* of $\mathrm{tp}_{L'}(a/\varnothing)$, we have $a^* \not\mathop{\smile\hskip-0.9em ^| \ }^{f}_{\varnothing} b_1b_2$ in $\mathcal{M}$, witnessed by the formula $b_1 < x < b_2$.

Remark B.5. It is not possible to strengthen the conclusion of Lemma B.2 to the following: For all small sets A, B, and C, such that $C = \mathrm{acl}_{L'}(C)$, and for any A'' such that $\mathrm{tp}_L(A''/C) = \mathrm{tp}_L(A/C)$ and $A'' \mathop{\smile\hskip-0.9em ^| \ }^{f}_{C} B$ in $\mathcal{M}$, there exists A' with $\mathrm{tp}_{L'}(A'/C) = \mathrm{tp}_{L'}(A/C)$ and $\mathrm{tp}_L(A'/BC) = \mathrm{tp}_L(A''/BC)$.

That is, while it is possible to find a realization A' of $\mathrm{tp}_{L'}(A/C)$ such that $\mathrm{tp}_L(A'/BC)$ is a nonforking extension of $\mathrm{tp}_L(A/C)$, it is not possible in general to obtain an arbitrary nonforking extension of $\mathrm{tp}_L(A/C)$ in this way.

For a counterexample, consider the theories T and T' from Example 2.15. T has stable forking and geometric elimination of imaginaries. Let a and b be elements of the clique defined by P in $\mathcal{M}'$, and let $C = \varnothing$ (which is algebraically closed in $\mathcal{M}'$). Let a'' be any element such that $\mathcal{M}' \models \neg a''Eb$, and note that $a'' \mathop{\smile\hskip-0.9em ^| \ }^{f}_{\varnothing} b$ and $\mathrm{tp}_L(a''/\varnothing) = \mathrm{tp}_L(a/\varnothing)$ (there is only one 1-type over the empty set with respect to T). But for any a' with $\mathrm{tp}_{L'}(a'/\varnothing) = \mathrm{tp}_{L'}(a/\varnothing)$, $\mathcal{M}' \models P(a')$, so $a'Eb$, and $\mathrm{tp}_L(a'/b) \neq \mathrm{tp}_L(a''/b)$.

We have seen that the hypotheses of stable forking (and hence simplicity) and geometric elimination of imaginaries in T are sufficient to ensure that $\mathop{\smile\hskip-0.9em ^| \ }^{f}$ has full existence over algebraically closed sets in T', with no further assumptions on T'. But we would also like $\mathop{\smile\hskip-0.9em ^| \ }^{f}$ to be a stationary independence relation in T.

In a simple theory T, $\mathop{\smile\hskip-0.9em ^| \ }^{f}$ satisfies stationarity over $\mathrm{acl}^{\mathrm{eq}}$-closed sets if and only if T is stable [Cas11, Chapter 11]. And a stable theory has weak elimination of imaginaries if and only if it has geometric elimination of imaginaries and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over acl-closed sets [CF04, Propositions 3.2 and 3.4]. So we have proven the following proposition, under very natural hypotheses.

Proposition B.6. *Suppose T is stable with weak elimination of imaginaries. Then $\mathop{\smile\!\!\!\!\!\!|}^f$ is a stationary and extendable independence relation in T.*

Appendix C. Abstract independence without base monotonicity

Throughout this section, we assume T is a complete theory, $\mathfrak{M} \models T$ is a monster model, and $\mathop{\smile\!\!\!\!\!\!|}^*$ is a ternary relation on subsets of $\mathfrak{M}$ satisfying:

1. **Invariance**: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $ABC \equiv A'B'C'$, then $A' \mathop{\smile\!\!\!\!\!\!|}^*_{C'} B'$.
2. **Monotonicity**: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B'$.
3. **Symmetry**: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$.
4. **Transitivity**: Suppose $C \subseteq B \subseteq A$. If $A \mathop{\smile\!\!\!\!\!\!|}^*_B D$ and $B \mathop{\smile\!\!\!\!\!\!|}^*_C D$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C D$.
5. **Normality**: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $AC \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
6. **Full existence**: For any A, B, and C, there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
7. **Finite character**: If $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all finite $A' \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$.
8. **Strong local character**: For every cardinal λ, there exists a cardinal κ such that for all A with $|A| = \lambda$, all B, and all $D \subseteq B$, there exists $D \subseteq C \subseteq B$ with $|C| \leq \max(|D|, \kappa)$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$.

This list of axioms is very similar to Adler's axioms for independence relations in [Adl09], with the following differences:

- We do not assume base monotonicity.
- We assume symmetry. Adler proves that symmetry follows from his other axioms, but this proof uses base monotonicity.
- Our formulation of local character is stronger and serves as a partial replacement for base monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $C \subseteq B' \subseteq B$, we don't necessarily have $A \mathop{\smile\!\!\!\!\!\!|}^*_{B'} B$. But by strong local character there is $B' \subseteq B'' \subseteq B$, with B'' not too much bigger than B', such that $A \mathop{\smile\!\!\!\!\!\!|}^*_{B''} B$.
- We assume full existence instead of extension. But extension follows from our axioms (see Remark C.4 below).

