EXTERNALLY DEFINABLE QUOTIENTS AND NIP EXPANSIONS OF THE REAL ORDERED ADDITIVE GROUP

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ABSTRACT. Let $\mathcal{R}$ be an NIP expansion of $(\mathbb{R}, <, +)$ by closed subsets of $\mathbb{R}^n$ and continuous functions $f : \mathbb{R}^m \to \mathbb{R}^n$. Then $\mathcal{R}$ is generically locally o-minimal. This follows from a more general theorem on NIP expansions of locally compact groups, which itself follows from a result on quotients of definable sets in $\aleph_1$-saturated NIP structures by equivalence relations which are both externally definable and $\bigwedge$-definable. We also show that $\mathcal{R}$ is strongly dependent if and only if $\mathcal{R}$ is either o-minimal or $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal for some $\alpha > 0$.

Suppose $\mathcal{R}$ expands $(\mathbb{R}, +, \times)$. Hieronymi [17] has shown that if $E \subseteq \mathbb{R}^n$ is closed, discrete, and $\mathcal{R}$-definable, $f : E \to \mathbb{R}$ is $\mathcal{R}$-definable and $f(E)$ is somewhere dense then $\mathcal{R}$ defines $\mathbb{Z} \subseteq \mathbb{R}$. Hieronymi's theorem has many applications which combine to form a general theory of first order expansions of $(\mathbb{R}, +, \times)$. The main conjecture in this theory is that if $\mathcal{R}$ does not define $\mathbb{Z}$ then every subset of $\mathbb{R}^n$ which is definable in the reduct of $\mathcal{R}$ generated by all closed $\mathcal{R}$-definable sets either has interior or is nowhere dense. (Observe that this conjecture generalizes Hieronymi's theorem.) We show that if $\mathcal{S}$ is an NIP expansion of $(\mathbb{R}, <, +)$ and $X, Y \subseteq \mathbb{R}^n$ are definable in the reduct of $\mathcal{S}$ generated by all closed $\mathcal{S}$-definable sets then X either has interior or is nowhere dense in Y. (There are many examples of NIP expansions of $(\mathbb{R}, <, +)$ which define dense and co-dense sets such as the expansion of $(\mathbb{R}, +, \times)$ by a unary predicate defining the real algebraic numbers.) Our proof is on general NIP-theoretic grounds and has many applications, some of which we describe.

Let $\mathcal{M}$ be a structure. Suppose M^n is equiped with a topology for all $n \geq 1$. In this paper we will always equip M^n with the product topology when $n \geq 2$, but the basic definitions are naturally formulated in the more general context. We say that $\mathcal{M}$ is **noiseless** if every definable subset of every M^n either has interior or is nowhere dense. We say that $\mathcal{M}$ is **noiseless in one variable** if every definable subset of M either has interior or is nowhere dense. We say that $\mathcal{M}$ is **strongly noiseless** if the induced structure on any definable $Y \subseteq M^n$ is noiseless. Hence $\mathcal{M}$ is strongly noiseless if whenever X, Y are definable subsets of M^n then X is either nowhere dense in Y or has interior in Y. We say that $\mathcal{M}$ is **strongly noiseless in one variable** if whenever $X, Y \subseteq M$ are definable then X is either nowhere dense in Y or has interior in Y. (These definitions are essentially due to Chris Miller.)

The **open core** $\mathcal{M}^\circ$ of $\mathcal{M}$ is the structure on M whose primitive n-ary relations are the closures of $\mathcal{M}$-definable subsets of M^n. If $\mathcal{M}$ defines a basis for the topology on each M^n then $\mathcal{M}^\circ$ is the reduct of $\mathcal{M}$ generated by all closed $\mathcal{M}$-definable sets. If the topology on M^n is Hausdorff, $X \subseteq M^m$ is $\mathcal{M}$-definable and either open or

Date: October 17, 2021.

closed, and $f : X \to M^n$ is continuous and $\mathcal{M}$-definable, then f is definable in $\mathcal{M}^\circ$. We say $\mathcal{M}$ is **generated by closed sets** or is an **expansion by closed sets** if it is interdefinable with $\mathcal{M}^\circ$. A subset of a topological space is **constructible** if it is a boolean combination of closed sets. Miller and Dougherty [9] show that if $\mathcal{M}$ defines a basis for the topology on each M^n then $\mathcal{M}^\circ$ is interdefinable with the reduct of $\mathcal{M}$ generated by all constructible $\mathcal{M}$-definable sets.

