

INTERPOLATIVE FUSIONS

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ABSTRACT. We define the interpolative fusion T_{\cup}^* of a family $(T_i)_{i \in I}$ of first-order theories over a common reduct T_{\cap} , a notion that generalizes many examples of random or generic structures in the model-theoretic literature. When each T_i is model-complete, T_{\cup}^* coincides with the model companion of $T_{\cup} = \bigcup_{i \in I} T_i$. By obtaining sufficient conditions for the existence of T_{\cup}^* , we develop new tools to show that theories of interest have model companions.

1. INTRODUCTION

It is often desirable to decompose a mathematical structure into simpler components and analyze the structure in terms of how the components behave and interact. In this paper, we take the components to be reducts of the structure, and we are interested in situations when these reducts interact in a definably random fashion modulo some common agreements. By Theorem 1.1 below, “definably random” in our sense agrees with “generic” in the sense of Robinson: e.g., if the first-order theory of each reduct is model-complete, then the original structure satisfies the model companion of the union of these theories.

In this paper, we introduce interpolative structures and interpolative fusions as an abstract framework for studying structures and theories that exhibit definably random/generic interactions between certain reducts. We observe that many examples in model theory can be put into this framework, and we obtain sufficient conditions for first-order logic to be able to capture the aforementioned randomness/genericity. This yields new strategies to show that certain theories have model companions. In subsequent papers, we will develop more machinery to determine model-theoretic properties of the structure or theory from those of its reducts.

Throughout, I is an index set, L_{\cap} is a first-order language, and $(L_i)_{i \in I}$ is a family of first order languages, all with the same set S of sorts, such that $L_i \cap L_j = L_{\cap}$ for all distinct $i, j \in I$. Let T_i be a (possibly incomplete) L_i -theory for each $i \in I$, and assume that each T_i has the same set T_{\cap} of L_{\cap} -consequences. (This assumption is quite mild: given an arbitrary family of L_i -theories $(T_i)_{i \in I}$, we can extend each T_i with the set of all L_{\cap} -consequences of $\bigcup_{i \in I} T_i$.) Set

$$L_{\cup} = \bigcup_{i \in I} L_i \quad \text{and} \quad T_{\cup} = \bigcup_{i \in I} T_i,$$

and assume that T_{\cup} is consistent. (Alternatively, we could just assume that T_{\cap} is consistent, as these two assumptions are equivalent by Robinson joint consistency; see Corollary 2.2.) Finally, \mathcal{M}_{\cup} is an L_{\cup} -structure, \mathcal{M}_{\square} is the L_{\square} -reduct of \mathcal{M}_{\cup} , and X_{\square} ranges over \mathcal{M}_{\square} -definable sets (with parameters) for $\square \in I \cup \{\cap\}$.

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Suppose $J \subseteq I$ is finite and $X_i \subseteq M^n$ is \mathcal{M}_i -definable for all $i \in J$. Then $(X_i)_{i \in J}$ is **separated** if there is a family $(X^i)_{i \in J}$ of \mathcal{M}_\cap -definable subsets of M^n such that

$$X_i \subseteq X^i \text{ for all } i \in J, \text{ and } \bigcap_{i \in J} X^i = \emptyset.$$

We say \mathcal{M}_\cup is **interpolative** if for all families $(X_i)_{i \in J}$ such that $J \subseteq I$ is finite and $X_i \subseteq M^n$ is \mathcal{M}_i -definable for all $i \in J$, $(X_i)_{i \in J}$ is separated if and only if $\bigcap_{i \in J} X_i = \emptyset$. If the class of interpolative models of T_\cup is elementary with theory T_\cup^* , then we say that T_\cup^* is the **interpolative fusion** (of $(T_i)_{i \in I}$ over T_\cap). We also say that “ T_\cup^* exists” if the class of interpolative T_\cup -models is elementary; this is not automatic, as the definition of an interpolative structure is not first-order.

The name “interpolative fusion” comes from the following resemblance with Craig’s interpolation theorem: When $I = \{1, 2\}$, \mathcal{M}_\cup is interpolative if and only if for all $X_1 \subseteq X_2$, there is an X_\cap such that $X_1 \subseteq X_\cap$ and $X_\cap \subseteq X_2$. A natural generalization of Craig’s theorem allows us to deduce the following theorem in Section 2.1.

Theorem 1.1. *Suppose each T_i is model-complete. Then $\mathcal{M}_\cup \models T_\cup$ is interpolative if and only if it is existentially closed in the class of T_\cup -models. Hence, T_\cup^* is precisely the model companion of T_\cup , if either of these exists.*

Theorem 1.1 can be seen as offering a semantic/geometric characterization of the existentially closed T_\cup -models and providing a path toward obtaining the model companion of T_\cup . For readers who see the notion of interpolative structures as directly reflecting definable randomness and thus fundamental on its own right, Theorem 1.1 can also be read as an explanation for the model-theoretically tame behavior of interpolative structures under favorable conditions. When the T_i are not model-complete, we can still view interpolative models as “relatively existentially closed” and T_\cup^* as the “relative model companion” of T_\cup ; see Theorem 2.7.

Example 1.2. The following are natural examples of interpolative fusions. See Section 2.3 for more details.

- (1) The expansion of a theory T by a generic predicate P [CP98]. This is the interpolative fusion of T with the theory of an infinite and co-infinite predicate P over the theory of an infinite set. More generally, the model companion of the union of any two model complete theories in disjoint languages, as treated by Winkler in [Win75].
- (2) Algebraically closed fields with multiple independent valuations [vdD, Joh16]. This is the interpolative fusion of multiple copies of ACVF (with distinct valuation symbols) over ACF. More generally, the model companion of the theory of fields expanded by multiple structures (valuations, derivations, automorphisms, etc.).
- (3) The group of integers with multiple p -adic valuations [Ad19]. This is similar to the previous example, using a distinct symbol for each p -adic valuation.
- (4) Algebraically closed fields with generic multiplicative circular orders [Tra17]. This is the interpolative fusion of ACF and the theory of circularly ordered multiplicative groups of models of ACF over the theory of multiplicative groups of models of ACF. If \mathbb{F} is the algebraic closure of a finite field and \triangleleft is any multiplicative circular order on \mathbb{F}^\times , then $(\mathbb{F}, \triangleleft)$ is a model of this theory.

The model companion T^* of a theory T of interest is usually not of the form T_{\cup} for any nontrivial choice of $(T_i)_{i \in I}$. However, we can sometimes construct a family $(T_i)_{i \in I}$ of model-complete theories so that T is existentially bi-interpretable with T_{\cup} . One can then deduce that T_{\cup}^* exists and is existentially bi-interpretable with T^* , i.e., T^* is essentially an interpolative fusion. See Section 2.2 for relevant material, in particular, the precise definition of existential bi-interpretation and the fact that existential bi-interpretations preserve existence of model companions.

Example 1.3. Each of the following examples is existentially bi-interpretable with an interpolative fusion. See Section 2.3 for details.

- (1) ACFA, the theory of existentially closed difference fields [CH99]. More generally, the expansion of any theory by a generic automorphism [CP98].
- (2) DCF₀, the theory of differentially closed fields of characteristic 0 [Rob59]. More generally, theories of existentially closed \mathcal{D} -fields in the sense of Moosa and Scanlon [MS14].
- (3) The random graph. More generally, the Fraïssé limits of the classes of finite directed graphs, tournaments, n -hypergraphs, k -colored graphs etc.
- (4) Generic Skolemizations, as defined in [Win75] and further studied in [KR18].

In the examples above, we can deduce existence of T_{\cup}^* from the fact that T has a model companion with results proven in Sections 2.1 and 2.2, but applications of these results can go in the other direction as well. The proof that a theory T of interest has a model companion T^* often involves obtaining a semantic/geometric characterization of the existentially closed models of T and showing that this semantic/geometric characterization is first-order axiomatizable. In many cases the semantic/geometric characterization is close to the notion of an interpolative structure. So we may hope to prove the existence of T_{\cup}^* directly, and then recover the existence of T^* .

The bulk of the technical work of this paper, Section 3 onward, concerns general machinery that aims to actualize the last statement of the preceding paragraph. This also provides the reader with the following new strategy to show that a theory T has a model companion T^* :

- (1) Find $(T_i)_{i \in I}$ so that T_{\cup} is existentially bi-interpretable with T ;
- (2) Use the machinery developed here to conclude that T_{\cup}^* exists and then, with Corollary 2.16, deduce that T^* exists.

In each of the examples above, the collection of definable subsets of T_{\cap} -models is equipped with an ordinal-valued dimension satisfying some natural conditions. We refer to this setting as “pseudo-topological” and investigate it in Section 3. We say that an arbitrary set A is **pseudo-dense** in X_{\cap} if A intersects every \mathcal{M}_{\cap} -definable $Y_{\cap} \subseteq X_{\cap}$ such that $\dim Y_{\cap} = \dim X_{\cap}$, and we say that X_{\cap} is a **pseudo-closure** of A if $A \subseteq X_{\cap}$ and A is pseudo-dense in X_{\cap} . We say that \mathcal{M}_i **has pseudo-closures in \mathcal{M}_{\cap}** if every \mathcal{M}_i -definable set has a pseudo-closure, and we say T_i **has pseudo-closures in T_{\cap}** if the same situation holds for every T_i -model. We say that T_i **defines pseudo-denseness over T_{\cap}** if pseudo-denseness is uniformly definable.

We obtain the following general conditions for the existence of interpolative fusions:

Theorem 1.4. *Suppose \dim is an ordinal dimension on T_{\cap} , T_i defines pseudo-denseness over T_{\cap} , and T_i has pseudo-closures in T_{\cap} for all $i \in I$. Then T_{\cup}^* exists.*

Without using the condition that T_i defines pseudo-denseness over T_\cap for $i \in I$ in the preceding theorem, we can still show that $\mathcal{M}_\cup = T_\cup$ is interpolative if and only if $\bigcap_{i \in J} X_i \neq \emptyset$ whenever $J \subseteq I$ is finite, X_i is \mathcal{M}_i -definable for all $i \in J$, and there is some \mathcal{M}_\cap -definable set X_\cap such that each X_i is pseudo-dense in X_\cap . This property is first-order axiomatizable when pseudo-denseness is definable, and it yields a natural system of “pseudo-topological” axioms for T_\cup^* . The pseudo-topological axioms are essentially identical with known axiomatizations in many examples.

In Sections 4, 5, and 6, we focus on more specific settings applicable to the examples listed above. We show that under further hypotheses on the theories and the notion of dimension, the general conditions of Theorem 1.4 specialize to more familiar notions.

Section 4 treats several settings where T_\cap is equipped with natural topology compatible with the aforementioned dimension; the use of the term “pseudo-topological” is motivated by consideration of these special cases. When T_\cap is o-minimal and \dim is the canonical o-minimal dimension, for example, any theory extending T_\cap defines pseudo-denseness, and T_i has pseudo-closures in T_\cap if and only if T_\cap is an *open core* of T_i , i.e., the closure of any \mathcal{M}_i -definable set is already \mathcal{M}_\cap -definable. This gives the following result.

Theorem 1.5. *Suppose T_\cap is o-minimal. If T_\cap is an open core of each T_i then T_\cup^* exists.*

In Section 5, we show that if T_\cap is \aleph_0 -stable, and \dim is Morley rank, then any theory extending T_\cap has pseudo-closures in T_\cap . The **induced dimension** of a definable set X in a model of T_i is the Morley rank of any pseudo-closure of X . Assuming further that T_\cap defines multiplicity, we show that T_i defines pseudo-denseness if and only if T_i uniformly defines induced dimension. Theorem 1.6 applies to the example of algebraically closed fields with independent valuations.

Theorem 1.6. *Suppose T_\cap is \aleph_0 -stable and defines multiplicity. If each T_i defines induced dimension, then T_\cup^* exists.*

In Section 6 we consider the case when T_\cap is \aleph_0 -stable, \aleph_0 -categorical, and weakly eliminates imaginaries. We prove that T_i defines pseudo-denseness if and only if T_i eliminates \exists^∞ . This applies to the examples of generic predicates, generic Skolemizations, and the random graph, hypergraph, and tournament. It also generalizes Winkler’s result on model companions of disjoint unions of theories [Win75].

Theorem 1.7. *Suppose T_\cap is complete, \aleph_0 -stable, and \aleph_0 -categorical. If T_i^{eq} eliminates \exists^∞ for all i , then T_\cup^* exists. If T_\cap weakly eliminates imaginaries and each T_i eliminates \exists^∞ , then T_\cup^* exists.*

In [vdD, 3.1.20] van den Dries notes a similarity between his main result and Winkler’s theorem and claims that this similarity “. . . suggests a common generalization of Winkler’s and my results”. We believe the present paper provides a moral answer to this suggestion but perhaps not the final answer, as our results do not in fact generalize the main result of [vdD].

1.1. Conventions and notation. Throughout, m and n range over the natural numbers (containing 0), and k and l range over the integers. We work in multi-sorted first-order logic. Our semantics allows empty sorts and empty structures.

Our syntax includes logical constants \top and \perp interpreted as true and false, respectively. We view constant symbols as 0-ary function symbols.

Throughout, L is a language with S the set of sorts. Concepts like variables, functions, formulas, etc. are by default with respect to L . Suppose \mathcal{M} is an L -structure. We use the corresponding capital letter M to denote the S -indexed family $(M_s)_{s \in S}$ of underlying sets of the sorts of \mathcal{M} . By $A \subseteq M$, we mean $A = (A_s)_{s \in S}$ with $A_s \subseteq M_s$ for each $s \in S$. If $A \subseteq M$, then a tuple of elements (possibly infinite) in A is a tuple whose each component is in A_s for some $s \in S$. If $x = (x_j)_{j \in J}$ is a tuple of variables (possibly infinite), we let $A^x = \prod_{j \in J} A_{s(x_j)}$ where $s(x_j)$ is the sort of the variable x_j . If $\varphi(x, y)$ is an L -formula and $b \in M^y$, we let $\varphi(\mathcal{M}, b)$ be the set defined in \mathcal{M} by the $L(b)$ -formula $\varphi(x, b)$. We call such $\varphi(\mathcal{M}, b)$ a definable set in \mathcal{M} or an \mathcal{M} -definable set. Hence, “definable” means “definable, possibly with parameters”. If we wish to exclude parameters, we write “ \emptyset -definable”.

Whenever we consider multiple reducts of a structure, we decorate these reducts with the same decorations as their languages. For example, if $L_0 \subseteq L_1$ are languages, we denote an L_1 -structure by \mathcal{M}_1 , and we denote its reduct $\mathcal{M}_1|_{L_0}$ to L_0 by \mathcal{M}_0 . In this situation, we write “in \mathcal{M}_0 ” to denote that we are evaluating some concept in the reduct.

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2. INTERPOLATIVE FUSIONS AND MODEL COMPANIONS

2.1. Basic results. This section clarifies the relationship between interpolative fusions and model companions of unions of theories. We make use of the definitions and notation set in the introduction.

The name “interpolative fusion” is inspired by a connection to the classical Craig interpolation theorem, which we state below; a proof is given, for example, in [Hod93, Theorem 6.6.3]. It is well-known that in the context of first-order logic, the Craig interpolation theorem is equivalent to Robinson’s joint consistency theorem.

Theorem 2.1. *Suppose L_1 and L_2 are first order languages with intersection L_\cap and φ_i is an L_i -sentence for $i \in \{1, 2\}$. If $\models (\varphi_1 \rightarrow \varphi_2)$ then there is an L_\cap -sentence ψ such that $\models (\varphi_1 \rightarrow \psi)$ and $\models (\psi \rightarrow \varphi_2)$. Equivalently: $\{\varphi_1, \varphi_2\}$ is inconsistent if and only if there is an L_\cap -sentence ψ such that $\models (\varphi_1 \rightarrow \psi)$ and $\models (\varphi_2 \rightarrow \neg\psi)$.*

Our first result is an easy generalization of Theorem 2.1, applicable to our setting.