Example C.1. Suppose T_0 is a reduct of T which is simple with stable forking and geometric elimination of imaginaries, and let $\mathfrak{M}_0$ be the corresponding reduct of $\mathfrak{M}$. As in Section 2.7, we define:

$$A \mathop{\smile\!\!\!\!\!\!|}^r_C B \iff \mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}(C)} \mathrm{acl}(BC) \text{ in } \mathfrak{M}_0.$$

where acl is the algebraic closure operator in $\mathfrak{M}$. Then $\mathop{\smile\!\!\!\!\!\!|}^r$ satisfies our axioms.

Invariance, monotonicity, symmetry, normality, and finite character are clear from the definition and the corresponding properties for $\mathop{\smile\!\!\!\!\!\!|}^f$ in simple theories.

Transitivity: Suppose $C \subseteq B \subseteq A$. If $A \mathop{\smile\!\!\!\!\!\!|}^r_B D$ and $B \mathop{\smile\!\!\!\!\!\!|}^r_C D$, we have

$$\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}(B)} \mathrm{acl}(BD) \quad \text{and} \quad \mathrm{acl}(B) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}(C)} \mathrm{acl}(CD) \quad \text{in} \quad \mathfrak{M}_0.$$

Since $\mathrm{acl}(CD) \subseteq \mathrm{acl}(BD)$, by monotonicity for $\mathop{\smile\!\!\!\!\!\!|}^f$, $\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}(B)} \mathrm{acl}(CD)$ in $\mathfrak{M}_0$, so by transitivity for $\mathop{\smile\!\!\!\!\!\!|}^f$, $\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!\!|}^f_{\mathrm{acl}(C)} \mathrm{acl}(CD)$ in $\mathfrak{M}_0$, so $A \mathop{\smile\!\!\!\!\!\!|}^r_C D$.

Full existence: Let

$$p(x) = \mathrm{tp}(\mathrm{acl}(AC)/\mathrm{acl}(C)) \quad \text{and} \quad q(y) = \mathrm{tp}(\mathrm{acl}(BC)/\mathrm{acl}(C)).$$

By B.2 (this is where we use the assumptions of stable forking and geometric elimination of imaginaries), there are realizations $\widehat{A}$ of $p(x)$ and $\widehat{B}$ of $q(y)$ such that $\widehat{A} \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \widehat{B}$ in $\mathfrak{M}_0$. Since $\widehat{B} \equiv_{\mathrm{acl}(C)} \mathrm{acl}(BC)$, we can move $\widehat{B}$ back to $\mathrm{acl}(BC)$ by an automorphism σ fixing $\mathrm{acl}(C)$. Let $A' \subseteq \sigma(\widehat{A})$ be the image under σ of the copy of A in $\widehat{A}$, so $A' \equiv_{\mathrm{acl}(C)} A$. In particular, $A' \equiv_C A$, and $\sigma(\widehat{A}) = \mathrm{acl}(A'C)$, so $\mathrm{acl}(A'C) \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(BC)$, so $A' \mathop{\smile\!\!\!\!\!\mid}^r_C B$.

Strong local character: Since T_0 is simple, there is a cardinal $\kappa(T)$ such that for all finite a and all B, there exists $B' \subseteq B$ with $|B'| \leq \kappa(T)$ and $a \mathop{\smile\!\!\!\!\!\mid}^f_{B'} B$ in $\mathfrak{M}_0$. Given the cardinal λ, let $\kappa = \max(\kappa(T), \lambda, |L|)$. I claim first that for all A with $|A| \leq \max(\lambda, |L|)$, for all B, and for all $D \subseteq B$, there exists C with $D \subseteq C \subseteq B$ and $|C| \leq \max(|D|, \kappa)$ such that $A \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(B)$ in $\mathfrak{M}_0$. Indeed, for every finite tuple a from A, we can find $B_a \subseteq \mathrm{acl}(B)$ with $|B_a| \leq \kappa(T)$ such that $a \mathop{\smile\!\!\!\!\!\mid}^f_{B_a} \mathrm{acl}(B)$ in $\mathfrak{M}_0$. Letting $B' = \bigcup_a B_a$, by base monotonicity and finite character for $\mathop{\smile\!\!\!\!\!\mid}^f$, $A \mathop{\smile\!\!\!\!\!\mid}^f_{B'} \mathrm{acl}(B)$, and $|B'| \leq \kappa$, since there are $|A|$-many finite tuples from A. Now we obtain $B'' \subseteq B$ with $B' \subseteq \mathrm{acl}(B'')$ by replacing each element $b \in B'$ with a finite tuple $b_1, \ldots, b_n \in B$ such that $b \in \mathrm{acl}(b_1, \ldots, b_n)$. We still have $|B''| \leq \kappa$, and $A \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(B'')} \mathrm{acl}(B)$. Finally, let $C = B'' \cup D$, so $D \subseteq C \subseteq B$, and $|C| \leq \max(|D|, \kappa)$. By base monotonicity for $\mathop{\smile\!\!\!\!\!\mid}^f$, $A \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(B)$.