Theorem A. *Suppose G is a group and $\mathcal{G}$ is an expansion of G such that $\mathcal{G}$ defines*

(1) a basis for a locally compact Hausdorff group topology on G,
(2) a family $(X_a : a \in G^n)$ of subsets of G such that for any compact $X \subseteq G$ there is $a \in G^n$ such that X_a is compact and contains X.

If $\mathcal{G}$ is NIP *then $\mathcal{G}^\circ$ is strongly noiseless.*

For example if $\mathcal{G}$ is an expansion of $(\mathbb{R}, <, +)$ then we take the definable basis to be $\{(-t, t) : t \in> 0\}$ and $X_t = [-t, t]$ for $t > 0$. Note that if the group topology on G is compact then (2) is trivially satisfied. Theorem A is a consequence of Theorem B, which is of independent interest. Suppose that $\mathcal{N}$ is λ-saturated, X is an $\mathcal{N}$-definable set, and E is a $\bigwedge$-definable equivalence relation on X (i.e. E is an intersection $< \lambda$ definable sets). The logic topology on X/E is defined by declaring a subset of X/E to be closed if and only if it is the image of a $\bigwedge$-definable subset of X under the quotient map $X \to X/E$. The logic topology is Hausdorff, it is discrete if and only if E is definable, and it is compact if and only if X/E is small. Recall that a compact Hausdorff topological space admits a compatible uniform structure which is unique up to uniform equivalence. When X/E is not small then we can still define a canonical uniform structure on X/E which we refer to as the logic uniformity, see Section 3.3 below.

Theorem B. *Suppose that $\mathcal{N}$ is an $\aleph_1$-saturated* NIP *structure and X is an $\mathcal{N}$-definable set. Let E be an equivalence relation on X which is both $\bigwedge$-definable and externally definable. Suppose that the Shelah completion $\mathcal{N}^{\mathrm{Sh}}$ of $\mathcal{N}$ defines a basis for the logic uniformity on X/E. Then the structure induced on X/E by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless.*

The "Shelah completion" is usually referred to as the "Shelah expansion". We adopt the term "completion" because we believe it is more suggestive.

We now describe the proof of Theorem A from Theorem B. Let $\mathcal{G} \prec \mathcal{N}$ be $|G|^+$-saturated. We let **Fin** be the subgroup of "finite" elements of $\mathcal{N}$ and **Inf** be the subgroup of "infinitesimal" elements of $\mathcal{N}$. We show that **Fin** and **Inf** are both externally definable. Using familiar ideas from nonstandard analysis we see that **Fin**/**Inf** can be identified with G and the quotient map **Fin** $\to$ **Fin**/**Inf** can be identified with the usual standard part map. We show that $\mathcal{G}^\circ$ is a reduct of the structure induced on G by $\mathcal{N}^{\mathrm{Sh}}$. Our proof does not rely on the group structure on G, only on the induced uniform structure. For this reason our proof goes through in the more general setting of locally compact Hausdorff uniform spaces, see Section 6. The general result is reasonably sharp, see Section 7.

We now describe consequences of Theorem A over the reals. Suppose $\mathcal{R}$ is an expansion of $(\mathbb{R}, <)$. We say that $\mathcal{R}$ is **locally o-minimal** if for every $\mathcal{R}$-definable $X \subseteq \mathbb{R}$ and $a \in X$ there is an open interval I containing a such that $I \cap X$ is definable in $(\mathbb{R}, <)$. We say that $\mathcal{R}$ is **generically locally o-minimal** if for every $\mathcal{R}$-definable $X \subseteq \mathbb{R}$ there is a dense definable open $V \subseteq X$ such that for all $a \in V$ there is an open interval I containing a such that $I \cap X$ is definable in $(\mathbb{R}, <)$. It is known that an expansion of $(\mathbb{R}, <, +)$ is strongly noiseless if and only if it is generically locally o-minimal (see Theorem 8.1 below). We obtain Theorem C.