Corollary 2.2. *For each $i \in I$, let $\Sigma_i(x)$ be a set of L_i -formulas. If $\bigcup_{i \in I} \Sigma_i(x)$ is inconsistent, then there is a finite subset $J \subseteq I$ and an L_\cap -formula $\varphi^i(x)$ for each $i \in J$ such that:*

$$\Sigma_i(x) \models \varphi^i(x) \text{ for all } i \in J, \text{ and } \{\varphi^i(x) \mid i \in J\} \text{ is inconsistent.}$$

Proof. By introducing a new constant symbol for each free variable, we reduce to the case when x is the empty tuple of variables. We may also assume that the sets Σ_i are closed under conjunction. By compactness, if $\bigcup_{i \in I} \Sigma_i$ is inconsistent, then there is a nonempty finite subset $J \subseteq I$ and a formula $\varphi_i \in \Sigma_i$ for all $i \in J$ such that $\{\varphi_i \mid i \in J\}$ is inconsistent.

We argue by induction on the size of J . For the sake of notational simplicity, we suppose $J = \{1, \dots, n\}$. If $n = 1$, then $\{\varphi_1\}$ is inconsistent, and we choose φ^1 to be the contradictory L_\cap -formula \perp . Suppose $n \geq 2$. Then $(\varphi_1 \wedge \dots \wedge \varphi_{n-1})$ is an $(L_1 \cup \dots \cup L_{n-1})$ -sentence and the set

$$\{(\varphi_1 \wedge \dots \wedge \varphi_{n-1}), \varphi_n\} \text{ is inconsistent.}$$

Applying Theorem 2.1, we get a sentence ψ in $L_n \cap (L_1 \cup \dots \cup L_{n-1}) = L_\cap$ such that

$$\models (\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \rightarrow \psi \quad \text{and} \quad \models \varphi_n \rightarrow \neg\psi.$$

Then $\varphi_i \wedge \neg\psi$ is an L_i -sentence for all $1 \leq i \leq n-1$, and $\{\varphi_i \wedge \neg\psi \mid 1 \leq i \leq n-1\}$ is inconsistent. Applying induction, we choose for each $1 \leq i \leq n-1$ an L_\cap -sentence θ^i such that

$$\models (\varphi_i \wedge \neg\psi) \rightarrow \theta^i \text{ for all } 1 \leq i \leq n-1, \text{ and } \models \neg(\theta^1 \wedge \dots \wedge \theta^{n-1}).$$

Finally, set φ^i to be $(\psi \vee \theta^i)$ for $1 \leq i \leq n-1$, and set φ^n to be $\neg\psi$. It is easy to check that all the desired conditions are satisfied. \square

Corollary 2.3 follows immediately from Corollary 2.2 and generalizes Robinson's joint consistency theorem.

Corollary 2.3. *Let $p(x)$ be a complete L_\cap -type, and for all $i \in I$, let $p_i(x)$ be a complete L_i -type such that $p(x) \subseteq p_i(x)$. Then $\bigcup_{i \in I} p_i(x)$ is consistent.*

Corollary 2.2 also allows us to show that families of definable sets that are not separated have ‘‘potentially’’ non-empty intersection.

Lemma 2.4. *Let \mathcal{M}_\cup be an L_\cup -structure, and suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is \mathcal{M}_i -definable for all $i \in J$. The family $(X_i)_{i \in J}$ is separated if and only if for every L_\cup -structure \mathcal{N}_\cup such that $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$, $\bigcap_{i \in J} X_i(\mathcal{N}_\cup) = \emptyset$.*

Proof. Suppose $(X_i)_{i \in J}$ is separated. Then there are \mathcal{M}_\cap -definable X^1, \dots, X^n such that $X_i \subseteq X^i$ for all $i \in J$ and $\bigcap_{i \in J} X^n = \emptyset$. Suppose \mathcal{N}_\cup is a T_\cup -model satisfying $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$. Then $X_i(\mathcal{N}_\cup) \subseteq X^i(\mathcal{N}_\cup)$ for all $i \in J$ and $\bigcap_{i \in J} X^i(\mathcal{N}_\cup) = \emptyset$, so also $\bigcap_{i \in J} X_i(\mathcal{N}_\cup) = \emptyset$.

Conversely, suppose that $\bigcap_{i \in J} X_i(\mathcal{N}_\cup) = \emptyset$ for every L_\cup -structure \mathcal{N}_\cup such that $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$. For each $i \in J$, let $\varphi_i(x, b)$ be an $L_i(M)$ -formula defining X_i . Then the partial type

$$\bigcup_{i \in I} \text{Ediag}(\mathcal{M}_i) \cup \bigcup_{i \in J} \varphi_i(x, b) \quad \text{is inconsistent.}$$

By compactness, there is a finite subset $J' \subseteq I$ with $J \subseteq J'$, a finite tuple $c \in M^y$ and a formula $\psi_i(b, c) \in \text{Ediag}(\mathcal{M}_i)$ for each $i \in J'$ such that

$$\{\psi_i(b, c) \mid i \in J'\} \cup \{\varphi_i(x, b) \mid i \in J\} \quad \text{is inconsistent.}$$

Let φ_i be the true formula \top when $i \in J' \setminus J$, and define $\varphi'_i(x, y, z) = \varphi_i(x, y) \wedge \psi_i(y, z)$ for all $i \in J'$. Note that since $\mathcal{M}_i \models \psi_i(b, c)$,

$$\varphi_i(\mathcal{M}_\cup, b) = \varphi'_i(\mathcal{M}_\cup, b, c).$$

Applying Corollary 2.2, we obtain an inconsistent family $\{\theta_i(x, y, z) \mid i \in J'\}$ of L_\cap -formulas such that $\models \varphi'_i(x, y, z) \rightarrow \theta_i(x, y, z)$ for each $i \in J'$. It follows that

$$\varphi_i(\mathcal{M}_\cup, b, c) \subseteq \theta_i(\mathcal{M}_\cup, b, c) \text{ for all } i \in J', \text{ and } \bigcap_{i \in J'} \theta_i(\mathcal{M}_\cup, b, c) = \emptyset.$$

But since $\varphi_i(\mathcal{M}_\cup, b, c) = M^x$ when $i \in J' \setminus J$, also $\theta_i(\mathcal{M}_\cup, b, c) = M^x$ when $i \in J' \setminus J$. So $\bigcap_{i \in J} \theta_i(\mathcal{M}_\cup, b, c) = \emptyset$ and $(\theta_i(\mathcal{M}_\cup, b, c))_{i \in J}$ separates $(X_i)_{i \in J}$. \square

Remark 2.5. If we change languages in a way that does not change the class of definable sets (with parameters), then the class of interpolative L_\cup -structures is not affected. In particular:

- (1) An interpolative structure \mathcal{M}_\cup remains so after adding new constant symbols naming elements of M to each of the languages L_\square for $\square \in I \cup \{\cup, \cap\}$.
- (2) Suppose L_\square^\diamond is an expansion by definitions of L_\square for $\square \in I \cup \{\cap\}$, $L_i^\diamond \cap L_j^\diamond = L_\cap^\diamond$ for distinct i and j in I , and $L_\cup^\diamond = \bigcup_{i \in I} L_i^\diamond$ is the resulting expansion by definitions of L_\cup . Then any L_\cup -structure \mathcal{M}_\cup has a canonical expansion $\mathcal{M}_\cup^\diamond$ to an L_\cup^\diamond -structure. And \mathcal{M}_\cup is an interpolative L_\cup -structure if and only if $\mathcal{M}_\cup^\diamond$ is an interpolative L_\cup^\diamond -structure.
- (3) An interpolative \mathcal{M}_\cup -structure remains so after replacing each function symbol f in each of the languages L_\square for $\square \in I \cup \{\cup, \cap\}$ by a relation symbol R_f , interpreted as the graph of the interpretation of f in \mathcal{M}_\cup .
- (4) Suppose \mathcal{M}_\cup is an L_\cup -structure. Moving to $\mathcal{M}_\cap^{\text{eq}}$ involves the introduction of new sorts and function symbols for quotients by L_\cap -definable equivalence relations on M . For all $\square \in I \cup \{\cup, \cap\}$, let $L_\square^{\cap\text{-eq}}$ be the language expanding L_\square produced by adding new symbols for L_\square -definable equivalence relations, and let $\mathcal{M}_\square^{\cap\text{-eq}}$ be the natural expansion of \mathcal{M}_\square to $L_\square^{\cap\text{-eq}}$. Then \mathcal{M}_\cup is interpolative if and only if $\mathcal{M}_\cup^{\cap\text{-eq}}$ is interpolative. This follows from the fact that if X_\square is an $\mathcal{M}_\square^{\cap\text{-eq}}$ -definable set in one of the new sorts, corresponding to the quotient of M^x by an L_\cap -definable equivalence relation, then the preimage of X_\square under the quotient is \mathcal{M}_\square -definable.

We now show that interpolative models of T_\cup can be thought of as “relatively existentially closed” models of T_\cup , and the interpolative fusion T_\cup^* can be thought of as the “relative model companion” of T_\cup .

Recall that a theory T is **inductive** if the class of models of T is closed under directed unions. Equivalently, T admits an axiomatization by $\forall\exists$ -sentences.

Fact 2.6 ([Hod93] Theorem 8.3.6). Suppose T is inductive. Then T has a model companion if and only if the class of existentially closed T -models is elementary. If these equivalent conditions hold, then the model companion of T is the theory of existentially closed T -models.

Theorem 2.7. *Suppose $\mathcal{M}_\cup \models T_\cup$.*

- (1) \mathcal{M}_\cup is an interpolative structure if and only if for all \mathcal{N}_\cup such that $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$,

$$\mathcal{N}_\cup \models \exists x \varphi_\cup(x) \quad \text{implies} \quad \mathcal{M}_\cup \models \exists x \varphi_\cup(x)$$

whenever $\varphi_\cup(x)$ is a Boolean combination of L_i -formulas with parameters from M .

- (2) If each L_i is relational and each T_i is model-complete, then the interpolative models of T_\cup are exactly the existentially closed models, and the interpolative fusion of T_\cup is precisely the model companion of T_\cup , if either of these exists.
- (3) There exists an interpolative structure \mathcal{N}_\cup such that $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$, and $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$.
- (4) If T_\cup^* exists, $\mathcal{M}_\cup \models T_\cup^*$, $\mathcal{N}_\cup \models T_\cup^*$, and $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$, then $\mathcal{M} \preceq \mathcal{N}$.

Proof. Part (1) follows immediately from Lemma 2.4 and the definition of interpolative structure.

For part (2), since each T_i is model-complete, whenever $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$ are both models of T_\cup , we have

$$\mathcal{M}_i \preceq \mathcal{N}_i \quad \text{for all } i \in I.$$

As L_\cup is relational, no atomic formula contains symbols from distinct languages, and hence every quantifier-free L_\cup -formula is a Boolean combination of L_i -formulas. Therefore, it follows from (1) that \mathcal{M}_\cup is interpolative if and only if it is existentially closed in the class of T_\cup -models. Each T_i is model-complete and hence inductive, so T_\cup is also inductive. By Fact 2.6 that T_\cup^* is the model companion of T_\cup , if either either of these exists.

For parts (3) and (4), applying Remark 2.5, we can assume by Morleyizing that each T_i admits quantifier elimination and each L_i is relational.

Now (3) follows from the well-known fact that every model of an inductive theory embeds in an existentially closed model [Hod93, Theorem 8.2.1]. In particular, for all $\mathcal{M}_\cup \models T_\cup$, there exists \mathcal{N}_\cup such that $\mathcal{M}_\cup \subseteq \mathcal{N}_\cup$ and $\mathcal{N}_\cup \models T_\cup$ is existentially closed. Then \mathcal{N}_\cup is interpolative by (2), and $\mathcal{M}_i \leq \mathcal{N}_i$ for all i by quantifier elimination.

For (4), if T_\cup^* exists, then T_\cup^* is model-complete by (2). Since \mathcal{M}_i is an L_i -substructure of \mathcal{N}_i for all $i \in I$, \mathcal{M}_\cup is an L_\cup -substructure of \mathcal{N}_\cup , so $\mathcal{M}_\cup \leq \mathcal{N}_\cup$. \square

In the rest of this section, we will do a bit more work to improve Theorem 2.7(2) by removing the hypothesis that the languages L_i are relational.

A formula is **atomic flat** if it is of the form $x_1 = x_2$, $R(x_1, \dots, x_n)$, or $f(x_1, \dots, x_n) = x_{n+1}$, where R is an n -ary relation symbol and f is an n -ary function symbol. Here x_1, \dots, x_{n+1} are arbitrary variables, which need not be distinct. A **flat literal** is an atomic flat formula or the negation of an atomic flat formula. A **flat formula** is a conjunction of finitely many flat literals. An **Eb-formula** is a formula of the form $\exists y \varphi(x, y)$, where $\varphi(x, y)$ is flat and $\models \forall x \exists^{\leq 1} y \varphi(x, y)$. Here x and y may be tuples of variables.

Remark 2.8. The class of Eb-formulas is closed (up to equivalence) under finite conjunction: the conjunction of the Eb-formulas $\exists y_1 \varphi_1(x, y_1)$ and $\exists y_2 \varphi_2(x, y_2)$ is equivalent to the Eb-formula

$$\exists y_1 y_2 (\varphi_1(x, y_1) \wedge \varphi_2(x, y_2)).$$

Lemma 2.9 is essentially [Hod93, Thm 2.6.1]. Hodges uses “unnested” for “flat”.

Lemma 2.9. *Every literal (atomic or negated atomic formula) is logically equivalent to an Eb-formula.*

Proof. We first show that for any term $t(x)$, with variables $x = (x_1, \dots, x_n)$, there is an associated Eb-formula $\varphi_t(x, y)$ such that $\varphi_t(x, y)$ is logically equivalent to $t(x) = y$. We apply induction on terms. For the base case where $t(x)$ is the variable x_k , we let $\varphi_t(x, y)$ be $x_k = y$. Now suppose $t_1(x), \dots, t_m(x)$ are terms and f is an m -ary function symbol. Then $\varphi_{f(t_1, \dots, t_m)}$ is the Eb-formula equivalent to

$$\exists z_1 \dots z_m \left[\bigwedge_{i=1}^m \varphi_{t_i}(x, z_i) \wedge (f(z_1, \dots, z_m) = y) \right].$$

We now show that every atomic or negated atomic formula is equivalent to an Eb-formula. Suppose $t_1(x), \dots, t_m(x)$ are terms and R is either an m -ary relation symbol or $=$ (in the latter case, we have $m = 2$). Then the atomic formula

$R(t_1(x), \dots, t_m(x))$ is equivalent to

$$\exists y_1 \dots y_m \left[\bigwedge_{i=1}^m \varphi_{t_i}(x, y_i) \wedge R(y_1, \dots, y_m) \right].$$

Negated atomic formulas can be treated similarly. \square

Corollary 2.10. *Every quantifier-free formula is logically equivalent to a finite disjunction of $E\mathfrak{b}$ -formulas.*

Proof. Suppose $\varphi(x)$ is quantifier-free. Then $\varphi(x)$ is equivalent to a formula in disjunctive normal form, i.e., a finite disjunction of finite conjunctions of literals. Applying Lemma 2.9 to each literal and using Remark 2.8, we find that $\varphi(x)$ is equivalent to a finite disjunction of $E\mathfrak{b}$ -formulas. \square

Remark 2.11. Any flat literal L_{\cup} -formula is an L_i -formula for some $i \in I$. As a consequence, if $\varphi(x)$ is a flat L_{\cup} -formula, then there is some finite $J \subseteq I$ and a flat L_i -formula $\varphi_i(x)$ for all $i \in J$ such that $\varphi(x)$ is logically equivalent to $\bigwedge_{i \in J} \varphi_i(x)$.