Now suppose we are given A with $|A| = \lambda$, B, and $D \subseteq B$. Build a sequence $(C_i)_{i \in \omega}$ such that for all $i \in \omega$, $|C_i| \leq \max(|D|, \kappa)$, $D \subseteq C_i \subseteq C_{i+1} \subseteq B$, and $\mathrm{acl}(AC_i) \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C_{i+1})} \mathrm{acl}(B)$. For the base case, set $C_0 = D$, and for the inductive step, we can use the claim in the last paragraph, since $|\mathrm{acl}(AC_i)| \leq \max(\lambda, |L|)$.

Let $C = \bigcup_{i \in \omega} C_i$. Then $|C| \leq \max(|D|, \kappa)$ and $D \subseteq C \subseteq B$. I claim that $A \mathop{\smile\!\!\!\!\!\mid}^r_C B$, i.e., $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(B)$. By finite character for $\mathop{\smile\!\!\!\!\!\mid}^f$, it suffices to show that for every finite tuple a from $\mathrm{acl}(AC)$, $a \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(B)$. But a is already contained in $\mathrm{acl}(AC_i)$ for some $i \in \omega$, and by base monotonicity for $\mathop{\smile\!\!\!\!\!\mid}^f$, $\mathrm{acl}(AC_i) \mathop{\smile\!\!\!\!\!\mid}^f_{\mathrm{acl}(C)} \mathrm{acl}(B)$, since $\mathrm{acl}(C_{i+1}) \subseteq \mathrm{acl}(C) \subseteq \mathrm{acl}(B)$.

Example C.2. If T is NSOP_1, then Kim-independence $\mathop{\smile\!\!\!\!\!\mid}^K$ is only defined over models. However, if we restrict our axioms to the cases where all the sets in the base of $\mathop{\smile\!\!\!\!\!\mid}^*$ are models, then they are all satisfied by $\mathop{\smile\!\!\!\!\!\mid}^K$ in NSOP_1 theories. Strong local character follows from [KRS19, Theorem 1.1] and transitivity from [KR21, Theorem 3.4]. Moreover, in all known examples, $\mathop{\smile\!\!\!\!\!\mid}^K$ agrees over models with an independence relation defined over arbitrary sets, which satisfies all our axioms. Recently, some progress has been made toward showing this holds in general, see [DKR19]. For an example in which $\mathop{\smile\!\!\!\!\!\mid}^K$ does not arise from forking independence in a simple reduct, take the independence relation $\mathop{\smile\!\!\!\!\!\mid}^I$ in the theory $T_{m,n}$ of generic $K_{m,n}$-free incidence structures $(m, n \geq 2)$ defined in [CK19].

In such a situation, where $\mathop{\smile\!\!\!\!\!\mid}^*$ agrees with Kim-independence over models, we can view the main results of this section (reasonable extension, Theorem C.8, the reasonable chain condition, Theorem C.13, and the reasonable independence theorem, Theorem C.15) as saying that we can obtain certain instances of base monotonicity. For example, reasonable extension says that for all $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and for all c, there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\!\!\!\!\!\mid}^K_M bc$, and $a' \mathop{\smile\!\!\!\!\!\mid}^*_{Mb} c$. The last assertion would be immediate in the presence of base monotonicity.

Question C.3. Does every NSOP_1 theory admit an independence relation $\mathop{\smile\hskip -0.9em ^| \ }^*$ satisfying our axioms, such that $\mathop{\smile\hskip -0.9em ^| \ }^*_M = \mathop{\smile\hskip -0.9em ^| \ }^K_M$ for all $M \prec \mathbb{M}$?

We now return to the case of general $\mathop{\smile\hskip -0.9em ^| \ }^*$. When working with this independence relation, we will almost always use the key Lemmas C.6 and C.7 below, instead of appealing directly to the axioms.

Remark C.4. Note that by monotonicity, normality, and transitivity, we have the following form of transitivity: If $A \mathop{\smile\hskip -0.9em ^| \ }^*_{BC} D$ and $B \mathop{\smile\hskip -0.9em ^| \ }^*_C D$, then $A \mathop{\smile\hskip -0.9em ^| \ }^*_C D$.

Though we will not need to use it, it may be worth noting that extension follows from our axioms: Suppose $A \mathop{\smile\hskip -0.9em ^| \ }^*_C B$ and $B \subseteq B'$. By full existence there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\hskip -0.9em ^| \ }^*_{BC} B'$. By invariance, symmetry, and the form of transitivity just observed, $A' \mathop{\smile\hskip -0.9em ^| \ }^*_C B'$.