Theorem C. *An* NIP *expansion of* $(\mathbb{R}, <, +)$ *by closed sets is generically locally o-minimal.*

Theorem C allows us to lift much of o-minimality to NIP expansions of $(\mathbb{R}, <, +)$ by closed sets. This will be done in another paper. Theorem C fails over $(\mathbb{R}, <)$. If $f : [0, 1] \to [0, 1]$ is the classical Cantor function, aka "the devil's staircase", then $(\mathbb{R}, <, f)$ is dp-minimal and noisy, see Section 7.

We obtain a complete description of strongly dependent expansions of $(\mathbb{R}, <, +)$ by closed sets. We say that $\mathcal{N}$ is **$\mathcal{M}$-minimal** if $\mathcal{N}$ is an expansion of $\mathcal{M}$ and every $\mathcal{N}$-definable subset of M is $\mathcal{M}$-definable. (If $\mathcal{M} = (\mathbb{R}, <)$ then $\mathcal{M}$-minimality is o-minimality.) In Section 9 we prove Theorem D by combining Theorem C, results of Dolich and Goodrick [7] on strongly dependent expansions of ordered abelian groups, work of Kawakami, Takeuchi, Tanaka, and Tsuboi [18] on locally o-minimal structures, and a recent result of Bès and Choffrut [2] on $(\mathbb{R}, <, +, \mathbb{Z})$.

Theorem D. *Suppose* $\mathcal{R}$ *is an expansion of* $(\mathbb{R}, <, +)$*. The following are equivalent.*

1. $\mathcal{R}$ *is a strongly dependent expansion by closed sets.*
2. $\mathcal{R}$ *is strongly dependent and noiseless.*
3. $\mathcal{R}$ *is either o-minimal or* $(\mathbb{R}, <, +, \alpha\mathbb{Z})$*-minimal for some* $\alpha > 0$*.*
4. $\mathcal{R}$ *is either o-minimal or interdefinable with* $(\mathbb{R}, <, +, \mathcal{B}, \alpha\mathbb{Z})$ *for some collection* $\mathcal{B}$ *of bounded sets such that* $(\mathbb{R}, <, +, \mathcal{B})$ *is o-minimal.*
5. $\mathcal{R}$ *is either o-minimal or locally o-minimal and interdefinable with* $(\mathcal{S}, \alpha\mathbb{Z})$ *for some o-minimal expansion* $\mathcal{S}$ *of* $(\mathbb{R}, <, +)$*.*

In each case above α *is unique up to rational multiples. Suppose* $\mathcal{R}$ *is* $(\mathbb{R}, <, +, \alpha\mathbb{Z})$*-minimal. Then every definable subset of* $\mathbb{R}^n$ *is a finite union of sets of the form* $\bigcup_{b \in B} \alpha b + A$ *for some* $(\mathbb{Z}, <, +)$*-definable* $B \subseteq \mathbb{Z}^n$ *and* $\mathcal{R}$*-definable* $A \subseteq [0, \alpha)^n$*. It follows that* $\mathcal{R}$ *is bi-interpretable with the disjoint union of the induced structure on* $[0, \alpha)$ *(which is o-minimal) and* $(\mathbb{Z}, <, +)$*.*

Thus a non o-minimal strongly dependent expansion by closed sets has a canonical decomposition into an "integer part" and an o-minimal "fractional part".

The straightforward analogues of Theorems C and D fail for NIP expansions of archimedean ordered abelian groups. In Section 11 we treat the correct analogue. Suppose $(R, <, +)$ is an archimedean ordered abelian group and $\mathcal{R}$ is a NIP expansion of $(R, <, +)$. We define a canonical completion $\mathcal{R}^{\square}$ of $\mathcal{R}$. We believe that this $\mathcal{R}^{\square}$ will be important in the study of $\mathcal{R}$. We expect that if $\mathcal{R}$ is a natural expansion by closed sets then the structure induced on R by $\mathcal{R}^{\square}$ is interdefinable with $\mathcal{R}^{\mathrm{Sh}}$. Theorem E summarizes our results on the completion.