We obtain a restatement of Theorem 1.1 from the introduction:

Theorem 2.12. *Suppose each T_i is model-complete. Then $\mathcal{M}_{\cup} \models T_{\cup}$ is interpolative if and only if it is existentially closed in the class of T_{\cup} -models. Hence, T_{\cup}^* is precisely the model companion of T_{\cup} , if either of these exists.*

Proof. We prove the first statement. Let $\mathcal{M}_{\cup} \models T_{\cup}$ be existentially closed. Suppose $J \subseteq I$ is finite and $\varphi_i(x)$ is an $L_i(M)$ -formula for each $i \in J$ such that $(\varphi_i(\mathcal{M}_{\cup}))_{i \in J}$ is not separated. We may assume each $\varphi_i(x)$ is existential, as T_i is model-complete. Lemma 2.4 gives a T_{\cup} -model \mathcal{N}_{\cup} extending \mathcal{M}_{\cup} such that $\mathcal{N}_{\cup} \models \exists x \bigwedge_{i \in J} \varphi_i(x)$. As \mathcal{M}_{\cup} is existentially closed and each φ_i is existential, we have $\mathcal{M}_{\cup} \models \exists x \bigwedge_{i \in J} \varphi_i(x)$. Thus \mathcal{M}_{\cup} is interpolative.

Now suppose $\mathcal{M}_{\cup} \models T_{\cup}$ is interpolative. Suppose $\psi(x)$ is a quantifier-free $L_{\cup}(M)$ -formula and \mathcal{N}_{\cup} is a T_{\cup} -model extending \mathcal{M}_{\cup} such that $\mathcal{N}_{\cup} \models \exists x \psi(x)$. Applying Corollary 2.10, $\psi(x)$ is logically equivalent to a finite disjunction of $E\mathfrak{b}$ -formulas $\bigvee_{k=1}^n \exists y_k \psi_k(x, y_k)$. Then for some k , $\mathcal{N}_{\cup} \models \exists x \exists y_k \psi_k(x, y_k)$. By Remark 2.11, the flat $L_{\cup}(M)$ -formula $\psi_k(x, y_k)$ is equivalent to a conjunction $\bigwedge_{i \in J} \varphi_i(x, y_k)$ where $J \subseteq I$ is finite and $\varphi_i(x, y_k)$ is a flat $L_i(M)$ -formula for each $i \in J$. So $\mathcal{N}_{\cup} \models \exists x \exists y_k \bigwedge_{i \in J} \varphi_i(x, y_k)$. As each T_i is model-complete, we have $\mathcal{M}_i \preceq \mathcal{N}_i$ for all $i \in I$. By Lemma 2.4, the sets defined by $\varphi_i(x, y_k)$ are not separated, and since \mathcal{M}_{\cup} is interpolative, $\mathcal{M}_{\cup} \models \exists x \exists y_k \bigwedge_{i \in J} \varphi_i(x, y_k)$. So $\mathcal{M}_{\cup} \models \exists x \psi(x)$. This shows \mathcal{M}_{\cup} is existentially closed.

Each T_i is model-complete and hence inductive, so T_{\cup} is inductive. Using Fact 2.6, we get the second statement as a consequence of the first statement. \square

2.2. Existential interpretations. In this section, we recall some standard facts about interpretations. We pay special attention to the case of existential bi-interpretations between inductive theories, which preserve the existence of model companions. This allows us to sometimes identify the model companion T^* of a theory T of interest with an interpolative fusion of simpler theories, via a bi-interpretation. We keep the notational conventions from the introduction.

Let T be an L -theory and T' be an L' -theory. An **interpretation** of T' in T , $F: T' \rightsquigarrow T$, consists of the following data:

- (1) For every sort s' in L' , an L -formula $\varphi_{s'}(x_{s'})$ and an L -formula $E_{s'}(x_{s'}, x_{s'}^*)$.

- (2) For every relation symbol R' in L' of type (s'_1, \dots, s'_n) in L' , an L -formula $\varphi_{R'}(x_{s'_1}, \dots, x_{s'_n})$.
- (3) For every function symbol f' in L' of type $(s'_1, \dots, s'_n) \rightarrow s'$ in L' , an L -formula $\varphi_{f'}(x_{s'_1}, \dots, x_{s'_n}, x_{s'})$.

We then require that for every model $\mathcal{M} \models T$, the formulas above define an L' -structure $\mathcal{M}' \models T'$ in the natural way. See [Hod93, Section 5.3] for details. For every sort s' in L' , the underlying set $M'_{s'}$ of the s' sort in \mathcal{M}' is the quotient of $\varphi_{s'}(\mathcal{M})$ by the equivalence relation defined by $E_{s'}$. We write $\pi_{s'}$ for the surjective quotient map $\varphi_{s'}(\mathcal{M}) \rightarrow M'_{s'}$. We sometimes denote \mathcal{M}' by $F(\mathcal{M})$.

An interpretation $F: T \rightsquigarrow T'$ is an **existential interpretation** if for each sort s' in L' , the L -formula $\varphi_{s'}(x_{s'})$ is T -equivalent to an existential formula, and all other formulas involved in the interpretation and their negations (i.e., the formulas $E_{s'}$, $\neg E_{s'}$, $\varphi_{R'}$, $\neg \varphi_{R'}$, $\varphi_{f'}$, and $\neg \varphi_{f'}$) are also T -equivalent to existential formulas.

Lemma 2.13. *Suppose $F: T \rightsquigarrow T'$ is an existential interpretation. Let $\varphi'(y)$ be a quantifier-free L' -formula, where $y = (y_1, \dots, y_n)$ and y_i is a variable of sort s'_i . Then there is an existential L -formula $\widehat{\varphi}(x_{s'_1}, \dots, x_{s'_n})$ such that for every $\mathcal{M} \models T$ and every tuple $a = (a_1, \dots, a_n)$ with $a_i \in \varphi_{s'_i}(\mathcal{M})$, $\mathcal{M} \models \widehat{\varphi}(a)$ if and only if $F(\mathcal{M}) \models \varphi'(\pi_{s'_1}(a_1), \dots, \pi_{s'_n}(a_n))$.*

Proof. By Corollary 2.10, $\varphi'(y)$ is equivalent to a finite disjunction of Eb -formulas. In the proof of [Hod93, Theorem 5.3.2], Hodges gives an explicit translation from L' -formulas to L -formulas. We apply this translation and observe that because F is an existential interpretation, the translation of a finite disjunction of Eb -formulas is an existential L -formula. \square

A **bi-interpretation** (F, G, η, η') between T and T' consists of an interpretation $F: T \rightsquigarrow T'$ and an interpretation $G: T' \rightsquigarrow T$, together with L -formulas and L' -formulas (one for each sort in L and L' , respectively) defining for each $\mathcal{M} \models T$ and each $\mathcal{N}' \models T'$ isomorphisms

$$\eta_{\mathcal{M}} : \mathcal{M} \rightarrow G(F(\mathcal{M})) \quad \text{and} \quad \eta'_{\mathcal{N}'} : \mathcal{N}' \rightarrow F(G(\mathcal{N}')).$$

See [Hod93, Section 5.4] for the precise definition. Such a bi-interpretation is **existential** if F and G are each existential interpretations, and moreover the L -formulas and L' -formulas defining the isomorphisms are existential. If there is an existential bi-interpretation between T and T' , we say that T and T' are existentially bi-interpretable. The next lemma is [Hod93, Exercise 5.4.3].

Lemma 2.14. *Suppose $F: T \rightsquigarrow T'$ is existential. Then F induces a functor from the category of models of T and embeddings to the category of models of T' and embeddings. Suppose moreover that (F, G, η, η') is an existential bi-interpretation between T and T' . Then the induced functors form an equivalence of categories; in particular, for every L -embedding $f : \mathcal{M} \rightarrow \mathcal{N}$, the following diagram, which expresses that η is a natural isomorphism from the identity functor to $G \circ F$, commutes:*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\eta_{\mathcal{N}}} & G(F(\mathcal{N})) \\ f \uparrow & & \uparrow G(F(f)) \\ \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & G(F(\mathcal{M})) \end{array}$$

Proposition 2.15. *Suppose (F, G, η, η') is an existential bi-interpretation between T and T' . Then \mathcal{M} is an existentially closed model of T if and only if $F(\mathcal{M})$ is an existentially closed model of T' .*

Proof. It suffices to show that if $F(\mathcal{M})$ is an existentially closed model of T' , then \mathcal{M} is an existentially closed model of T . Indeed, by symmetry it follows that if $G(\mathcal{N}')$ is an existentially closed model of T , then \mathcal{N}' is an existentially closed model of T' . And then, since $\eta_{\mathcal{M}} : \mathcal{M} \rightarrow G(F(\mathcal{M}))$ is an isomorphism, if \mathcal{M} is existentially closed, then $F(\mathcal{M})$ is existentially closed.

So assume that $F(\mathcal{M})$ is an existentially closed model of T' . Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding of T -models, and let $\varphi(y)$ be a quantifier-free formula with parameters from \mathcal{M} that is satisfied in \mathcal{N} . By commutativity of the diagram in Lemma 2.14, after moving the parameters of $\varphi(y)$ into $G(F(\mathcal{M}))$ by the isomorphism $\eta_{\mathcal{M}}$, we find that $\varphi(y)$ is satisfied in $G(F(\mathcal{N}))$, and it suffices to show that it is satisfied in $G(F(\mathcal{M}))$.

By Lemma 2.13, there is an existential L' -formula $\widehat{\varphi}'(x)$ with parameters from $F(\mathcal{M})$ such that $F(\mathcal{N}) \models \widehat{\varphi}'(a)$ if and only if $G(F(\mathcal{N})) \models \varphi(b)$, where b is the image of a under the appropriate π_s quotient maps. Writing $\widehat{\varphi}'(x)$ as $\exists z \psi'(x, z)$, we have $F(\mathcal{N}) \models \psi'(a, c)$ for some c , where a is any preimage of the tuple from $G(F(\mathcal{N}))$ satisfying $\varphi(y)$. But since $F(\mathcal{M})$ is existentially closed, there are some a^* and c^* in $F(\mathcal{M})$ such that $F(\mathcal{M}) \models \psi'(a^*, c^*)$, so $F(\mathcal{M}) \models \widehat{\varphi}'(a^*)$, and it follows that $\varphi(y)$ is satisfied in $G(F(\mathcal{M}))$, as desired. \square

Corollary 2.16. *Suppose T and T' are inductive, and (F, G, η, η') is an existential bi-interpretation between T and T' . Then T has a model companion T^* if and only if T' has a model companion $(T')^*$. Further, (F, G, η, η') induces an existential bi-interpretation between T^* and $(T')^*$ when they exist.*

In particular, if T_i is model-complete for all $i \in I$, T is an inductive theory, and (F, G, η, η') is an existential bi-interpretation between T and T_{\cup} , then T has a model companion T^ if and only if the interpolative fusion T_{\cup}^* exists. Further, (F, G, η, η') induces an existential bi-interpretation between T^* and T_{\cup}^* when they exist.*

Proof. Suppose T has a model companion T^* . By [Hod93, Theorem 5.3.2], for every L -sentence $\varphi \in T^*$, there is an L' -sentence φ' , such that for all $\mathcal{M}' \models T'$, $\mathcal{M}' \models \varphi'$ if and only if $G(\mathcal{M}') \models \varphi$. Let $(T')^* = T' \cup \{\varphi' \mid \varphi \in T^*\}$. Then $\mathcal{M}' \models (T')^*$ if and only if $\mathcal{M}' \models T'$ and $G(\mathcal{M}') \models T^*$. By Proposition 2.15 and Fact 2.6, $\mathcal{M}' \models (T')^*$ if and only if \mathcal{M}' is an existentially closed model of T' . So $(T')^*$ is the model companion of T' by Fact 2.6. Proposition 2.15 further implies that $\mathcal{M} \models T^*$ if and only if $F(\mathcal{M}) \models (T')^*$. So (F, G, η, η') induces an existential bi-interpretation between the model companions.

The application to interpolative fusions then follows immediately from the first statement and Theorem 2.12. \square

2.3. Examples. We now give a more detailed description of the items of Examples 1.2 and 1.3. The first four examples are interpolative fusions from the literature. The last four are well-known theories that we show are bi-interpretible with interpolative fusions of simpler theories. All of these examples are natural, as each is either the model companion of a natural theory or is the theory of an interpolative structure found in the wild.

Generic predicates and disjoint unions of theories. The simplest case of the notion of interpolative fusion is when the L_i are pairwise disjoint. This case was treated by Winkler [Win75]. In our notation, Winkler showed that if T_\cap is the empty language, T_\cap is the theory of an infinite set, and each T_i is model-complete and eliminates \exists^∞ , then T_\cup has a model companion, which is T_\cup^* by Theorem 1.1. This result easily generalizes via Morleyization to show that T_\cup^* exists when each T_i eliminates \exists^∞ and T_\cap is the theory of an infinite set.

If $I = \{1, 2\}$ and T_2 is the theory of an infinite set equipped with an infinite and co-infinite unary predicate, then T_\cup^* is well-known as the expansion of T_1 by a generic unary predicate (see [CP98]).

Algebraically closed fields with independent valuations. Let T_\cap be the theory of algebraically closed fields and T_i be the theory of an algebraically closed field equipped with a non-trivial valuation v_i for each $i \in I$. (We let L_i be the language of fields extended by a unary predicate for the valuation ring of v_i for each $i \in I$.) Then T_\cup has a model companion T_\cup^* , which is the theory of an algebraically closed field K equipped with a family $(v_i)_{i \in I}$ of pairwise independent valuations. The theory T_\cup^* is studied in [vdD] and [Joh16].

The group of integers with p -adic valuations. Given integers k, l we write $k \leq_p l$ if the p -adic valuation of k is no greater than the p -adic valuation of l . Let I be the set of primes, T_\cap be the theory of $(\mathbb{Z}, +)$, and T_p be the theory of $(\mathbb{Z}, +, \leq_p)$ for $p \in I$. Then the theory of $(\mathbb{Z}, +, (\leq_p)_{p \in I})$ is the model companion T_\cup^* of T_\cup , see [Ad19]. In particular, $(\mathbb{Z}, +, (\leq_p)_{p \in I})$ is an interpolative structure.

The algebraic closure of a finite field with a multiplicative circular orders. Let $(\mathbb{F}, +, \times)$ range over algebraic closures of finite fields, and let \triangleleft range over multiplicatively invariant circular orders on \mathbb{F}^\times (see [Tra17] for the definition). With $I = \{1, 2\}$, T_\cap the common theory of all (\mathbb{F}, \times) , T_1 the common theory of all $(\mathbb{F}, +, \times)$, and T_2 the common theory of all $(\mathbb{F}, \times, \triangleleft)$, it follows from [Tra17] that T_\cup has a model companion T_\cup^* , which is the common theory of all $(\mathbb{F}, +, \times, \triangleleft)$. In this case, T_2 is not model-complete, so the model-completeness assumption in Theorem 1.1 is sufficient but not necessary. In fact, T_\cup^* is the model companion of the theory of algebraically closed fields with a multiplicative circular order. The initial motivation of this paper was to find a common generalization of this example, algebraically closed fields with independent valuations, and generic predicates.

Generic automorphisms. Let T_0 be a one-sorted model-complete theory, and let T be the theory whose models are (\mathcal{M}_0, σ) , where $\mathcal{M}_0 \models T_0$ and σ is an automorphism of \mathcal{M}_0 . If T_0 is ACF, then T has a model companion T^* , which is called ACFA [CH99]. In general, the question of existence of T^* is subtle.