Definition C.5. A sequence $(b_\alpha)_{\alpha<\mu}$ is $\mathop{\smile\hskip -0.9em ^| \ }^*$-independent over C if $b_\alpha \mathop{\smile\hskip -0.9em ^| \ }^*_C b_{<\alpha}$ for all $0 < \alpha < \mu$. The sequence is $\mathop{\smile\hskip -0.9em ^| \ }^*$-independent over C in the type $p(y)$ if additionally b_α realizes p for all $\alpha < \mu$. The sequence is a $\mathop{\smile\hskip -0.9em ^| \ }^*$-Morley sequence over C if additionally it is C-indiscernible.

Lemma C.6. *Let $p(y)$ be a type over C, and let μ be a cardinal. Then there exists a $\mathop{\smile\hskip -0.9em ^| \ }^*$-Morley sequence over C in p of length μ.*

Proof. Define a sequence $(c_i)_{i\in\omega}$ by induction: pick $b = c_0$ realizing p, and for all $n > 0$, pick $c_n \equiv_C c_0$ such that $c_n \mathop{\smile\hskip -0.9em ^| \ }^*_C c_{<n}$ by full existence. Now let $(b_\alpha)_{\alpha<\mu}$ be a C-indiscernible sequence based on $(c_i)_{i\in\omega}$. By finite character, symmetry, and invariance, $b_\alpha \mathop{\smile\hskip -0.9em ^| \ }^*_C b_{<\alpha}$ for all $0 < \alpha < \mu$. □

Lemma C.7. *Let λ be a cardinal, and let κ be the cardinal provided for λ by strong local character. If μ is a regular cardinal greater than $\max(\kappa, |C|)$, and $(b_\alpha)_{\alpha<\mu}$ is an $\mathop{\smile\hskip -0.9em ^| \ }^*_C$-independent sequence over C, then for any a with $|a| = \lambda$, there exists $\beta < \mu$ such that $a \mathop{\smile\hskip -0.9em ^| \ }^*_C b_{\beta'}$ for all $\beta \leq \beta' < \mu$.*

Proof. By strong local character, there exists B with $C \subseteq B \subseteq C(b_\alpha)_{\alpha<\mu}$ and $|B| = \max(\kappa, |C|)$ such that $a \mathop{\smile\hskip -0.9em ^| \ }^*_B C(b_\alpha)_{\alpha<\lambda}$. Then there exists $\beta < \mu$ such that $B \subseteq C(b_\alpha)_{\alpha<\beta}$, and by monotonicity, for all $\beta \leq \beta' < \mu$, $b_{\beta'} \mathop{\smile\hskip -0.9em ^| \ }^*_C B$, so by symmetry $B \mathop{\smile\hskip -0.9em ^| \ }^*_C b_{\beta'}$. By monotonicity again, $a \mathop{\smile\hskip -0.9em ^| \ }^*_B b_{\beta'}$, and by transitivity, $a \mathop{\smile\hskip -0.9em ^| \ }^*_C b_{\beta'}$. □

In applying Lemma C.7, we will typically just write that a Morley sequence is "long enough", meaning that its length μ satisfies the hypotheses. The only exception is in the proof of Theorem C.15 below, where we have to be a bit more careful with our choice of cardinal.

From now on, we assume T is NSOP_1 and examine the relationship between $\mathop{\smile\hskip -0.9em ^| \ }^K$ and $\mathop{\smile\hskip -0.9em ^| \ }^*$. Our ultimate goal is to prove the "reasonable independence theorem" (Theorem C.15 below). In [KR18], Ramsey and the first-named author proved this theorem in the special case $\mathop{\smile\hskip -0.9em ^| \ }^* = \mathop{\smile\hskip -0.9em ^| \ }^a$, and our proof closely follows that in [KR18].

Theorem C.8 (Reasonable extension)**.** *For all $a \mathop{\smile\hskip -0.9em ^| \ }^K_M b$ and for all c, there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\hskip -0.9em ^| \ }^K_M bc$, and $a' \mathop{\smile\hskip -0.9em ^| \ }^*_{Mb} c$.*

Proof. Let $(c_\alpha)_{\alpha<\mu}$ be a long enough $\mathop{\smile\hskip -0.9em ^| \ }^*_{Mb}$-independent sequence in $\mathrm{tp}(c/Mb)$. By extension for $\mathop{\smile\hskip -0.9em ^| \ }^K$, there exists $a'' \equiv_{Mb} a$ such that $a'' \mathop{\smile\hskip -0.9em ^| \ }^K_M b(c_\alpha)_{\alpha<\mu}$. By Lemma C.7, there exists $\beta < \mu$ with $a'' \mathop{\smile\hskip -0.9em ^| \ }^*_{Mb} c_\beta$. By monotonicity for Kim-independence, we also have $a'' \mathop{\smile\hskip -0.9em ^| \ }^K_M bc_\beta$. Let σ be an automorphism moving c_β to c and fixing Mb, and let $a' = \sigma(a'')$. Then $a' \mathop{\smile\hskip -0.9em ^| \ }^K_M bc$ and $a' \mathop{\smile\hskip -0.9em ^| \ }^*_{Mb} c$. □

Next, we wish to build $\mathop{\smile\hspace{-0.9em}|}^*$-Morley sequences over Ma which are also Kim-independent from a over M. Lemma C.10, which handles the inductive step of the construction, uses the improved independence theorem from [KR18].