Theorem E. *Let* R *be a dense subgroup of* $(\mathbb{R}, +)$*,* $\mathcal{R}$ *be an* NIP *expansion of* $(R, <, +)$*, and* $\mathcal{R} \prec \mathcal{N}$ *be* $|\mathbb{R}|^+$*-saturated. Let* **Fin** *be the convex hull of* R *in* N *and* **Inf** *be the set of* $a \in N$ *such that* $|a| < b$ *for all* $b \in R, b > 0$*. Identify* **Fin**/**Inf** *with* $\mathbb{R}$ *and identify the quotient map* $\mathbf{Fin} \to \mathbb{R}$ *with the usual standard part map.*

As **Fin** *and* **Inf** *are* $\mathcal{N}^{\mathrm{Sh}}$*-definable we regard* $\mathbb{R}$ *as an imaginary sort of* $\mathcal{N}^{\mathrm{Sh}}$*. The following structures are interdefinable.*

(1) The structure $\mathcal{R}^{\square}$ *on* $\mathbb{R}$ *with an* n*-ary relation symbol defining the closure in* $\mathbb{R}^n$ *of every subset of* R^n *which is externally definable in* $\mathcal{R}$*.*
(2) The structure on $\mathbb{R}$ *with an* n*-ary relation symbol defining the image under the standard part map* $\mathbf{Fin}^n \to \mathbb{R}^n$ *of* $\mathbf{Fin}^n \cap X$ *for every* $\mathcal{N}$*-definable* $X \subseteq N^n$*.*
(3) The open core of the structure induced on $\mathbb{R}$ *by* $\mathcal{N}^{\mathrm{Sh}}$*.*

Furthermore $\mathcal{R}^{\square}$ *is generically locally o-minimal and if* $\mathcal{R}$ *is strongly dependent then* $\mathcal{R}^{\square}$ *is either o-minimal or* $(\mathbb{R}, <, +, \alpha\mathbb{Z})$*-minimal for some* $\alpha > 0$*. The structure induced on* R *by* $\mathcal{R}^{\square}$ *is a reduct of* $\mathcal{R}^{\mathrm{Sh}}$*. If* $\mathcal{R}$ *is strongly dependent and noiseless then the structure induced on* R *by* $\mathcal{R}^{\square}$ *is interdefinable with* $\mathcal{R}^{\mathrm{Sh}}$*.*

Theorem E is related to work of Laskowski and Steinhorn [20] and work of Wencel [33, 34] in the o-minimal and weakly o-minimal settings, respecively.

1. NOTATION AND CONVENTIONS

Throughout m, n, k, d are natural numbers, i, j are integers, and λ, α are real numbers. We let $\mathbb{R}_{>0}$ be the set of positive real numbers. Given a subset X of $A \times B$ and $a \in A$ we let X_a be $\{b \in B : (a, b) \in X\}$.

Throughout all structures are first order. When we say that something is definable in a structure we mean that is definable possibly with parameters from that structure. Two structures on a common domain M are **interdefinable** if they define the same subsets of all M^n. We regard interdefinable structures as the same. Let $\mathcal{M}$ be a structure with domain M. The structure **induced** on $A \subseteq M^m$ by $\mathcal{M}$ is the structure with domain A whose primitive n-ary relations are all sets of the form $X \cap A^n$ for $\mathcal{M}$-definable $X \subseteq M^{mn}$. The structure induced on A eliminates quantifiers if every subset of A^n definable in the induced structure is of the form $Y \cap A^n$ for $\mathcal{M}$-definable $Y \subseteq M^{mn}$. (We will commonly encounter this situation.)

We say that a family of $\mathcal{C}$ of sets is **subdefinable** if it is a subfamily of a definable family of sets. A subdefinable basis for a topology on a definable set X is a subdefinable family of sets forming a basis for a topology on X.

We let $\mathrm{Cl}(X)$ be the closure and $\mathrm{Bd}(X)$ be the boundary of a subset X of a topological space.