Suppose $\mathcal{M} = (\mathcal{M}_0, \sigma)$ is a model of T . Then \mathcal{M} is bi-interpretable with a two-sorted structure $(\mathcal{M}_0, \mathcal{N}_0; \iota_1, \iota_2)$, where ι_1 and ι_2 are isomorphisms $\mathcal{M}_0 \rightarrow \mathcal{N}_0$. In one direction, we can take $\mathcal{N}_0 = \mathcal{M}_0$, $\iota_1 = \text{id}$, and $\iota_2 = \sigma$. In the other direction, we can take $\sigma = \iota_1^{-1} \circ \iota_2$. Note also that $(\mathcal{M}_0, \mathcal{N}_0; \iota_1)$ and $(\mathcal{M}_0, \mathcal{N}_0; \iota_2)$ are both bi-interpretable with \mathcal{M}_0 .

Let $I = \{1, 2\}$, let T_\cap be the two-sorted theory of two disjoint elementarily equivalent T_0 -models, and let T_i be the two-sorted theory of two T_0 -models with an isomorphism ι_i between them. Then T_\cup is the theory of two T_0 -models with two isomorphisms ι_1 and ι_2 between them. As described in the previous paragraph,

T and T_{\cup} are existentially bi-interpretable. By Corollary 2.16, if T^* exists, then T_{\cup}^* exists and is existentially bi-interpretable with T^* . Furthermore, T_1 and T_2 are both existentially bi-interpretable with T_0 . So, for example, ACFA is existentially bi-interpretable with an interpolative fusion of two theories, each of which is existentially bi-interpretable with ACF.

Differentially closed fields and free \mathcal{D} -fields. Let T be the theory of differential fields of characteristic 0 whose underlying field is algebraically closed. The theory DCF_0 of differentially closed fields is the model companion T^* of T [Rob59].

Let (K, ∂) be a model of T . Let $D = K[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers over K , and $\pi : D \rightarrow K$ be the residue map $\pi(a + b\varepsilon) = a$. Then (K, ∂) is existentially bi-interpretable with the two-sorted structure $(K, D; \pi, \sigma_1, \sigma_2)$ where $\sigma_1 : K \rightarrow D$ is given by $a \mapsto a + 0\varepsilon$ and $\sigma_2 : K \rightarrow D$ is given by $a \mapsto a + \partial(a)\varepsilon$. It can also be verified that $(K, D; \pi, \sigma_1)$ and $(K, D; \pi, \sigma_2)$ are isomorphic and mutually existentially interpretable with K .

Let $I = \{1, 2\}$, $T_{\cap} = \text{Th}(K, D; \pi)$, $T_1 = \text{Th}(K, D; \pi, \sigma_1)$, and $T_2 = \text{Th}(K, D; \pi, \sigma_2)$. Then T_{\cup} is existentially bi-interpretable with T , so T_{\cup}^* exists and is existentially bi-interpretable with $T^* = \text{DCF}_0$. Moreover, T_2 is a copy of T_1 , and both are existentially bi-interpretable with ACF_0 . Thus, DCF_0 is existentially bi-interpretable with the interpolative fusion of two theories each of which is existentially bi-interpretable with ACF_0 .

ACFA and DCF_0 admit a common generalization, the theories of existentially closed \mathcal{D} -fields of Moosa and Scanlon [MS14]. These theories are also essentially interpolative fusions; they will be discussed in future work.

The random graph and related structures. Let T be the theory of infinite graphs with infinitely many edges. The theory of the random graph is the model companion T^* of T . Let S_V be the quotient $\{(v_1, v_2) \in V^2 : v_1 \neq v_2\} / \sim$, where the equivalence relation \sim is defined by $(v_1, v_2) \sim (v'_1, v'_2)$ if and only if $\{v_1, v_2\} = \{v'_1, v'_2\}$. Let $\pi_V : \{(v_1, v_2) \in V^2 : v_1 \neq v_2\} \rightarrow S_V$ be the quotient map seen as a relation on $V^2 \times S_V$, and E_V the image of E under π_V , considered as a relation on S_V . Then $(V, S_V; \pi_V, E_V)$ is existentially bi-interpretable with $(V; E)$.

Now suppose $I = \{1, 2\}$, T_{\cap} is the common theory of (V, S_V) , T_1 is the common theory of $(V, S_V; \pi_V)$, and T_2 is the common theory of $(V, S_V; E_V)$. It can be checked that T_1 and T_2 are model-complete. Then T_{\cup} is existentially bi-interpretable with T . Hence T_{\cup}^* exists, and is existentially bi-interpretable T^* . It can also be shown that T_1 and T_2 are interpretable in the theory of infinite sets, so the theory of random graphs is up to existential bi-interpretation the interpolative fusion of two theories interpretable in the theory of infinite sets. Similar ideas also work with directed graphs, tournaments, n -hypergraphs, etc.

Generic Skolemizations. Suppose L_0 is a one-sorted language and T_0 is a model-complete L_0 -theory. Let $\varphi(x, y)$ be an L_0 -formula with y a single variable and x a tuple of variables of length $n > 0$, such that $T_0 \models \forall x \exists^{\geq k} y \varphi(x, y)$ for all k . Let $L = L_0 \cup \{f\}$ with f a new n -ary function symbol, and $T = T_0 \cup \{\forall x \varphi(x, f(x))\}$. Then T is the “ φ -Skolemization” of T_0 . Winkler showed that T has a model companion T^* , the “generic φ -Skolemization” of T_0 , when T_0 eliminates \exists^{∞} [Win75]. Generic Skolemizations were also studied more recently in [KR18].

Let $\mathcal{M} = (\mathcal{M}_0, f)$ range over the models of T . For each such \mathcal{M} , let $E \subseteq M^{n+1}$ be defined in \mathcal{M}_0 by φ , let $p_x : E \rightarrow M^n$ and $p_y : E \rightarrow M$ be the projection on the first n coordinates and the last coordinate, respectively, and let $g : M^n \rightarrow E$ be the function $a \mapsto (a, f(a))$. Note that p_x is an infinite-to-one surjection onto M^n , and g is a section of p_x . Then the two-sorted structure $(\mathcal{M}_0, E; p_x, p_y)$ is existentially bi-interpretable with \mathcal{M}_0 , and $(\mathcal{M}_0, E; p_x, p_y, g)$ is existentially bi-interpretable with \mathcal{M} .

Let $I = \{1, 2\}$, let T_\cap the common theory of all $(M, E; p_x)$, let T_1 be the common theory of all $(\mathcal{M}_0, E; p_x, p_y)$, and let T_2 be the common theory of all $(M, E; p_x, g)$. Then T_1 and T_2 are model-complete, and T_\cup is existentially bi-interpretable with T . Hence T_\cup^* exists and is bi-interpretable with T^* when T_0 eliminates \exists^∞ . It can also be checked that T_1 is existentially bi-interpretable with T_0 , and T_2 is interpretable in the theory of an infinite set.

3. PSEUDO-TOPOLOGICAL BASE

In many examples, including most of those in Section 2.3, the existence of the interpolative fusion can be explained by a simple idea: in the presence of a well-behaved notion of dimension on L_\cap -definable sets, a family $(X_i)_{i \in I}$ of L_i -definable sets is not separated (as defined in Section 2.1) if and only if they are simultaneously “dense” in some L_\cap -definable set.

In this section, we provide a general framework abstracting these examples. Under very general hypotheses, we can use this framework to prove the existence of the interpolative fusion and provide an explicit $\forall\exists$ -axiomatization.

Throughout the rest of the paper, we introduce the following additional notational conventions: L' is a first-order language extending L with the same sorts as L , \mathcal{M}' is an L' -structure and \mathcal{M} is its L -reduct, T' is an L' -theory, and T is the set of L -consequences of T' . Moreover, we assume the existence of a function \dim that assigns an ordinal or the formal symbol $-\infty$ to each \mathcal{M} -definable set whenever $\mathcal{M} \models T$, so that for all \mathcal{M} -definable $X, X_1, X_2 \subseteq M^x$:

- (1) $\dim X_1 \cup X_2 = \max\{\dim X_1, \dim X_2\}$;
- (2) $\dim X = -\infty$ if and only if $X = \emptyset$;
- (3) if X is finite then $\dim X = 0$;
- (4) $\dim X = \dim X(\mathcal{N})$ for any elementary extension \mathcal{N} of \mathcal{M} .

We call such a function \dim an **ordinal dimension** on T . Examples include Morley rank on an \aleph_0 -stable theory, U-rank on a superstable theory, SU-rank on a supersimple theory, and o-minimal dimension on an o-minimal theory. In fact, most natural examples of tame theories are equipped with a dimension, which is often canonical.

Let X be a definable subset of M^x and A an arbitrary subset of M^x . Then A is **pseudo-dense** in X if A intersects every nonempty definable $X' \subseteq X$ such that $\dim X' = \dim X$. We call X a **pseudo-closure** of A if $A \subseteq X$ and A is pseudo-dense in X . Lemma 3.1 collects a few easy facts about pseudo-denseness, the proofs of which we leave to the readers.

Lemma 3.1. *Let X and X' be \mathcal{M} -definable subsets of M^x , and let A be an arbitrary subset of M^x . Then:*

- (1) *When $\dim X = 0$, A is pseudo-dense in X if and only if $X \subseteq A$.*

- (2) If A is pseudo-dense in X , $X' \subseteq X$, and $\dim X' = \dim X$, then A is pseudo-dense in X' .
- (3) If $X^1, \dots, X^n \subseteq X$ are \mathcal{M} -definable, with $\dim X^i = \dim X$ for all i , and

$$\dim X \setminus (X^1 \cup \dots \cup X^n) < \dim X,$$

then A is pseudo-dense in X if and only if A is pseudo-dense in each X^i .

If in addition X is a pseudo-closure of A , then:

- (4) $A \subseteq X'$ implies $\dim X \leq \dim X'$.
- (5) If X' is another pseudo-closure of A , then $\dim(X \triangle X') < \dim X = \dim X'$.
- (6) If $A \subseteq X' \subseteq X$ then X' is a pseudo-closure of A .

We next introduce equivalent characterizations of interpolative structures under extra assumptions. Let \mathcal{C} be a collection of \mathcal{M} -definable sets. Then X admits a **\mathcal{C} -decomposition** if there is a finite family $(X_j)_{j \in J}$ from \mathcal{C} such that

$$\dim \left(X \triangle \bigcup_{j \in J} X^j \right) < \dim X,$$

and X admits a **\mathcal{C} -patching** if there is a finite family $(X^j, Y^j, f^j)_{j \in J}$ such that for all $j, j' \in J$:

- (1) Y^j is in \mathcal{C} .
- (2) $f^j : X^j \rightarrow Y^j$ is an \mathcal{M} -definable bijection.
- (3) And finally, $\dim \left(X \triangle \bigcup_{j \in J} X^j \right) < \dim X$.

We say that \mathcal{C} is a **pseudo-cell collection** for \mathcal{M} if either every \mathcal{M} -definable set admits a \mathcal{C} -decomposition or \dim is preserved under \mathcal{M} -definable bijections and every \mathcal{M} -definable set admits a \mathcal{C} -patching. Examples include the collection of irreducible varieties in an algebraically closed field and the collection of cells in an o-minimal structure.

Suppose \dim is an ordinal dimension on $\text{Th}(\mathcal{M}_\cap)$ and \mathcal{C} is a collection of \mathcal{M}_\cap -definable sets. We say \mathcal{M}_\cup is **\mathcal{C} -weakly interpolative** if for all finite $J \subseteq I$, $X_\cap \in \mathcal{C}$, and $(X_i)_{i \in J}$, where X_i is \mathcal{M}_i -definable and pseudo-dense in X_\cap , we have $\bigcap_{i \in J} X_i \neq \emptyset$. If \mathcal{C} is the collection of all \mathcal{M}_\cap -definable sets then we say that \mathcal{M}_\cup is **weakly interpolative**. It is easy to see that if \mathcal{C} is a collection of pseudo-cells, then \mathcal{M}_\cup is weakly interpolative if and only if \mathcal{M}_\cup is \mathcal{C} -weakly interpolative.

Suppose \mathcal{M}' is an expansion of \mathcal{M} . Then \mathcal{M}' has **pseudo-closures in \mathcal{M}** (with respect to the ordinal dimension \dim on \mathcal{M}) if every \mathcal{M}' -definable set admits an \mathcal{M} -definable pseudo-closure. For later use, we say that T' has **pseudo-closures in T** if every T' -model has pseudo-closures in its L -reduct.

Proposition 3.2. *Suppose $J \subseteq I$ is finite and $X_i \subseteq M^x$ is \mathcal{M}_i -definable for all $i \in J$. If there is an \mathcal{M}_\cap -definable set X in which each X_i is pseudo-dense, then $(X_i)_{i \in J}$ is not separated. The converse implication holds provided \mathcal{M}_i has pseudo-closures in \mathcal{M}_\cap for all $i \in J$. It follows that if \mathcal{C} is a collection of \mathcal{M}_\cap -definable sets, then:*

- (1) If \mathcal{M}_\cup is interpolative, then \mathcal{M}_\cup is \mathcal{C} -weakly interpolative. In particular, if \mathcal{M}_\cup is interpolative, then \mathcal{M}_\cup is weakly interpolative.
- (2) Suppose moreover that each \mathcal{M}_i has pseudo-closures in \mathcal{M}_\cap . If \mathcal{M}_\cup is weakly interpolative, or if \mathcal{M}_\cup is \mathcal{C} -weakly interpolative and \mathcal{C} is a collection of pseudo-cells, then \mathcal{M}_\cup is interpolative.

Proof. We only prove the first two claims, since then (1) and (2) follow easily.

For the first statement, suppose X is a nonempty \mathcal{M}_\cap -definable subset of M^x in which each X_i is pseudo-dense, and $(X^i)_{i \in J}$ is a family of \mathcal{M}_\cap -definable sets satisfying $X_i \subseteq X^i$ for each $i \in J$. As X_i is pseudo-dense in X and disjoint from $X \setminus X^i$, we have $\dim X \setminus X^i < \dim X$ for all $i \in J$. Hence,

$$\dim \bigcup_{i \in J} (X \setminus X^i) < \dim X.$$

Thus $\dim \bigcap_{i \in J} X^i \geq \dim X$, so $\bigcap_{i \in J} X^i$ is nonempty.

For the converse, assume \mathcal{M}_i has pseudo-closures in \mathcal{M}_\cap for each $i \in J$, and suppose X_i is an \mathcal{M}_i -definable set for each $i \in J$. We want to show that if there is no \mathcal{M}_\cap -definable set Z in which each X_i is pseudo-dense, then $(X_i)_{i \in J}$ is separated. Simplifying notation, we assume $J = \{1, \dots, n\}$. We show $(X_i)_{i=1}^n$ is separated by applying simultaneous transfinite induction to d_1, \dots, d_n where d_i is the dimension of any pseudo-closure of X_i .