Theorem C.9 ([KR18], Theorem 2.13)**.** *If $a_0 \mathop{\smile\hspace{-0.9em}|}^K_M b$, $a_1 \mathop{\smile\hspace{-0.9em}|}^K_M c$, $b \mathop{\smile\hspace{-0.9em}|}^K_M c$ and $a_0 \equiv_M a_1$, then there exists a with $a \equiv_{Mb} a_0$, $a \equiv_{Mc} a_1$, $a \mathop{\smile\hspace{-0.9em}|}^K_M bc$, $b \mathop{\smile\hspace{-0.9em}|}^K_M ac$, and $c \mathop{\smile\hspace{-0.9em}|}^K_M ab$.*

Lemma C.10. *If $a \mathop{\smile\hspace{-0.9em}|}^K_M b$ and $a \mathop{\smile\hspace{-0.9em}|}^K_M c$, then there exists $b' \equiv_{Ma} b$ such that $a \mathop{\smile\hspace{-0.9em}|}^K_M b'c$ and $b' \mathop{\smile\hspace{-0.9em}|}^*_{Ma} c$.*

Proof. We build a sequence $(c_i)_{i<\omega}$ by induction, such that for all $i < \omega$, the following conditions hold:

1. $c_i \equiv_{Ma} c$.
2. $c_i \mathop{\smile\hspace{-0.9em}|}^K_M ac_{<i}$.
3. $c_i \mathop{\smile\hspace{-0.9em}|}^*_{Ma} c_{<i}$.

Set $c_0 = c$, and the induction step follows from Theorem C.8.

Let $I = (c_\alpha)_{\alpha<\kappa}$ be a long enough $\mathop{\smile\hspace{-0.9em}|}^*$-Morley sequence over Ma based on $(c_i)_{i<\omega}$. I claim that there exists $b'' \equiv_{Ma} b$ such that $a \mathop{\smile\hspace{-0.9em}|}^K_M b''c_\alpha$ for all $\alpha < \kappa$. By compactness, it suffices to show that for all $n < \omega$, there exists $b_n \equiv_{Ma} b$ such that $a \mathop{\smile\hspace{-0.9em}|}^K_M b_n c_i$ for all $i < n$. We argue by induction on n, additionally ensuring that $b_n \mathop{\smile\hspace{-0.9em}|}^K_M ac_{<n}$ for all n.

In the base case, we may take $b_0 = b$. Suppose we are given b_n. By extension for Kim-independence, choose $b'_n \equiv_M b_n$ with $b'_n \mathop{\smile\hspace{-0.9em}|}^K_M c_n$. Then since we also have $b_n \mathop{\smile\hspace{-0.9em}|}^K_M ac_{<n}$ and $c_n \mathop{\smile\hspace{-0.9em}|}^K_M ac_{<n}$, we may apply the strengthened independence theorem (Theorem C.9), to find b_{n+1} such that $b_{n+1} \equiv_{Mac_{<n}} b_n$, $b_{n+1} \equiv_{Mc_n} b'_n$, $b_{n+1} \mathop{\smile\hspace{-0.9em}|}^K_M ac_{\leq n}$, and $ac_{<n} \mathop{\smile\hspace{-0.9em}|}^K_M b_{n+1}c_n$. In particular, we also have $b_{n+1} \equiv_{Ma} b_n \equiv_{Ma} b$ and $a \mathop{\smile\hspace{-0.9em}|}^K_M b_{n+1}c_i$ for all $i < n+1$.

Having obtained our b'' as described above, by Lemma C.7 there exists $\beta < \kappa$ such that $b'' \mathop{\smile\hspace{-0.9em}|}^*_{Ma} c_\beta$. Let σ be an automorphism fixing Ma and moving c_β to c, and let $b' = \sigma(b'')$. □

Theorem C.11. *If $a \mathop{\smile\hspace{-0.9em}|}^K_M b$, then for any cardinal κ, there exists a $\mathop{\smile\hspace{-0.9em}|}^*$-Morley sequence over Ma, $I = (b_\alpha)_{\alpha<\kappa}$, with $b_0 = b$ and $a \mathop{\smile\hspace{-0.9em}|}^K_M I$.*

Proof. We mimic the proof of Lemma C.6, building a sequence $(c_i)_{i<\omega}$ in $\mathrm{tp}(b/Ma)$ by induction, such that $c_i \equiv_{Ma} b$, $c_i \mathop{\smile\hspace{-0.9em}|}^*_{Ma} c_{<i}$, and additionally $a \mathop{\smile\hspace{-0.9em}|}^K_M c_{\leq i}$ for all $i < \omega$. Set $c_0 = b$, and for the induction step we can use Lemma C.10, since $a \mathop{\smile\hspace{-0.9em}|}^K_M c_{<i}$ and $a \mathop{\smile\hspace{-0.9em}|}^K_M b$.