We will sometimes work in a multi-sorted setting. Suppose L is a language with S the set of sorts and $\mathcal{M}$ is an L-structure. Then we let M denote the S-indexed family $(M_s)_{s \in S}$ of underlying sets of the sorts of $\mathcal{M}$. If $x = (x_j)_{j \in J}$ is a tuple of variables, we let $M^x = \prod_{j \in J} M_{s(x_j)}$ where $M_{s(x_j)}$ is the sort of the variable x_j. If $\varphi(x, y)$ is an L-formula and $b \in M^y$, we let $\varphi(M^y, b)$ be the set defined by $\varphi(x, b)$.

Suppose $\mathcal{N}$ is $\aleph_1$-saturated. We say that a set A is **small** when $\mathcal{M}$ is $|A|^+$-saturated, equivalently: when $\mathcal{M}$ is λ-saturated for some $\lambda \geq |A|$. A set is $\bigwedge$-definable if it is an intersection of a small family of definable sets. Such sets are often said to be "type-definable".

2. NIP

We recall relevant background on NIP structures. Let $\mathcal{M}$ be a possibly multi-sorted first order structure and $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$.

We first discuss externally definable sets. A subset X of M^x is **externally definable** (in $\mathcal{M}$) if there is an $\mathcal{N}$-definable subset Y of N^x such that $X = M^x \cap Y$. A routine saturation argument shows that the collection of externally definable sets does not depend on choice of $\mathcal{N}$. The proof of Fact 2.1 is left to the reader.

Fact 2.1. *Suppose X is an $\mathcal{M}$-definable set and $<$ is an $\mathcal{M}$-definable linear order on X. Then every $<$-convex subset of X is externally definable in $\mathcal{M}$,*

We will need the following easy fact whose verification we leave to the reader.

Fact 2.2. *Suppose $Y \subseteq N^x$ is externally definable in $\mathcal{N}$. Then $Y \cap M^x$ is externally definable in $\mathcal{M}$.*

We say that $\mathcal{M}$ is **Shelah complete** if every externally definable set is definable. The **Shelah completion** $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ is the expansion of $\mathcal{M}$ by all externally definable subsets of all M^x. If $\mathcal{M}$ is one-sorted then $\mathcal{M}^{\mathrm{Sh}}$ is the structure induced on M by $\mathcal{N}$. Fact 2.3 is due to Shelah [26], see also Chernikov and Simon [5, Corollary 1.10].

Fact 2.3. *Suppose $\mathcal{M}$ is* NIP*. Then every $\mathcal{M}^{\mathrm{Sh}}$-definable set is externally definable in $\mathcal{M}$. It follows that $\mathcal{M}^{\mathrm{Sh}}$ is* NIP *when $\mathcal{M}$ is* NIP *and $\mathcal{M}^{\mathrm{Sh}}$ is strongly dependent when $\mathcal{M}$ is strongly dependent.*

The one-sorted case of Fact 2.3 asserts that the structure induced on M by $\mathcal{N}$ eliminates quantifiers. An easy application of Fact 2.3 shows that the Shelah completion of an NIP structure is Shelah complete, so our terminology is reasonable.

Lemma 2.4 is an easy consequence of Fact 2.3, we leave the proof to the reader.

Lemma 2.4. *Suppose that $\mathcal{M}$ is* NIP*, $\mathcal{M}$ is Shelah complete, and $\mathcal{O} \prec \mathcal{M}$. Then every $\mathcal{O}^{\mathrm{Sh}}$-definable set is of the form $O^n \cap X$ for some $\mathcal{M}$-definable $X \subseteq M^n$.*

We record another theorem of Chernikov and Simon [6, Corollary 9]. The right to left implication is a saturation exercise and does not require NIP.