Let X^i be a pseudo-closure of X_i for each i and let

$$Z = X^1 \cap \dots \cap X^n.$$

If $\dim X^j = -\infty$ for some $j \in J$, then X^j and Z are both empty, so $(X^i)_{i=1}^n$ separates $(X_i)_{i=1}^n$. If $\dim X^i = \dim Z$ for each i , then Lemma 3.1(2) shows each X_i is pseudo-dense in Z , contradiction. After re-arranging the X_i if necessary we suppose $\dim Z < \dim X^1$. Let $Y_1 = X_1 \cap Z$. As $(X_i)_{i=1}^n$ cannot be simultaneously pseudo-dense in an \mathcal{M}_\cap -definable set, it follows that Y_1, X_2, \dots, X_n cannot be simultaneously pseudo-dense in an \mathcal{M}_\cap -definable set. As the dimension of any pseudo-closure of Y_1 is strictly less than the dimension of X^1 , an application of the inductive hypothesis provides \mathcal{M}_\cap -definable sets Y^1, \dots, Y^n separating Y_1, X_2, \dots, X_n . It is easy to see

$$Y^1 \cup (X^1 \setminus Z), Y^2 \cap X^2, \dots, Y^n \cap X^n$$

separates X_1, \dots, X_n , which completes the proof. \square

Let T be an L -theory equipped with an ordinal dimension \dim and \mathcal{C} an arbitrary collection of definable sets in T -models. We say that \mathcal{C} is a **pseudo-cell collection** for T if for all $\mathcal{M} \models T$, $\mathcal{C} \cap \text{Def}(\mathcal{M})$ is a pseudo-cell collection for \mathcal{M} . We say that T **defines \mathcal{C} -membership** if for every L -formula $\varphi(x, y)$ there is an L -formula $\gamma(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$,

$$\varphi(\mathcal{M}, b) \text{ is in } \mathcal{C} \text{ if and only if } \mathcal{M} \models \gamma(b).$$

We say that T' **defines pseudo-denseness over \mathcal{C}** if for every L' -formula $\varphi'(x, y)$ and every L -formula $\varphi(x, z)$, there is an L' -formula $\delta'(y, z)$ such that if $\mathcal{M}' \models T'$ and $c \in M'^y$ with $\varphi(\mathcal{M}', c) \in \mathcal{C}$, then

$$\varphi'(\mathcal{M}', b) \text{ is pseudo-dense in } \varphi(\mathcal{M}', c) \text{ if and only if } \mathcal{M}' \models \delta'(b, c).$$

If \mathcal{C} is the collection of all \mathcal{M}_\cap -definable sets then we say that T' **defines pseudo-denseness over T** .

We say T **defines dimension** if for every ordinal α , and every L -formula $\varphi(x, y)$, there is an L -formula $\delta_\alpha(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$

$$\dim \varphi(\mathcal{M}, b) = \alpha \quad \text{if and only if} \quad \mathcal{M} \models \delta_\alpha(b).$$

Note that if T defines dimension then, by compactness, for every formula $\varphi(x, y)$, there are finitely many ordinals $\alpha_1, \dots, \alpha_n$ such that for all $\mathcal{M} \models T$ and $b \in M^y$, we have $\dim(\varphi(\mathcal{M}, b)) \in \{\alpha_1, \dots, \alpha_n\}$.

Proposition 3.3. *Suppose \mathcal{C} is a pseudo-cell collection, T defines \mathcal{C} -membership and dimension, and T' defines pseudo-denseness over \mathcal{C} . Then T' defines pseudo-denseness over T .*

Proof. We only treat the case where \dim is preserved under definable bijection and every definable set admits a \mathcal{C} -patching. The other case is similar and easier. Let $\varphi'(x, z')$ be an L' -formula and $\varphi(x, z)$ be an L -formula. We will produce an L' -formula $\delta'(z, z')$ such that whenever $\mathcal{M}' \models T'$ and $\mathcal{M} = \mathcal{M}' \upharpoonright L$, we have

$$\mathcal{M}' \models \delta(c, c') \text{ if and only if } \varphi'(\mathcal{M}', c') \text{ is pseudo dense in } \varphi(\mathcal{M}, c).$$

Applying compactness we obtain n and an L -formula $\psi(x, y, w)$ such that for all $\mathcal{M}' \models T'$ and $\mathcal{M} = \mathcal{M}' \upharpoonright L$ we have the following

- (1) For all $d \in M^w$, the formula $\psi(x, y, d)$ defines a function f_d from a subset X_d of M^x to a subset Y_d of M^y .
- (2) For each $c \in M^z$, there exists $J \subseteq \{1, \dots, n\}$ and $(d_j)_{j \in J}$ from M^w , such that $(X_{d_j}, Y_{d_j}, f_{d_j})_{j \in J}$ as defined in (1) is a \mathcal{C} -patching of $\varphi(\mathcal{M}, c)$.

Now we have that $\varphi'(\mathcal{M}, c')$ is pseudo-dense in $\varphi(\mathcal{M}, c)$ if and only if there are J , and $(d_j)_{j \in J}$ as in (1) and (2) such that $f_{d_j}(\varphi'(\mathcal{M}, c') \cap X_{d_j})$ is pseudo-dense in Y_{d_j} for all $j \in J$ such that $\dim X_{d_j} = \dim \varphi(\mathcal{M}, c)$. From the analysis above, it is easy to see that we can choose the desired formula $\delta'(z, z')$. \square

With all the pieces in place, we can prove the main theorem of this section, which has Theorem 1.4 from the introduction as a special case.

Theorem 3.4. *Suppose \mathcal{C} is a collection of definable sets of T_\cap -models such that T_\cap defines \mathcal{C} -membership, and each T_i defines pseudo-denseness over \mathcal{C} . Then we have the following:*

- (1) *The class of \mathcal{C} -weakly interpolative T_\cup -models is elementary.*
- (2) *If \mathcal{C} is a pseudo-cell collection for T_\cap , then the class of weakly interpolative T_\cup -models is elementary.*
- (3) *If, in addition, each T_i has pseudo-closures in T_\cap , then the interpolative fusion T_\cup^* exists.*

In particular, taking \mathcal{C} to be the collection of all definable sets in T_\cap -models:

- (4) *If T_i defines pseudo-denseness over T_\cap for all $i \in I$, then the class of weakly interpolative T_\cup -models is elementary.*
- (5) *If, in addition, T_i has pseudo-closures in T_\cap for all $i \in I$, then T_\cup^* exists.*

Proof. It suffices to prove (1). Then (2) follows from (1) and the fact, noted above, that \mathcal{C} is a pseudo-cell collection for T_\cap , then \mathcal{M}_\cup is weakly interpolative if and only if \mathcal{M}_\cup is \mathcal{C} -weakly interpolative. Now (3) follows from Proposition 3.2. Finally, (4) and (5) are special cases of (2) and (3), respectively.

Let $\varphi_\cap(x, y)$ be an L_\cap -formula, let $J \subseteq I$ be finite, and let $\varphi_i(x, z_i)$ be an L_i -formula for each $i \in J$. Let $\gamma_\cap(y)$ be an L_\cap -formula defining \mathcal{C} -membership for $\varphi_\cap(x, y)$, and let $\delta_i(y, z_i)$ be an L_i -formula defining pseudo-denseness over \mathcal{C} for $\varphi_\cap(x, y)$ and $\varphi_i(x, z_i)$ for each $i \in J$. For simplicity, we assume $J = \{1, \dots, n\}$. Then we have the following axiom:

$$\forall y, z_1, \dots, z_n \left(\left(\gamma_\cap(y) \wedge \bigwedge_{i=1}^n \delta_i(y, z_i) \right) \rightarrow \exists x \bigwedge_{i=1}^n \varphi_i(x, z_i) \right).$$

Then T_{\cup} , together with one axiom of the above form for each choice of $\varphi_{\cap}(x, y)$, J , and $\varphi_i(x, z_i)$ for $i \in J$ as above, axiomatizes the class of \mathcal{C} -weakly interpolative T_{\cup} -models. \square

We refer to the axioms obtained in the proof of Theorem 3.4 as the **pseudo-topological axioms** for T_{\cup}^* .

At present, we know that Theorem 3.4 applies to all examples described in Example 1.2 and 1.3 except for the integers with p -adic valuations and differentially closed fields. We do not know if Theorem 3.4 applies to differentially closed fields, as in this case we do not have a good understanding of definable sets in the base theory.

We now show that Theorem 3.4 does not apply to the example of the integers with p -adic valuations. Let \dim be the canonical ordinal dimension on the additive group of integers, which coincides with U-rank, acl-dimension, etc., see [Con18].

Proposition 3.5. *Suppose $(\mathbb{Z}; +, \times)$ is an \aleph_1 -saturated elementary extension of $(\mathbb{Z}; +, \times)$. Fix a prime p and let v_p be the p -adic valuation on \mathbb{Z} . Given $a, b \in \mathbb{Z}$ we declare $a \preceq_p b$ if and only if $v_p(a) \leq v_p(b)$. Fix N in \mathbb{Z} such that $v_p(N) \geq n$ for all $n \in \mathbb{N}$ and let*

$$E = \{z \in \mathbb{Z} \mid N \preceq_p z\}.$$

Then E does not have a pseudo-closure in $(\mathbb{Z}; +)$. So $(\mathbb{Z}; +, \preceq_p)$ does not have pseudo-closures in $(\mathbb{Z}; +)$.

Proof. The quantifier elimination for $\text{Th}(\mathbb{Z}; +)$ implies that every $(\mathbb{Z}; +)$ -definable subset of \mathbb{Z} is a finite union of sets of the form $(k\mathbb{Z} + l) \setminus F$ for $k, l \in \mathbb{Z}$ and finite F . So if E has a pseudo-closure in $(\mathbb{Z}; +)$, then E is pseudo-dense in $k\mathbb{Z} + l$ for some $k, l \in \mathbb{Z}$ with $k \neq 0$.

So we fix $k, l \in \mathbb{Z}$ with $k \neq 0$ and show that E is not pseudo-dense in $k\mathbb{Z} + l$. As $\dim(k\mathbb{Z} + l) = 1$, it is enough to show that E is disjoint from some infinite definable subset of $k\mathbb{Z} + l$.

If $l \neq 0$, let $n = v_p(l)$, so $l = p^n m$ for some $m \in \mathbb{Z}$ which is coprime to p . Then $p^{n+1}k\mathbb{Z} + l = p^n(pk\mathbb{Z} + m)$ is an infinite definable subset of $k\mathbb{Z} + l$. Every element of this set has p -adic valuation at most $n \in \mathbb{N}$, so it is disjoint from E .

If $l = 0$, consider $k\mathbb{Z} \setminus pk\mathbb{Z}$. This is an infinite definable subset of $k\mathbb{Z}$, and if $a \in k\mathbb{Z} \setminus pk\mathbb{Z}$, then $v_p(a) = v_p(k) \in \mathbb{N}$. So E is disjoint from $k\mathbb{Z} \setminus pk\mathbb{Z}$. \square

Remark 3.6. One can apply the “quasi-coset” decompositions given in [Con18, Theorem 4.10] to show that $\{(k, l) \in \mathbb{Z}^2 : k \preceq_p l\}$ does not have a pseudo-closure in \mathbb{Z}^2 . This presents some technical difficulties, so we do not include it here. As every $(\mathbb{Z}; +, \preceq_p)$ -definable subset of \mathbb{Z} is $(\mathbb{Z}; +)$ -definable [Ad19], we must pass to an elementary extension to obtain a unary set without a pseudo-closure.

4. TAME TOPOLOGICAL BASE

We consider in this section specializations of Theorem 3.4 to settings in which the base theory T_{\cap} admits a well-behaved definable topology. Throughout, we maintain the notational conventions described at the beginning of Section 3, and we explore the degree to which the pseudo-topological notions defined there agree with natural topological notions. We show, among other things, that if \mathcal{M} is o-minimal, then \mathcal{M}' has pseudo-closures in \mathcal{M} if and only if the closure of every \mathcal{M}' -definable set

is \mathcal{M} -definable. This equivalence only depends on two well-known facts from o-minimality. One of these is known as the frontier inequality, and we refer to the other as the residue inequality. Whenever these inequalities hold in our abstract setting, we automatically obtain definability of pseudo-denseness.

A **definable topology** \mathcal{T} on \mathcal{M} consists of a topology \mathcal{T}_x on each M^x , such that $\{\varphi(\mathcal{M}, b) : b \in M^y\}$ is a basis for \mathcal{T}_x , for some L -formula $\varphi(x, y)$. Note that we also obtain a definable topology on every model of $\text{Th}(\mathcal{M})$. For the rest of Section 4, we suppose \mathcal{T} is a definable topology on \mathcal{M} and \dim is an ordinal dimension on $T = \text{Th}(\mathcal{M})$ such that T defines dimension.

Let A be a subset of M^x . We denote by $\text{cl}(A)$ the closure of A with respect to \mathcal{T}_x . The **frontier** of A , $\text{fr}(A)$, is defined as $\text{cl}(A) \setminus A$. Since \mathcal{T} is a definable topology, the interior, closure, and frontier of a definable subset of M^x are all definable.

In general there need be no connection between pseudo-denseness and \mathcal{T} -denseness. We give conditions under which the two naturally relate. We say \mathcal{M} satisfies the **frontier inequality** if

$$\dim \text{fr}(X) < \dim X \quad \text{for all definable } X.$$

This is a strong assumption, which in particular implies, by a straight-forward induction on dimension, that every definable set is a Boolean combination of open definable sets.

Lemma 4.1. *Suppose \mathcal{M} satisfies the frontier inequality and $X' \subseteq X$ are \mathcal{M} -definable sets. If $\dim X' = \dim X$, then X' has nonempty interior in X .*

Proof. If X' has empty interior in X , then $X \setminus X'$ is dense in X , and so $X' \subseteq X \subseteq \text{cl}(X \setminus X')$. In particular, $X' \subseteq \text{fr}(X \setminus X')$. The frontier inequality implies $\dim X' < \dim X \setminus X' \leq \dim X$. \square

Lemma 4.2. *The following are equivalent:*

- (1) \mathcal{M} satisfies the frontier inequality.
- (2) If $A \subseteq M^x$ is dense in a definable $X \subseteq M^x$ then A is pseudo-dense in X .

Proof. Suppose that \mathcal{M} satisfies the frontier inequality and that $A \subseteq M^x$ is dense in a definable set $X \subseteq M^x$. Suppose $X' \subseteq X$ is definable and $\dim X' = \dim X$. Lemma 4.1 implies that X' has nonempty interior in X . Thus A intersects X' . It follows that A is pseudo-dense in X .

Conversely, assume (2), and let $X \subseteq M^x$ be definable. Since X is dense in $\text{cl}(X)$, X is also pseudo-dense in $\text{cl}(X)$. As $X \cap \text{fr}(X) = \emptyset$, we have $\dim \text{fr}(X) < \dim \text{cl}(X)$. It follows that $\dim X = \dim \text{cl}(X)$, so the frontier inequality holds. \square

Pseudo-density does not, in general, imply density. For example, if $X \subseteq M^x$ is an infinite definable set and $p \in M^x$ does not lie in $\text{cl}(X)$, then X is pseudo-dense in $X \cup \{p\}$ but not dense in $X \cup \{p\}$. However, the converse to (2) does hold for certain definable sets, which we call pure dimensional.

Let $X \subseteq M^x$ be definable. Given $p \in X$, we define

$$\dim_p X = \min\{\dim(U \cap X) : U \text{ is a definable neighborhood of } p\}.$$

We say that X is **pure dimensional** if $\dim_p X = \dim X$ for all $p \in X$. Equivalently, X is pure dimensional if and only if $\dim U = \dim X$ for all nonempty definable open subsets of X .

Lemma 4.3. *Suppose $X \subseteq M^x$ is definable. Then the following are equivalent:*

- (1) X is pure dimensional.
- (2) If a subset A of M^x is pseudo-dense in X , then A is dense in X .

Proof. Suppose X is not pure dimensional. Let U be a definable nonempty open subset of X such that $\dim U < \dim X$. Then $X \setminus U$ is pseudo-dense in X and not dense in X .

Suppose X is pure dimensional and A is pseudo-dense in X . Suppose U is a nonempty open subset of X . Then there is a definable nonempty open subset U' of U . Then $\dim U' = \dim X$, so A intersects U' . Hence A is dense in X . \square

Proposition 4.4 gives another characterization of pure dimensional sets. We will not use this characterization, so we leave its proof to the reader.