Now let $I = (b_\alpha)_{\alpha<\kappa}$ be a $\mathop{\smile\hspace{-0.9em}|}^*$-Morley sequence over Ma based on $(c_i)_{i<\omega}$. After an automorphism fixing Ma, we may assume $b_0 = b$. And $a \mathop{\smile\hspace{-0.9em}|}^K_M I$ by invariance and the finite character of Kim-independence. □

Theorem C.12 is a chain condition for Kim-independence.

Theorem C.12 ([KR20] Proposition 3.20)**.** *Suppose $a \mathop{\smile\hspace{-0.9em}|}^K_M b$ and $I = (b_\alpha)_{\alpha<\kappa}$ is a q-Morley sequence over M with $b_0 = b$, for some global M-invariant type $q(y)$ extending $\mathrm{tp}(b/M)$. Then there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\hspace{-0.9em}|}^K_M I$, and I is Ma'-indiscernible.*

The next theorem strengthens the conclusion that I is Ma'-indiscernible to the conclusion that I is $\mathop{\smile\hspace{-0.9em}|}^*$-Morley over Ma', with the caveat that the indexing of I has to be reversed. For convenience in the application of this theorem, we assume to start with that I is a reverse q-Morley sequence.

Let $p(x)$ be a global M-invariant type. Then a **reverse p-Morley sequence over M** is a sequence $(a_\alpha)_{\alpha<\kappa}$ such that a_α realizes $p|_{Ma_{>\alpha}}$ for all $\alpha < \kappa$. Note any two reverse p-Morley sequences of the same length realize the same type over M, since for any $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$, $\operatorname{tp}(a_{\alpha_n} a_{\alpha_{n-1}} \dots a_{\alpha_1}/M) = p^{\otimes n}|_M$.

Theorem C.13 (Reasonable chain condition)**.** *Suppose $a \mathop{\smile\!\!\!\!|}^K_M b$ and $I = (b_\alpha)_{\alpha<\kappa}$ is a reverse q-Morley sequence over M with $b_0 = b$, for some global M-invariant type $q(y)$ extending $\operatorname{tp}(b/M)$. Then there exists a' such that $a' \equiv_{Mb} a$, $a' \mathop{\smile\!\!\!\!|}^K_M I$, and I is a $\mathop{\smile\!\!\!\!|}^*$-Morley sequence over Ma'.*

Proof. We show by induction on n that there exists $(c_0, \dots, c_n) \models q^{\otimes(n+1)}|_M$ such that $c_i \equiv_{Ma} b$ for all $i \le n$, $c_i \mathop{\smile\!\!\!\!|}^*_{Ma} c_{>i}$ for all $0 \le i < n$, and $a \mathop{\smile\!\!\!\!|}^K_M (c_i)_{i\le n}$.

When $n = 0$, taking $c_0 = b$ suffices. So suppose we are given the tuple $(c_0, \dots, c_n)$ by induction. By Theorem C.11, there is a long enough $\mathop{\smile\!\!\!\!|}^*$-Morley sequence $J = (d_{0,\alpha})_{\alpha<\kappa}$ over Ma in $\operatorname{tp}(b/Ma)$, such that $a \mathop{\smile\!\!\!\!|}^K_M J$.

Let $(d_1, \dots, d_{n+1})$ realize $q^{\otimes(n+1)}|_{MJ}$. Since $d_{0,\alpha} \models q|_M$ for all α, we have $(d_{0,\alpha}, d_1, \dots, d_{n+1}) \models q^{\otimes(n+2)}|_M$ for all α. Let $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$ be such that $\sigma(c_i) = d_{i+1}$ for all $i \le n$, and let $a' = \sigma(a)$. Now $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!|}^K_M J$ (by choice of J), $a' \mathop{\smile\!\!\!\!|}^K_M d_1 \dots d_{n+1}$ (by invariance and induction), and $d_1 \dots d_{n+1} \mathop{\smile\!\!\!\!|}^K_M J$ (since $\operatorname{tp}(d_1 \dots d_{n+1}/MJ)$ extends to a global M-invariant type).

Applying the independence theorem, we find a'' with $a'' \equiv_{MJ} a$, $a'' \equiv_{Md_1\dots d_{n+1}} a'$, and $a'' \mathop{\smile\!\!\!\!|}^K_M J d_1 \dots d_{n+1}$. By invariance, we have $d_i \mathop{\smile\!\!\!\!|}^*_{Ma''} d_{>i}$ for all $1 \le i < n+1$, and by Lemma C.7, there is some $\beta < \kappa$ such that $d_1 \dots d_{n+1} \mathop{\smile\!\!\!\!|}^*_{Ma''} d_{0,\beta}$. Defining $d_0 = d_{0,\beta}$, by symmetry we have $d_i \mathop{\smile\!\!\!\!|}^*_{Ma''} d_{>i}$ for all $0 \le i < n+1$.