Fact 2.5. *Suppose $\mathcal{M}$ is* NIP*. Let X be a subset of M^x. Then X is externally definable in $\mathcal{M}$ if and only if there is an $\mathcal{M}$-definable family of $(X_a)_{a \in M^y}$ of subsets of M^x such that for every finite $A \subseteq X$ there is $a \in M^y$ such that $A \subseteq X_a \subseteq X$.*

We will need to use honest definitions at one point. Given an $\mathcal{M}$-definable $Z \subseteq M^x$ we let Z' be the subset of N^x defined by any formula defining Z. Suppose X is an externally definable subset of M^x. Then an $\mathcal{N}$-definable subset Y of N^x is an **honest definition** of X if $Y \cap M^x = X$ and $Y \subseteq Z'$ for every $\mathcal{M}$-definable $Z \subseteq M^x$ such that $X \subseteq Z$. Taking complements, we see that if Y is an honest definition of X then $Y \cap Z' = \emptyset$ for every $\mathcal{M}$-definable $Z \subseteq M^x$ such that $X \cap Z = \emptyset$. The following theorem is a corollary to Fact 2.5 [6, Proposition 1.6].

Fact 2.6. *If $\mathcal{M}$ is* NIP *then every externally definable set has an honest definition.*

3. Uniform structures and $\bigwedge$-definable equivalence relations

3.1. **Uniform structures.** We describe background on uniform structures. Let X be a set and declare $\Delta := \{(x,x) : x \in X\}$ Given $A, B \subseteq X^2$ we declare

$$A^{-1} := \{(x', x) : (x, x') \in A\}$$

and

$$A \circ B := \{(x, x') : \exists y\, (x, y) \in A, (y, x') \in B\}.$$

We say that $A \subseteq X^2$ is **symmetric** if $A^{-1} = A$. A **basis for a uniform structure** on X is a collection $\mathcal{B}$ of subsets of X^2 satisfying the following conditions.

(1) Every element of $\mathcal{B}$ contains Δ.
(2) Every element of $\mathcal{B}$ is symmetric.
(3) For all $U, U' \in \mathcal{B}$ there is $V \in \mathcal{B}$ such that $V \subseteq U \cap U'$.
(4) For every $U \in \mathcal{B}$ there is $V \in \mathcal{B}$ such that $V \circ V \subseteq U$.

Suppose $\mathcal{B}$ is a basis for uniform structure on X. The **uniform structure** $\overline{\mathcal{B}}$ on X generated by $\mathcal{B}$ is the collection of all subsets of X^2 which contain some element of $\mathcal{B}$. Suppose $\mathcal{B}$ is a basis for a uniform structure on X. We associate a topology on X to $\mathcal{B}$ by declaring a subset A of X to be open if and only if for every $a \in A$ there is $U \in \mathcal{B}$ such that $U_a \subseteq A$. This topology only depends on $\overline{\mathcal{B}}$. Note that $(U_a)_{U \in \mathcal{B}}$ is a neighbourhood basis for $a \in X$. (In general U_a need not be an open neighbourhood of a.) We say that a uniform structure is Hausdorff if the associated topology is Hausdorff. It is easy to see that $\overline{\mathcal{B}}$ is Hausdorff if and only if $\bigcap \mathcal{B} = \Delta$.

We leave the following lemma as a exercise.

Lemma 3.1. *Suppose $\mathcal{B}$ is a basis for a uniform structure on X. Let $\mathcal{C}$ be a collection of subsets of X^2 such that*

(1) Every $V \in \mathcal{C}$ is symmetric,
(2) Every $V \in \mathcal{C}$ contains some $U \in \mathcal{B}$,
(3) $\mathcal{B}$ is a subfamily of $\mathcal{C}$.

Then $\mathcal{C}$ is a basis for $\overline{\mathcal{B}}$.

We recall some uniform structures. Recall that a pseudo-metric space (X, d) consists of a set X and a symmetric function $d : X^2 \to \mathbb{R}$ such that $d(a, b) \geq 0$ and

$$d(a, c) \leq d(a, b) + d(b, c) \quad \text{for all } a, b, c \in X.$$

If (X, d) is a pseudo-metric space then the collection of sets of the form $\{(x, x') \in X^2 : d(x, x') < t\}$ for $t > 0$ is a basis for a uniform structure on X which induces the d-topology. If G is a topological group and $\mathcal{U}$ is a neighbourhood basis for the identity then the collection $\{(g, g') \in G^2 : g^{-1}g' \in U\}$, U ranging over $\mathcal{U}$, is a basis for a uniform structure on G which induces the group topology. Suppose τ is a compact Hausdorff topology on X and equip X^2 with the associated product topology. Then the collection of all symmetric open subsets of X^2 containing Δ forms a basis for a uniform structure on X. This is the unique uniform structure on X for which the associated topology is τ. If E is an equivalence relation on X then $\{E\}$ is a basis for a uniform structure on X. Finally, the discrete uniform structure on X is the uniform structure with basis $\{\Delta\}$.