Proposition 4.4. *Suppose $X \subseteq M^x$ is definable. If X is pure dimensional, then there are no definable sets X^1 and X^2 such that $X = X^1 \cup X^2$, X^1 and X^2 are closed in X , neither X^1 nor X^2 contains the other, and $\dim X^1 \neq \dim X^2$. If \mathcal{M} satisfies the frontier inequality, then the converse holds.*

For a definable $X \subseteq M^x$, we define the **essence** of X , $\text{es}(X)$, and the **residue** of X , $\text{rs}(X)$:

$$\begin{aligned} \text{es}(X) &= \{p \in X : \dim_p X = \dim X\} \\ \text{rs}(X) &= \{p \in X : \dim_p X < \dim X\} \end{aligned}$$

As \mathcal{T}_x admits a definable basis, and T defines dimension, it follows that $\text{es}(X)$ and $\text{rs}(X)$ are definable.

We say that \mathcal{M} satisfies the **residue inequality** if

$$\dim \text{rs}(X) < \dim X \quad \text{for all definable } X.$$

Lemma 4.5. *If \mathcal{M} satisfies the residue inequality, then for all definable $X \subseteq M^x$, $\text{es}(X)$ is pure dimensional.*

Proof. Let $p \in \text{es}(X)$, and let U be a definable neighborhood of p . We need to show that $\dim(U \cap \text{es}(X)) = \dim \text{es}(X)$. By the residue inequality, $\dim \text{rs}(X) < \dim X$. So $\dim \text{es}(X) = \dim X = \dim(U \cap X)$ because $p \in \text{es}(X)$. Thus,

$$\dim \text{es}(X) = \dim(U \cap X) = \dim((U \cap X) \setminus \text{rs}(X)) = \dim(U \cap \text{es}(X)). \quad \square$$

We will not use Proposition 4.6, but we include it here, since it provides additional motivation for the residue inequality.

Proposition 4.6. *\mathcal{M} satisfies the residue inequality if and only if every definable set is a finite disjoint union of pure dimensional definable sets.*

Proof. Suppose first that \mathcal{M} satisfies the residue inequality. Let $X \subseteq M^x$ be definable. We argue by induction on $\dim X$. If $\dim X = -\infty$, then $X = \emptyset$ and the conclusion holds vacuously. Otherwise, X is the disjoint union of $\text{es}(X)$ and $\text{rs}(X)$. By Lemma 4.5, $\text{es}(X)$ is pure dimensional, and by the residue inequality $\dim \text{rs}(X) < \dim X$, so by induction $\text{rs}(X)$ is a finite disjoint union of pure dimensional definable sets.

Conversely, for any definable set X , suppose that X is a disjoint union of pure dimensional definable sets Y_1, \dots, Y_m . We will show that $\dim \text{rs}(X) < \dim X$. We may assume without loss of generality that $1 \leq j \leq m$ is such that

$$\dim Y_k = \dim X \text{ when } k \leq j \quad \text{and} \quad \dim Y_k < \dim X \text{ when } k > j.$$

Let $p \in \text{rs}(X)$, and suppose for contradiction that $p \in Y_k$ for some $k \leq j$. Then since Y_k is pure dimensional, $\dim_p Y_k = \dim Y_k = \dim X$, so for any definable neighborhood U of p ,

$$\dim X = \dim(U \cap Y_k) \leq \dim(U \cap X) \leq \dim X.$$

So $\dim_p X = \dim X$, contradicting the fact that $p \in \text{rs}(X)$. Thus $\text{rs}(X) \subseteq \bigcup_{k>j} Y_k$, and $\dim \text{rs}(X) \leq \dim \bigcup_{k>j} Y_k < \dim X$. \square

We say \mathcal{T} is **dim-compatible** if \mathcal{M} satisfies both the frontier inequality and the residue inequality. Definability of the dimension and the topology ensure that dim-compatibility is an elementary property, i.e., the topology on any model of T is dim-compatible.

Proposition 4.7. *Suppose \mathcal{T} is dim-compatible. Suppose $X \subseteq M^x$ is definable and $A \subseteq M^x$. Then A is pseudo-dense in X if and only if A is dense in $\text{es}(X)$.*

Proof. Since $\dim \text{rs}(X) < \dim X$ and $\dim \text{es}(X) = \dim X$, A is pseudo-dense in X if and only if A is pseudo-dense in $\text{es}(X)$. The equivalence then follows from Lemma 4.2, Lemma 4.3, and Lemma 4.5. \square

Proposition 4.8. *Suppose \mathcal{T} is dim-compatible. Any expansion T' of T defines pseudo-denseness over T .*

Proof. Suppose \mathcal{M} is a T -model and \mathcal{M}' is a T' -model expanding \mathcal{M} . Suppose $(X_b)_{b \in M^y}$ and $(X'_c)_{c \in M^z}$ are families of subsets of M^x , which are \mathcal{M} -definable and \mathcal{M}' -definable, respectively. By Proposition 4.7, X'_c is pseudo-dense in X_b if and only if X'_c is dense in $\text{es}(X_b)$.

Using definability of the topology and dimension, essences of definable sets are uniformly definable, i.e., there is an \mathcal{M} -definable family $(Y_b)_{b \in M^y}$ such that $Y_b = \text{es}(X_b)$ for all $b \in M^y$. Thus X'_c is pseudo-dense in X_b if and only if X'_c is dense in Y_b . And using definability of the topology, the set of all (b, c) such that X'_c is dense in Y_b is definable. \square

Proposition 4.9. *Suppose \mathcal{T} is dim-compatible. Suppose \mathcal{M}' expands \mathcal{M} . Then \mathcal{M}' has pseudo-closures in \mathcal{M} if and only if the closure of any \mathcal{M}' -definable set is \mathcal{M} -definable.*

Proof. Suppose that the closure of any \mathcal{M}' -definable set is \mathcal{M} -definable. Then for any \mathcal{M}' -definable $X \subseteq M^x$, $\text{cl}(X)$ is a pseudo-closure of X by Lemma 4.2.

Conversely, suppose \mathcal{M}' has pseudo-closures in \mathcal{M} and $X' \subseteq M^x$ is \mathcal{M}' -definable. Let X be a pseudo-closure of X' . We apply induction to the dimension of X . If $\dim X = -\infty$, then X' is empty and trivially \mathcal{M} -definable. Now suppose $\dim X \geq 0$. We have

$$\text{cl}(X') = \text{cl}(X' \cap \text{es}(X)) \cup \text{cl}(X' \cap \text{rs}(X)).$$

Since X' is pseudo-dense in X , X' is dense in $\text{es}(X)$ by Proposition 4.7. It follows that $\text{cl}(X' \cap \text{es}(X)) = \text{cl}(\text{es}(X))$, which is \mathcal{M} -definable. As $(X' \cap \text{rs}(X)) \subseteq \text{rs}(X)$, any pseudo-closure of $(X' \cap \text{rs}(X))$ has dimension at most $\dim \text{rs}(X) < \dim X$.

So $\text{cl}(X' \cap \text{rs}(X))$ is \mathcal{M} -definable by induction. Thus $\text{cl}(X')$ is a union of two \mathcal{M} -definable sets and is therefore \mathcal{M} -definable. \square

We conclude this section by giving examples of structures with dim-compatible definable topologies. In those examples, \mathcal{T} and dim are canonical, so we do not describe them in detail. In each case, existence of pure dimensional decompositions (and hence the residue inequality, by Proposition 4.6) follows from the appropriate cell decomposition or “weak cell decomposition”. In different settings, cells (or “weak cells”) have different definitions, but they are easily seen to be pure dimensional in each case.

The most familiar case is when \mathcal{M} is an o-minimal expansion of a dense linear order, see [vdD98b]. Similarly, it follows from [SW19, Propositions 4.1 and 4.3] that if \mathcal{M} is a dp-minimal expansion of a divisible ordered abelian group, then the usual order topology is compatible with dp-rank (and dp-rank agrees with several other natural notions of dimension [SW19, Proposition 2.4]). This covers the case when \mathcal{M} is an expansion of an ordered abelian group with weakly o-minimal theory. It is shown in Johnson’s thesis [Joh16] that if \mathcal{M} is a dp-minimal, but not strongly minimal, expansion of a field, then \mathcal{M} admits a canonical non-discrete definable field topology. Taking the product topology on M^n for all n , it is shown in [SW19] that this topology is compatible with dp-rank. This covers the case of a C-minimal expansion of an algebraically closed field or a P-minimal expansion of a p -adically closed field. It was previously shown in [CKDL17] that P-minimal expansions of p -adically closed fields satisfy the frontier inequality and admit pure dimensional decompositions.

We say that T is an **open core** of T' if the closure of every T' -definable set in every T' -model \mathcal{M}' is $\mathcal{M} = \mathcal{M}'|L$ definable. Propositions 4.8 and 4.9 together yield Theorem 4.10, which generalizes Theorem 1.5 from the introduction.

Theorem 4.10. *If T_\cap admits a definable ordinal dimension dim and a dim-compatible definable topology, and T_\cap is an open core of each T_i , then T_\cup^* exists. In particular, if T_\cap is an o-minimal expansion of a dense linear order or a P-minimal expansion of a p -adically closed field, and T_\cap is an open core of each T_i , then T_\cup^* exists.*

We give an application of Theorem 4.10. Suppose T_\cap is a complete and model-complete o-minimal theory extending the theory of ordered abelian groups. For each $i \in I$, let T_i be the theory of a T -model \mathcal{N} equipped with a unary predicate R_i defining a dense elementary substructure of \mathcal{N} . Then T_i is model-complete by [vdD98a, Thm 1] and T_\cap is an open core of T_i [DMS10, Section 5]. Applying Theorem 4.10, we see that the theory T_\cup of a T -model \mathcal{N} equipped with a family $(R_i)_{i \in I}$ of unary predicates defining dense elementary substructures of \mathcal{N} has a model companion.

5. \aleph_0 -STABLE BASE

In almost all of the examples from Section 2.3, the base theory T_\cap is \aleph_0 -stable (in fact, T_\cap is almost always interpretable in ACF or the theory of an infinite set). In this section, we specialize Theorem 3.4 to this setting, where the natural ordinal dimension is Morley rank, and we obtain pseudo-closures for free.

We keep the additional notational conventions described at the beginning of Section 3. Throughout this section, T is \aleph_0 -stable, \dim is Morley rank, and mult is Morley degree.

Suppose X^1 and X^2 are \mathcal{M} -definable subsets of M^x . We will say that X^1 is **almost a subset** of X^2 and write $X^1 \subseteq_a X^2$ if

$$\dim(X^1 \setminus X^2) < \dim(X^1).$$

We will say that X^1 is **almost equal** to X^2 and write $X^1 =_a X^2$ if $X^1 \subseteq_a X^2$ and $X^2 \subseteq_a X^1$. It is easy to see that $=_a$ is an equivalence relation. An \mathcal{M} -definable subset X of M^x is **almost irreducible** if whenever $X = X^1 \cup X^2$ for \mathcal{M} -definable X^1 and X^2 , we have $X =_a X^1$ or $X =_a X^2$. Any \mathcal{M} -definable set of Morley degree one is almost irreducible, and the converse holds when $\text{Th}(\mathcal{M})$ defines Morley rank or when \mathcal{M} is \aleph_0 -saturated.

Proposition 5.1 is the main advantage of assuming that T_\cap is \aleph_0 -stable in our setting.

Proposition 5.1. *Let $\mathcal{M} \models T$. Every $A \subseteq M^x$ has a pseudo-closure. More precisely, an \mathcal{M} -definable set $X \subseteq M^x$ is a pseudo-closure of A if and only if $A \subseteq X$ and for all \mathcal{M} -definable $X' \subseteq M^x$ with $A \subseteq X'$,*

$$(\dim X, \text{mult} X) \leq_{\text{Lex}} (\dim X', \text{mult} X').$$

It follows that every expansion of T has pseudo-closures in T .

Proof. By standard properties of Morley rank and degree in \aleph_0 -stable theories, for any \mathcal{M} -definable X and X' , if $(\dim X', \text{mult} X') <_{\text{Lex}} (\dim X, \text{mult} X)$, then $\dim(X \setminus X') = \dim X$. If $X' \subseteq X$, then the converse is true.

Let X be a pseudo-closure of A , so $A \subseteq X$, and suppose for contradiction that there is some \mathcal{M} -definable $X' \subseteq M^x$ with $A \subseteq X'$ and $(\dim X', \text{mult} X') <_{\text{Lex}} (\dim X, \text{mult} X)$. Then $\dim(X \setminus X') = \dim X$, but $A \cap (X \setminus X') = \emptyset$, contradicting the fact that A is pseudo-dense in X .

Conversely, suppose $A \subseteq X$ and $(\dim X, \text{mult} X)$ is minimal in the lexicographic order among \mathcal{M} -definable sets containing A . Then for any \mathcal{M} -definable $X' \subseteq X$ with $\dim X' = \dim X$, $(\dim(X \setminus X'), \text{mult}(X \setminus X')) <_{\text{Lex}} (\dim X, \text{mult} X)$. It follows that $A \not\subseteq (X \setminus X')$, so $A \cap X' \neq \emptyset$. Hence X is a pseudo-closure of A . \square

Corollary 5.2 now follows immediately from Proposition 3.2.

Corollary 5.2. *If $\text{Th}(\mathcal{M}_\cap)$ is \aleph_0 -stable and \dim is Morley rank, then \mathcal{M}_\cup is interpolative if and only if it is weakly interpolative.*

In Proposition 3.5, we gave a concrete example of an expansion of $T = \text{Th}(\mathbb{Z}; +)$ that does not have pseudo-closures in T . It is well known that T is superstable but not \aleph_0 -stable, so this demonstrates that superstability is not sufficient for Proposition 5.1. For the reader who is still looking for a free ride outside of the \aleph_0 -stable context, Proposition 5.4 will dash this hope.

If \dim_1, \dim_2 are ordinal dimensions on an L^\diamond -theory T^\diamond then we say \dim_1 is **smaller than** \dim_2 if $\dim_1 X \leq \dim_2 X$ for all definable sets X .

Remark 5.3. The theory T^\diamond is \aleph_0 -stable if and only if it admits an ordinal dimension \dim such that for every T^\diamond -model \mathcal{M}^\diamond , \mathcal{M}^\diamond -definable set X , and family $(X_n)_{n \in \mathbb{N}}$ of pairwise disjoint \mathcal{M}^\diamond -definable subsets of X , we have $\dim X_n < \dim X$ for some n . If T^\diamond is \aleph_0 -stable, then Morley rank is the smallest ordinal dimension with this property.

Proposition 5.4. *Suppose L^\diamond is countable and \dim^\diamond is an ordinal dimension on a complete L^\diamond -theory T^\diamond . If T^\diamond is not \aleph_0 -stable, then there is an expansion of T^\diamond that does not have pseudo-closures in T^\diamond .*

Proof. Suppose T^\diamond is not \aleph_0 -stable. Applying Remark 5.3, we obtain a T^\diamond -model \mathcal{M}^\diamond , an \mathcal{M}^\diamond -definable set X with $\dim^\diamond X = \alpha$, and a sequence $(X_n)_{n \in \mathbb{N}}$ of pairwise disjoint \mathcal{M}^\diamond -definable subsets of X such that $\dim^\diamond X_n = \alpha$ for all n . Since X and each X_n are definable with parameters from a countable elementary submodel, we may assume \mathcal{M}^\diamond is countable.

Given $S \subseteq \mathbb{N}$, let $A_S = \bigcup_{n \in S} X_n$. We show that A_S does not have a pseudo-closure for uncountably many $S \subseteq \mathbb{N}$. Suppose $S \subseteq \mathbb{N}$ is nonempty and X' is a pseudo-closure of A_S . As $A_S \subseteq X$, we have $\dim^\diamond X' \leq \alpha$. As S is nonempty, we have $X_n \subseteq X'$ for some n , so $\dim^\diamond X' \geq \alpha$. Thus any pseudo-closure X' of A_S has $\dim^\diamond X' = \alpha$.