It remains to move a'' back to a by an automorphism $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$. The tuple $\sigma(d_0), \dots, \sigma(d_{n+1})$ has the desired properties: $a\sigma(d_0) \equiv_M a''d_0 \equiv_M ad_0 \equiv_M ab$, $a\sigma(d_i) \equiv_M a''d_i \equiv_M a'd_i \equiv ac_{i-1} \equiv ab$ for all $0 < i \le n+1$, $\sigma(d_i) \mathop{\smile\!\!\!\!|}^*_{Ma} \sigma(d_{>i})$ for all $0 \le i < n+1$, and $a \mathop{\smile\!\!\!\!|}^K_M \sigma(d_{\le(n+1)})$.

By compactness, and reversing our indexing, we can find a reverse q-Morley sequence over M, $(c_\alpha)_{\alpha<\kappa}$, such that $c_\alpha \equiv_{Ma} b$ for all $\alpha < \kappa$, $c_\alpha \mathop{\smile\!\!\!\!|}^*_{Ma} c_{<\alpha}$ for all $\alpha < \kappa$, and $a \mathop{\smile\!\!\!\!|}^K_M I'$. In fact, we can assume I' is Ma-indiscernible, by replacing it with an Ma-indiscernible sequence based on it. As I' and I are both reverse q-Morley sequences over M, we can move I' to I by an automorphism $\sigma \in \operatorname{Aut}(\mathbb{M}/M)$, and take $a' = \sigma(a)$. □

Combining the reasonable chain condition with the usual chain condition gives us Lemma C.14, which is the key ingredient in the proof of the reasonable independence theorem.

Lemma C.14. *Suppose $a \mathop{\smile\!\!\!\!|}^K_M b$, and let $p(x)$ and $q(y)$ be global invariant types such that $\operatorname{tp}(a/M) \subseteq p(x)$ and $\operatorname{tp}(b/M) \subseteq q(y)$. Then for any κ and κ', there exist sequences $I = (a_\alpha)_{\alpha<\kappa}$ and $J = (b_\beta)_{\beta<\kappa'}$ such that:*

1. *I is a reverse p-Morley sequence over M.*
2. *J is a reverse q-Morley sequence over M.*
3. *$a_0 = a$ and $b_0 = b$.*
4. *J is $\mathop{\smile\!\!\!\!|}^*$-Morley over Ma.*
5. *I is MJ-indiscernible.*
6. *$I \mathop{\smile\!\!\!\!|}^K_M J$.*

In particular, J is $\mathop{\smile\!\!\!\!|}^*$-Morley over Ma_α for all $\alpha < \kappa$, and $a_\alpha b_\beta \equiv_M ab$ for all $\alpha < \kappa$ and $\beta < \kappa'$.

Proof. First, note that the last claims follow from (3), (4) and (5). For all $\alpha < \kappa$, since J is $\perp^*$-Morley over Ma and $a_\alpha \equiv_{MJ} a$, by invariance J is $\perp^*$-Morley over Ma_α. And for all $\alpha < \kappa$ and $\beta < \kappa'$, $a_\alpha b_\beta \equiv_M ab_\beta \equiv_M ab$, since $b_0 = b$ and J is Ma-indiscernible.

So let $J' = (b'_\beta)_{\beta<\kappa'}$ be any reverse q-Morley sequence over M with $b'_0 = b$. By Theorem C.13, we can find $a' \equiv_{Mb} a$, $a' \perp^K_M J'$, and J' is a $\perp^*$-Morley sequence over Ma'. Now let $I' = (a'_\alpha)_{\alpha<\kappa}$ be any reverse p-Morley sequence over M with $a'_0 = a'$. Since $J \perp^K_M a'$, by Theorem C.12, we can find $J'' = (b''_\beta)_{\beta<\kappa'}$ such that $J'' \equiv_{Ma'} J'$ such that $J'' \perp^K_M I'$ and I' is MJ''-indiscernible.

Now $a'_0 b''_0 = a' b''_0 \equiv_M a' b'_0 = a'b \equiv_M ab$, so we can pick an automorphism σ fixing M and moving a'_0 to a and b''_0 to b. Let $I = \sigma(I')$ and $J = \sigma(J'')$. Conditions (1)-(6) follow from invariance. □

Theorem C.15 (Reasonable independence theorem). *If $a \perp^K_M b$, $a' \perp^K_M c$, $b \perp^K_M c$, and $a \equiv_M a'$, then there exists a'' such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$, and $a'' \perp^K_M bc$, and further $a'' \perp^*_{Mc} b$, $a'' \perp^*_{Mb} c$, and $b \perp^*_{Ma''} c$.*

Proof. Let κ_a, κ_b, and κ_c be the cardinals provided by strong local character for $|a|$, $|b|$, and $|c|$, respectively. Let μ_c be a regular cardinal greater than $\max(\kappa_a, |Mb|)$, let μ_b be a regular cardinal greater than $\max(\kappa_c, |Ma|, \mu_c)$, and let μ_a be a regular cardinal greater than $\max(\kappa_b, |Mc|, \mu_b, \mu_c)$.

Let $p(x)$, $q(y)$, and $r(z)$ be global M-invariant types extending $\mathrm{tp}(a/M) = \mathrm{tp}(a'/M)$, $\mathrm{tp}(b/M)$, and $\mathrm{tp}(c/M)$, respectively.

Apply Lemma C.14 to $q(y)$ and $r(z)$, obtaining sequences $B = (b_\beta)_{\beta<\mu_b}$ and $C = (c_\gamma)_{\gamma<\mu_c}$ where C is $\perp^*$-Morley over Mb and B is MC-indiscernible. Then apply it to $p(x)$ and $q(y)$, obtaining sequences $A = (a_\alpha)_{\alpha<\mu_a}$ and $\widehat{B} = (\widehat{b_\beta})_{\beta<\mu_b}$, where $\widehat{B}$ is $\perp^*$-Morley over Ma and A is $M\widehat{B}$-indiscernible. Finally, apply it to $r(z)$ and $p(x)$, obtaining sequences $\widehat{C} = (\widehat{c_\gamma})_{\gamma<\mu_c}$ and $A' = (a'_i)_{\alpha<\mu_a}$, where A' is $\perp^*$-Morley over Mc and $\widehat{C}$ is MA'-indiscernible.

Since B and $\widehat{B}$ are both reverse q-Morley sequences over M, and C and $\widehat{C}$ are both reverse r-Morley sequences over M, we may assume $\widehat{B} = B$ and $\widehat{C} = C$ at the expense of moving A and A' by automorphisms over M. Note that A and A' are both reverse p-Morley sequences over M, so $A \equiv_M A'$. We also have $A \perp^K_M B$, $A' \perp^K_M C$, and $B \perp^K_M C$.

So we can apply the independence theorem, obtaining a sequence $A'' = (a''_\alpha)_{\alpha<\mu_a}$ such that $A'' \equiv_{MB} A$, $A'' \equiv_{MC} A'$, and $A'' \perp^K_M BC$. This, together with the conclusions of Lemma C.14, ensure that for any $\alpha < \mu_a$, $\beta < \mu_b$, and $\gamma < \mu_c$, we have:

(1) $a''_\alpha \perp^K_M b_\beta c_\gamma$.
(2) $a''_\alpha b_\beta \equiv_M ab$, $a''_\alpha c_\gamma \equiv_M a'c$, and $b_\beta c_\gamma \equiv_M bc$.
(3) A is $\perp^*$-Morley over Mc_γ, B is $\perp^*$-Morley over Ma''_α, and C is $\perp^*$-Morley over Mb_β.

For all $\beta < \mu_b$ and $\gamma < \mu_c$, since A'' is $\perp^*$-Morley over Mc_γ and $\mu_a > \max(\kappa_b, |Mc_\gamma|)$, there exists a $\delta(\beta,\gamma) < \mu_a$ such that $b_\beta \perp^*_{Mc_\gamma} a''_{\delta'}$ for all $\delta' > \delta(\beta,\gamma)$. Now $\Delta = \{\delta(\beta,\gamma) \mid \beta < \mu_b, \gamma < \mu_c\}$ is a set of ordinals of cardinality $\leq \max(\mu_b, \mu_c)$. But μ_a is regular and greater than $\max(\mu_b, \mu_c)$, so Δ is not cofinal in μ_a. Let $\delta'' = \sup \Delta < \mu_a$, and let $a_* = a''_{\delta''}$.

Similarly, for all $\gamma < \mu_c$, since B is $\perp^*$-Morley over Ma_* and $\mu_b > \max(\kappa_c, |Ma_*|)$, there exists a $\delta(\gamma) < \mu_b$ such that $c_\gamma \perp^*_{Ma_*} b_{\delta'}$ for all $\delta' > \delta(\gamma)$. Now $\Delta = \{\delta(\gamma) \mid$

$\gamma < \mu_c\}$ is a set of ordinals of cardinality $\leq \mu_c$. But μ_b is regular and greater than μ_c, so Δ is not cofinal in μ_b. Let $\delta'' = \sup \Delta < \mu_b$, and let $b_* = b_{\delta''}$.

Finally, since C is $\mathop{\smile\!\!\!\!\!\!|}^*$-Morley over Mb_* and $\mu_c > \max(\kappa_a, |Mb_*|)$, there exists a $\delta < \mu_c$ such that $a_* \mathop{\smile\!\!\!\!\!\!|}^*_{Mb_*} c_\delta$. Let $c_* = c_\delta$.

In total, we have $a_* \mathop{\smile\!\!\!\!\!\!|}^K_M b_*c_*$, $a_*b_* \equiv_M ab$, $a_*c_* \equiv_M a'c$, $b_*c_* \equiv_M bc$, $a_* \mathop{\smile\!\!\!\!\!\!|}^*_{Mb_*} c_*$, $c_* \mathop{\smile\!\!\!\!\!\!|}^*_{Ma_*} b_*$, and $b_* \mathop{\smile\!\!\!\!\!\!|}^*_{Mc_*} a_*$. It remains to move b_*c_* back to bc by an automorphism σ fixing M, and let $a'' = \sigma(a_*)$. □

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