Now suppose $S, S' \subseteq \mathbb{N}$ are nonempty and $S \not\subseteq S'$. We show any pseudo-closure of A_S is not a pseudo-closure of $A_{S'}$. Fix $n \in S \setminus S'$ and suppose X' is a pseudo-closure of A_S . Then $\dim^\diamond X' = \alpha$, X_n is an \mathcal{M}^\diamond -definable subset of X' with $\dim^\diamond X_n = \alpha$, but X_n is disjoint from $A_{S'}$. Thus X' is not a pseudo-closure of $A_{S'}$.

Let \mathfrak{J} be an uncountable collection of nonempty subsets of \mathbb{N} such that $S \not\subseteq S'$ for all distinct $S, S' \in \mathfrak{J}$. If $S, S' \in \mathfrak{J}$ are distinct, then A_S and $A_{S'}$ cannot have a common pseudo-closure. As \mathcal{M}^\diamond and L are countable, there are only countably many \mathcal{M}^\diamond -definable sets, so there are uncountably many $S \in \mathfrak{J}$ such that A_S does not have a pseudo-closure. The expansion of \mathcal{M}^\diamond by a predicate defining any such A_S does not have pseudo-closures in \mathcal{M}^\diamond . It follows that the theory of this expansion does not have pseudo-closures in T^\diamond . \square

We next give a useful characterization of definability of pseudo-denseness over an \aleph_0 -stable theory. Proposition 5.1 motivates the following definition. Suppose M' is a model of T' , $\mathcal{M} = \mathcal{M}'|L$, and $X' \subseteq M^x$ is \mathcal{M}' -definable. Define

$$\dim' X' = \dim X \quad \text{and} \quad \text{mult}' X' = \text{mult} X$$

where X is any pseudo-closure of X' . Lemma 5.5 is an immediate consequence of Proposition 5.1.

Lemma 5.5. *For $A \subseteq M^x$ and \mathcal{M} -definable $X \subseteq M^x$, we have the following:*

- (1) *A is pseudo-dense in X if and only if we have both $\dim'(X \cap A) = \dim(X)$ and $\text{mult}'(X \cap A) = \text{mult}(X)$.*
- (2) *If X is almost irreducible, then A is pseudo-dense in X if and only if $\dim'(X \cap A) = \dim(X)$.*

In general \dim' might not be an ordinal dimension on T' , as $\dim'(X')$ might be different from $\dim'(X'(\mathcal{N}'))$ where \mathcal{N}' is an elementary extension of \mathcal{M}' . When T defines Morley rank, we can easily check that \dim' is an ordinal dimension on T' , which we will refer to as the **induced dimension** on T' .

We say T **defines multiplicity** (also known as having the DMP in the literature) if for all L -formulas $\varphi(x, y)$, ordinals α , and n , there is an L -formula $\mu_{\alpha, n}(y)$ such that for all $\mathcal{M} \models T$ and $b \in M^y$ we have that

$$\mathcal{M} \models \mu_{\alpha, n}(b) \text{ if and only if } \dim \varphi(\mathcal{M}, b) = \alpha \text{ and } \text{mult} \varphi(\mathcal{M}, b) = n.$$

In particular, if T defines multiplicity, then T defines Morley rank, and the induced dimension on T' is well-defined.

Proposition 5.6. *Suppose T defines multiplicity. Then T' defines pseudo-denseness over T if and only if T' defines induced dimension.*

Proof. Suppose T' defines pseudo-denseness and $\varphi'(x, y)$ is an L' -formula. Let $(X'_b)_{b \in M^y}$ be the family of subsets of M^x defined by $\varphi'(x, y)$. Using the assumption that T' defines pseudo-denseness and a standard compactness argument, we obtain a family $(X_c)_{c \in M^z}$ defined by a formula whose choice might depend on $\varphi'(x, y)$ but not on M' , such that for every $b \in M^y$, X'_b has a pseudo-closure that is a member of the family $(X_c)_{c \in M^z}$. It follows from Proposition 5.1 that $\dim'(X'_b) = \alpha$ for $b \in M^y$ if and only there is $c \in M^z$ such that X'_b is pseudo-dense in X_c and $\dim(X_c) = \alpha$. As T defines Morley rank and T' defines pseudo-denseness, it follows that T' defines induced dimension.

Now suppose T' defines induced dimension. Let \mathcal{C} be the collection of almost irreducible subsets of T -models. Then \mathcal{C} is a collection of pseudo-cells for T . As T defines multiplicity, T defines \mathcal{C} -membership. So by Proposition 3.3, it suffices to show T' defines pseudo-denseness over \mathcal{C} . Let $(X'_b)_{b \in M^y}$ and $(X_c)_{c \in M^z}$ be a families defined by an L' -formula $\varphi'(x, y)$ and an L -formula $\varphi(x, z)$. It follows from Proposition 5.1 that when X_c is in \mathcal{C} , X'_b is pseudo-dense in X_c if and only if $\dim'(X \cap X') = \dim(X)$. The desired conclusion follows. \square

Remark 5.7. If T defines Morley rank, then mult' is preserved under elementary extensions, so we may speak of induced multiplicity on T' . There is also an analogue of Proposition 5.6 that involves both \dim' and mult' : If T defines Morley rank, then T' defines pseudo-denseness if and only if T' defines induced dimension and induced multiplicity. We do not include it here as we do not have an application in mind.

We get the main result of this section, which is a restatement of Theorem 1.6:

Theorem 5.8. *Suppose T_\cap is \aleph_0 -stable and defines multiplicity. If each T_i defines induced dimension, then T_\cup^* exists.*

Proof. This is an immediate consequence of Theorem 3.4, Proposition 5.1, and Proposition 5.6. \square

Remark 5.9. Proposition 5.6 and Theorem 5.8 are mainly of interest because there are several situations where the natural dimension is induced dimension. Proposition 5.11 below presents a general class of such situations. In forthcoming work of the third author and Aschenbrenner it will be shown that T' defines induced dimension when T is ACF_0 and T' is either DCF_0 or the theory of $(\mathbb{C}; \mathbb{R})$ or $(\mathbb{C}_p; \mathbb{Q}_p)$.

The **algebraic dimension** $\text{adim}(X)$ of an \mathcal{M} -definable set X is the maximal k for which there is $a = (a_1, \dots, a_k)$ in the extension $X(\mathcal{M})$ of X to an \aleph_0 -saturated model \mathcal{M} such that (after permuting coordinates) a_1, \dots, a_k are acl-independent over M . It is well-known that algebraic dimension is an ordinal dimension on $\text{Th}(\mathcal{M})$, which coincides with Morley rank for strongly minimal theories. Fact 5.10 is also well known (see [CP98, Lemma 2.2]).

Fact 5.10. A theory defines algebraic dimension if and only if it eliminates \exists^∞ .

Proposition 5.11. *Suppose T is strongly minimal and acl' agrees with acl in all T' -models. Then T' defines induced dimension if and only if T' eliminates \exists^∞ .*

Proof. Suppose $\mathcal{M}' \models T'$, and $\mathcal{M} = \mathcal{M}'|L$. Since T is strongly minimal, $\dim = \text{adim}$. We write \dim' for the induced dimension on T' and adim' for the algebraic

dimension in \mathcal{M}' . Using Fact 5.10, both directions of the equivalence will be proved if we show that $\dim' = \text{adim}'$.

If X' is an arbitrary \mathcal{M}' -definable subset of M^x ,

$$\dim'(X') = \min\{\text{adim}(X) \mid X \subseteq M^x \text{ is } \mathcal{M}\text{-definable, and } X' \subseteq X\}.$$

As $\text{acl}' = \text{acl}$, whenever $a \in X'(\mathcal{M}')$ has k components that are acl' -independent over M , these components are also acl -independent over M , and we have $a \in X(\mathcal{M}')$ for any \mathcal{M} -definable X such that $X' \subseteq X$. Hence, $\text{adim}'(X') \leq \dim'(X')$.

Conversely, let $X \subseteq M^x$ be a pseudo-closure of X' , and $k = \text{adim}(X)$. Then X' is not contained in any \mathcal{M} -definable set of smaller dimension. Since the set of \mathcal{M} -definable sets of dimension less than k is closed under finite unions, by compactness there is some $a' \in X'(\mathcal{M}')$ that is not contained in any \mathcal{M} -definable set of dimension less than k . If a' does not have k components that are acl' -independent over M , then since $\text{acl}' = \text{acl}$, this dependence is witnessed by $a' \in Y$, where Y is \mathcal{M} -definable and $\text{adim}(Y) < k$. This contradicts the choice of a' . \square

When T is the theory of algebraically closed fields and T' is the theory of algebraically closed valued fields, T' eliminates \exists^∞ and acl' agrees with acl in all T' -models; see [vdD89] for details. Thus, using Proposition 5.11 and Theorem 5.8, we obtain a new proof of the existence of a model companion for the theory of algebraically closed fields with multiple valuations.

6. \aleph_0 -CATEGORICAL AND \aleph_0 -STABLE BASE

In this section, we generalize Winkler's result [Win75] on model companions of disjoint unions of theories to allow T_\cap to be any complete \aleph_0 -stable and \aleph_0 -categorical theory with weak elimination of imaginaries. Our generalization applies to several other examples from Section 2.3, such as the random graph and other combinatorial Fraïssé limits, and generic Skolemizations.

Throughout, we keep the additional notational conventions described at the beginning of Section 3 and further assume that T is \aleph_0 -categorical with only infinite models and \aleph_0 -stable. We write \dim for Morley rank on T and mult for Morley degree on T .

The \aleph_0 -stable assumption allows us to make extensive use of Proposition 5.1, which ensures that every subset of a model of T has a pseudo-closure. It also provides us with the following “inductive” procedure to check whether a subset is pseudo-dense in an almost irreducible set. Let \mathcal{M} be a model of T . Recall from Section 5 that \subseteq_a denotes the almost subset relation, and $=_a$ denotes the almost equality relation. A collection \mathcal{D} of almost irreducible subsets of M^x in $\mathcal{M} \models T$ is **representative** if \mathcal{D} contains a (not necessarily unique) representative for each almost equality class.

Lemma 6.1. *Suppose $X \subseteq M^x$ is almost irreducible, \mathcal{D} is a representative collection of almost irreducible subsets of M^x , and A is a subset of M^x . For $\alpha < \dim X$, let $\mathcal{D}_\alpha(A, X) \subseteq \mathcal{D}$ consist of all $Y \in \mathcal{D}$ such that:*

$$\dim Y = \alpha, \quad A \text{ is pseudo-dense in } Y, \quad \text{and } Y \subseteq_a X.$$

If $\mathcal{D}_\beta(A, X) = \emptyset$ for all $\alpha < \beta < \dim X$, then:

- (1) *If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then A is pseudo-dense in X .*

- (2) If $\mathcal{D}_\alpha(A, X)$ is finite up to almost equality, $X_\alpha^1, \dots, X_\alpha^n$ are representatives of the almost equality classes, and

$$A' = A \setminus \bigcup_{i=1}^n X_\alpha^i,$$

then $\mathcal{D}_\beta(A', X) = \emptyset$ for all $\alpha \leq \beta < \dim X$, and A is pseudo-dense in X if and only if A' is.

Proof. As \mathcal{M} is \aleph_0 -stable, $A \cap X$ has a pseudo-closure Y that is a subset of X by Proposition 5.1. Suppose $\mathcal{D}_\beta(A, X) = \emptyset$ for all $\alpha < \beta < \dim X$. Then either $\dim Y \leq \alpha$ or $\dim Y = \dim X$. If $\mathcal{D}_\alpha(A, X)$ is infinite up to almost equality, then $\dim Y > \alpha$, and so $\dim Y = \dim X$. The latter implies A is pseudo-dense in X by Proposition 5.1. Thus we get statement (1).

Now suppose $X_\alpha^1, \dots, X_\alpha^n$ and A' are as stated in (2). Since A' is a subset of A , $\mathcal{D}_\beta(A', X)$ is a subset of $\mathcal{D}_\beta(A, X)$ for all β . So in particular, $\mathcal{D}_\beta(A', X) = \emptyset$ for all $\alpha < \beta < \dim X$. Suppose X_α is an element of $\mathcal{D}_\alpha(A', X)$. Then A is also pseudo-dense in X_α and so $X_\alpha =_a X_\alpha^i$ for some $i \in \{1, \dots, n\}$. As $X_\alpha^i \cap A' = \emptyset$, X_α^i and X_α are both almost irreducible, and $\dim X_\alpha^i = \dim X_\alpha$, it follows from Lemma 3.1 that A' is not pseudo-dense in X_α , which is absurd. Thus,

$$\mathcal{D}_\beta(A', X) = \emptyset \quad \text{for all } \alpha \leq \beta < \dim X.$$

If A' is pseudo-dense in X , then clearly A is. Suppose A' is not pseudo-dense in X . Then $A' \cap X$ has a pseudo-closure Y' with $\dim Y' < \dim X$. It follows that A has a pseudo-closure Y that is a subset of $Y' \cup X_\alpha^1 \cup \dots \cup X_\alpha^n$. Then

$$\dim Y \leq \max(\dim Y', \alpha) < \dim X,$$

and so A is not pseudo-dense in X . We have thus obtained all the desired conclusions in (2). \square

Lemma 6.1 is hardly useful if the purpose is defining pseudo-denseness for a general \aleph_0 -stable theory. The issue is that many of the objects involved in the previous lemma are not definable. However, many of them are definable under the additional assumption of \aleph_0 -categoricity. We recall a number of facts about \aleph_0 -categorical and \aleph_0 -stable theories.

Fact 6.2. The first two statements below only require \aleph_0 -categoricity:

- (1) If T has only infinite models, then T is complete.
- (2) For all finite x , there are finitely many formula $\varphi(x)$ up to T -equivalence.
- (3) T defines multiplicity.
- (4) ([CHL85], Theorem 5.1) \mathcal{M} has finite Morley rank. That is, for all finite x , $\dim M^x < \omega$.
- (5) ([CHL85], Theorem 6.3) If x is a finite tuple of variables, and $p \in S^x(\mathcal{M})$, then p is definable over an element of $M^x \times M^x$.

We now prove a key lemma that does not hold outside of the \aleph_0 -categorical setting. An L -formula $\psi(x, z)$ is **representative** for the tuple of variables x if it defines in every $\mathcal{M} \models T$ a representative collection of almost irreducible sets.

Lemma 6.3. For each finite tuple of variables x , there is a representative formula $\psi(x, z)$.

Proof. Fix $\mathcal{M} \models T$ and a finite tuple x of variables. We claim that every almost irreducible subset X of M^x is almost equal to a subset X' of M^x such that X' is definable over an element of $M^w = M^x \times M^x$.

Let $p \in S^x(M)$ be the generic type of X and p^{eq} the unique element of $S^x(M^{\text{eq}})$ extending p . Fact 6.2(5), gives us $d \in M^w = M^x \times M^x$ such that p is definable over d . Then p^{eq} is definable over d and therefore stationary over $\text{acl}^{\text{eq}}(d)$. Hence,

$$q = p^{\text{eq}}|_{S^x(\text{acl}^{\text{eq}}(d))} \text{ has } \text{mult}(q) = 1.$$

Let $X'' \subseteq M^x$ be defined by a formula in q such that X'' has minimal Morley rank and degree. Then X'' is \mathcal{M}^{eq} -definable over $\text{acl}^{\text{eq}}(d)$ and $X'' =_a X$. Let X''_1, \dots, X''_k are the finitely many conjugates of X'' by $\text{Aut}(\mathcal{M}^{\text{eq}}/d)$, and set $X' = \bigcap_{i=1}^k X''_i$. It is easy to check that X' is definable over d and $X' =_a X$. This completes the proof of the claim.

By Fact 6.2(2) there are finitely many formulas $\psi_1(x, w), \dots, \psi_l(x, w)$ such that every L -formula in variables (x, w) is T -equivalent to one of these. Through routine manipulation, we can get a finite tuple z of variables and a formula $\psi(x, z)$ and such that for all $i \in I$ and $d \in M^w$, there is $c \in M^z$ with $\psi_i(\mathcal{M}, d) = \psi(\mathcal{M}, c)$. Using the claim, if $(X_c)_{c \in M^z}$ is the family of subsets of M^x defined by $\psi(x, z)$, then every almost irreducible X is almost equal to X_c for some $c \in M^z$.

Finally, by Fact 6.2(3), T defines multiplicity. So we can modify $\psi(x, z)$ to ensure that every member of the family $(X_c)_{c \in M^z}$ is almost irreducible. The result is a representative formula for x because by Fact 6.2(1), T is complete. \square

A function up-to-permutation from $Z \subseteq M^z$ to M^w is a relation $f \subseteq Z \times M^w$ satisfying the following two conditions:

- (1) For all $c \in Z$, there is $d \in M^w$ such that $(c, d) \in f$.
- (2) If (c, d) and (c, d') are both in f , then d is a permutation of d' .

A function up-to-permutation f determines an ordinary function $\tilde{f} : Z \rightarrow M^w / \sim$, where \sim is the equivalence relation defined by permutations. We are interested in f instead of \tilde{f} , as it is possible that f is \mathcal{M} -definable while \tilde{f} is only \mathcal{M}^{eq} -definable. For $C \subseteq Z$, we will write $f(Z)$ for the set

$$\{d \in M^w \mid \text{there is } c \in C \text{ such that } (c, d) \in f\}.$$

It is easy to observe that $|\tilde{f}(Z)| \leq |f(Z)| \leq |w|! |\tilde{f}(Z)|$ with \tilde{f} as above. In particular, $f(Z)$ is finite if and only if $\tilde{f}(Z)$ is.

Recall that T **weakly eliminates imaginaries** if for all $\mathcal{M} \models T$ and all $b \in \mathcal{M}^{\text{eq}}$, there exists $a \in \mathcal{M}$ such that $a \in \text{acl}^{\text{eq}}(b)$ and $b \in \text{dcl}^{\text{eq}}(a)$. The following Fact 6.4 only uses the assumption that T is complete.

Fact 6.4. Suppose T weakly eliminates imaginaries. For all $\mathcal{M} \models T$, 0-definable $Z \subseteq M^z$, and 0-definable equivalence relation $R \subseteq Z^2$, there is a tuple w of variables and a 0-definable function up-to-permutation from Z to M^w such that cRc' in Z if and only if $f(c) = f(c')$. Moreover, the choice of formula defining f can be made depending only on the choices of L -formulas defining Z and R but not on the choice of \mathcal{M} .

Below is the key proposition for this section. The main geometrical idea of the proof is already contained in Lemma 6.1, we just need to check that everything can be carried out definably.

Proposition 6.5. *Suppose T weakly eliminates imaginaries. Then T' defines pseudo-denseness over T if and only if T' eliminates \exists^∞ .*

Proof. Let $\mathcal{M}' \models T'$ and $\mathcal{M} = \mathcal{M}'|L$. In this proof, everything will be uniform in \mathcal{M}' : when we say “definable”, we mean “definable by a formula which does not depend on the choice of \mathcal{M}' ”. Alternatively, we can obtain this uniformity for free by working in a sufficiently saturated model.

For the forward direction, suppose T' defines pseudo-denseness. Let $(X'_b)_{b \in M^y}$ be a family of subsets of M^x defined by an L' -formula $\varphi'(x, y)$. Our job is to show that the set

$$\{b \in M^y : X'_b \text{ is infinite}\}$$

is definable by an L' -formula.

Using Lemma 6.3, we get a representative formula $\psi(x, z)$ for the tuple of variables x . Let $(X_c)_{c \in M^z}$ be the family defined by $\psi(x, z)$. Note that X'_b is pseudo-dense in each of the almost irreducible components of its pseudo-closure. So X'_b is infinite if and only if there is $c \in M^z$ such that

$$X'_b \text{ is pseudo-dense in } X_c \text{ and } \dim X_c > 0.$$

As T' defines pseudo-denseness, the set of pairs (b, c) with X'_b pseudo-dense in X_c is definable by an L' -formula. By Fact 6.2, T defines Morley rank, so the set of $c \in M^z$ with $\dim X_c > 0$ is definable by an L -formula. Hence, we get the desired conclusion.

For the backward implication, suppose T' eliminates \exists^∞ . Let $(X'_b)_{b \in M^y}$ and $(X_c)_{c \in M^z}$ be families of subsets of M^x defined by an L' -formula $\varphi'(x, y)$ and an L -formula $\psi(x, z)$, respectively. Set

$$P = \{(b, c) \in M^{(y, z)} \mid X'_b \text{ is pseudo-dense in } X_c\}.$$

We need to show that P is definable by an L' -formula.

We first reduce to the special case where $\psi(x, z)$ is a representative formula for the tuple of variables x . Using Lemma 6.3, we get a representative formula $\delta(x, w)$ for the tuple of variables x . Let $(X_d)_{d \in M^w}$ be the family of subsets of M^x defined by $\delta(x, w)$. For $b \in M^y$ and $c \in M^z$, $(b, c) \in P$ if and only if X'_b is pseudo-dense in X_d for all $d \in M^w$ with $X_d \subseteq_a X_c$ and $\dim X_d = \dim X_c$. Again, T defines Morley rank, so T defines the relation of being almost a subset. Hence, we can deduce the general case from this special case.

We decompose P into finitely many sets, which we will then show to be definable using induction. For $\gamma \leq \dim M^x$, set

$$P^\gamma = \{(b, c) \in P \mid \dim X_c = \gamma\}.$$

Then $P = \bigcup_{\gamma \leq \dim M^x} P^\gamma$. Therefore, as $\dim(M^x)$ is finite, it suffices to show that for all γ , P^γ is definable.

We will proceed by induction on γ , uniformly with respect to the choice of the formula $\varphi(x, y)$. For $\gamma = 0$, by Lemma 3.1(1), X'_b is pseudo-dense in a finite set if and only if X'_b contains that finite set. So $(b, c) \in M^{(y, z)}$ is in P^0 if and only if $\dim(X_c) = 0$ and $X_c \subseteq X'_b$. Hence, P^0 is definable by an L' -formula.

Now, assume $\gamma > 0$ and we have proven the statement for all smaller values of γ . Let $\mathcal{D} = (X_c)_{c \in M^z}$ be the representative collection of almost irreducible subsets of M^x defined by $\psi(x, z)$. Toward applying Lemma 6.1, set

$$D_{\alpha, b, c} = \{d \in M^z \mid (b, d) \in P^\alpha \text{ and } X_d \subseteq_a X_c\}$$

for each ordinal α and each $(b, c) \in M^y \times M^z$. In other words, if $\mathcal{D}_\alpha(X'_b, X_c)$ is defined as in Lemma 6.1, then d is in $D_{\alpha, b, c}$ if and only if X_d is in $\mathcal{D}_\alpha(X'_b, X_c)$. For all $\alpha < \gamma$, since T defines Morley rank and P^α is definable by the inductive hypothesis, the family $(D_{\alpha, b, c})_{(b, c) \in M^{(y, z)}}$ is definable by an L' -formula.

For $\alpha < \gamma$, set

$$P_\alpha^\gamma = \{(b, c) \in P^\gamma \mid D_{\beta, b, c} = \emptyset \text{ for all } \alpha < \beta < \gamma\}.$$

Then $P^\gamma = P_{\gamma-1}^\gamma$. Hence, we can get the desired conclusion by showing the stronger fact that P_α^γ is definable by an L' -formula for all $\alpha < \gamma \leq \dim M^x$.

Now we proceed by an inner induction on α . When $\alpha = 0$, we get from Lemma 6.1 that $(b, c) \in M^{(y, z)}$ is in P_0^γ if and only if

$$\dim X_c = \gamma, \quad D_{\beta, b, c} = \emptyset \text{ for all } 0 < \beta < \gamma, \quad \text{and } X'_b \text{ is infinite.}$$

Hence, P_0^γ can be defined by an L' -formula, by the assumption that T' eliminates \exists^∞ and the fact that T defines Morley rank.

Suppose $0 < \alpha < \gamma$ and we have shown the statement for all smaller values of α . As noted above, for all $\beta < \gamma$, the families $(D_{\beta, b, c})_{(b, c) \in M^y \times M^z}$ are definable by L' -formulas. Recall that T defines Morley rank and weakly eliminates imaginaries. By Fact 6.4, there is w and a L -definable function up-to-permutation f from Z to M^w , such that for all d_1 and d_2 in M^z ,

$$f(d_1) = f(d_2) \text{ if and only if } X_{d_1} =_a X_{d_2}.$$

In particular, the family $(f(D_{\alpha, b, c}))_{(b, c) \in Y \times Z_\gamma}$ can be defined by an L' -formula. As T' eliminates \exists^∞ , there is n such that

$$|f(D_{\alpha, b, c})| > n|w|! \text{ implies } f(D_{\alpha, b, c}) \text{ is infinite.}$$

Now let \tilde{Y} be the set of $\tilde{b} = (b, c, d_1, \dots, d_n)$ in $M^y \times M^z \times M^z \times \dots \times M^z$ such that the following properties hold:

- (1) $\dim X_c = \gamma$ and $D_{\beta, b, c} = \emptyset$ for all $\alpha < \beta < \gamma$.
- (2) $f(D_{\alpha, b, c})$ is finite.
- (3) $\dim X_{d_i} = \alpha$ and X'_b is pseudo-dense in X_{d_i} for $i \in \{1, \dots, n\}$.
- (4) If $\dim X_d = \alpha$ and X'_b is pseudo-dense in X_d for some $d \in M^z$, then $X_d =_a X_{d_i}$ for some $i \in \{1, \dots, n\}$.

By the inductive hypothesis and the fact that T defines Morley rank, \tilde{Y} is definable by an L' -formula. For each $\tilde{b} \in \tilde{Y}$, set

$$\tilde{X}'_{\tilde{b}} = X'_b \setminus \bigcup_{i=1}^n X_{d_i}.$$

For $\tilde{b} \in M^{\tilde{y}} \setminus \tilde{Y}$, let $\tilde{X}'_{\tilde{b}}$ be the empty set. Then, the family $(\tilde{X}'_{\tilde{b}})_{\tilde{b} \in M^{\tilde{y}}}$ is definable by an L' -formula $\tilde{\varphi}'(x, \tilde{y})$. We obtain $\tilde{P}_{\alpha-1}^\gamma$ from $\tilde{\varphi}'(x, \tilde{y})$ and $\psi(x, z)$ in the same fashion as we obtain $P_{\alpha-1}^\gamma$ from $\varphi'(x, y)$ and $\psi(x, z)$. The induction hypothesis, applied to the formula $\tilde{\varphi}'(x, \tilde{y})$, implies that $\tilde{P}_{\alpha-1}^\gamma$ is definable. From Lemma 6.1, (b, c) is in P_α^γ if and only if $\dim X_c = \gamma$ and $D_{\beta, b, c} = \emptyset$ for all $\alpha < \beta < \gamma$ and either of the following hold:

- (1) $f(D_{\alpha, b, c})$ is infinite.
- (2) There are d_1, \dots, d_n in M^z with $\tilde{b} = (b, c, d_1, \dots, d_n) \in \tilde{Y}$ and $(\tilde{b}, c) \in \tilde{P}_{\alpha-1}^\gamma$.

Thus P_α^γ is definable, which completes the proof. \square

We get the main result of this section, which is a restatement of Theorem 1.7:

Theorem 6.6. *Suppose T_\cap is complete, \aleph_0 -stable, and \aleph_0 -categorical. If T_\cap weakly eliminates imaginaries, and each T_i eliminates \exists^∞ , then T_\cup^* exists. If T_i^{eq} eliminates \exists^∞ for all i , then T_\cup^* exists.*

Proof. The first statement follows from Theorem 3.4, Proposition 5.1, and Proposition 6.5. The second statement then follows from the first statement and Remark 2.5(4): if T_i^{eq} eliminates \exists^∞ , so does $T_i^{\cap\text{-eq}}$, so we may assume T_\cap eliminates imaginaries. \square

The conditions of Theorem 6.6 are satisfied when L_\cap is the empty one-sorted language and T_\cap is the theory of infinite sets. So we recover Winkler's theorem on disjoint unions of theories [Win75]. Using similar ideas, we recover Winkler's theorem on generic Skolemizations; see Section 2.3.

The assumptions of Corollary 6.7 are very strong, but it is applicable more often than one might expect. For example, it applies to the random graph and many other combinatorial Fraïssé limits, as presented in Section 2.3.

Corollary 6.7. *Suppose T_\cap is complete and each T_i is interpretable in the theory of infinite sets. Then T_\cup^* exists.*

Proof. The theory of infinite sets is \aleph_0 -stable, \aleph_0 -categorical, and eliminates \exists^∞ in T^{eq} . Each of these three properties is preserved under interpretations, so if T_i is interpretable in the theory of infinite sets then T_i is \aleph_0 -stable, \aleph_0 -categorical, and T_i^{eq} eliminates \exists^∞ . So the result follows from Theorem 6.6. \square

The theory T_q of vector spaces over a finite field with q elements is \aleph_0 -stable, \aleph_0 -categorical, and weakly eliminates imaginaries. Thus any theory T' extending T_q defines pseudo-denseness if and only if it eliminates \exists^∞ . This does not generalize to vector spaces over characteristic zero fields, which are \aleph_0 -stable and weakly eliminate imaginaries, but are not \aleph_0 -categorical. For example, let T be the theory of torsion-free divisible abelian groups (vector spaces over \mathbb{Q}). Let T' be ACF_0 , note that T' is an expansion of T . Then T' does not define pseudo-denseness over T . Suppose \mathcal{M}' is an \aleph_1 -saturated model of T' . Let

$$L = \{(a, b, c) \in \mathbf{M}^3 \mid ab = c\}$$

and consider the definable family $\{L_a \mid a \in M\}$ where $L_a = \{(b, c) \in \mathbf{M}^2 \mid ab = c\}$. We leave the easy verification of Lemma 6.8 to the reader:

Lemma 6.8. *Fix $a \in M$. Then L_a is pseudo-dense in \mathbf{M}^2 if and only if $a \notin \mathbb{Q}$.*

As \mathbb{Q} is countable and infinite it cannot be a definable set in an \aleph_1 -saturated structure. Thus \mathcal{M}' does not define pseudo-denseness over $(\mathbf{M}; +)$. The same argument shows that any theory expanding T' does not define pseudo-denseness over T .

There is a natural rank rk on any \aleph_0 -categorical theory, described in [Sim18b, Section 2.3] and [CH03, Section 2.2.1]. This rank is known to agree with thorn rank on \aleph_0 -categorical structures, so it is an ordinal rank on rosy \aleph_0 -categorical theories. A special case of Theorem 4.10 is that any expansion of the theory DLO of dense linear orders without endpoints defines pseudo-denseness over DLO with respect to rk (which agrees with the usual o-minimal dimension over DLO). This fact, together with Proposition 6.5, and recent groundbreaking work on NIP \aleph_0 -categorical structures [Sim18b, Sim18a] motivates Question 6.9.

Question 6.9. Suppose T is NIP, \aleph_0 -categorical, and rosy. If T' eliminates \exists^∞ , then must T' define pseudo-denseness over T (with respect to rk)?

Unfortunately, rk does not necessarily agree with Morley rank on \aleph_0 -stable and \aleph_0 -categorical theories. One might hope that an approach to Question 6.9 would synthesize the ideas of Section 4 and Section 6.

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