Suppose $\mathcal{C}$ is a basis for a uniform structure on a set Y. Let f be a function $X \to Y$. Then f is uniformly continuous if for every $U \in \mathcal{C}$ there is $V \in \mathcal{B}$ such that for all $(a, b) \in V$ we have $(f(a), f(b)) \in U$. If f is uniformly continuous then f is a continuous map between the topologies associated to $\mathcal{B}$ and $\mathcal{C}$. We say that f is a **uniform equivalence** if f is bijective and f, f^{-1} are both uniformly continuous.

One can associate a quotient Hausdorff uniform structure to a general uniform structure in a canonical way. Note that $E := \bigcap \mathcal{B}$ is an equivalence relation on X. Let π be the quotient map $X \to X/E$. Given $U \in \mathcal{B}$ we let $\pi(U)$ to be the set of $(y, y') \in (X/E)^2$ such that there are $x \in \pi^{-1}(\{y\}), x' \in \pi^{-1}(\{y'\})$ satisfying $(x, x') \in U$. Let $\pi(\mathcal{B}) := (\pi(U))_{U \in \mathcal{B}}$. Then $\pi(\mathcal{B})$ is a basis for a Hausdorff uniform structure on X/E. Both E and the uniform structure on X/E depend only on $\overline{\mathcal{B}}$.

Suppose Y is a subset of X. Then $\mathcal{B}|_Y := \{U \cap Y^2 : U \in \mathcal{B}\}$ is a basis for a uniform structure on Y which we refer to as the induced uniform structure on Y.

We describe the product uniform structure on X^n. Given $U \in \mathcal{B}$ we let U_n be the set of $(x, x') \in X^n \times X^n$ such that $(x_k, x'_k) \in U$ for all $1 \leq k \leq n$ where $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n)$. Then $\mathcal{B}_n := \{U_n : U \in \mathcal{B}\}$ is a basis for a uniform structure on X^n which induces the product topology on X^n.

Lemma 3.2 will be used below. Given a subset A of X and $U \in \mathcal{B}$ we let $A[U]$ be $\bigcup_{a \in A} U_a$. Note that A lies in the interior of $A[U]$. The proof of Lemma 3.2 is easy and left to the reader.

Lemma 3.2. *Suppose $\mathcal{W}$ is a finite collection of nonempty open subsets of X and $A \subseteq X$ is not dense in any $W \in \mathcal{W}$. Then there is $U \in \mathcal{B}$ such that $W \setminus A[U]$ has interior for all $W \in \mathcal{W}$.*

3.2. **Subdefinable uniform structures.** We now suppose that $\mathcal{M}$ is a (possibly multisorted) structure. Given a definable set X, a (sub)definable basis for a uniform structure on X is a (sub)definable family of sets forming a basis for a uniform structure on X. If $\mathcal{B}$ is a subdefinable basis for a uniform structure on X then $\mathcal{B}_n$ is a subdefinable basis for the product uniform structure on X^n and $\mathcal{B}_n|_Y$ is a subdefinable basis for the induced uniform structure on a definable $Y \subseteq X$. We leave the routine proof of Lemma 3.3 to the reader.

Lemma 3.3. *Suppose $\mathcal{B}$ is a subdefinable basis for a uniform structure on an $\mathcal{M}$-definable set X. Then $E := \bigcap \mathcal{B}$ is externally definable in $\mathcal{M}$.*

Proposition 3.4. *Suppose $\mathcal{M}$ is* NIP. *Suppose X is an $\mathcal{M}$-definable set and $\mathcal{B}$ is a subdefinable basis for a uniform structure on X. Then $\mathcal{M}^{\mathrm{Sh}}$ defines a basis for $\overline{\mathcal{B}}$.*

Proof. Let $(B_a)_{a \in M^x}$ be a definable family of sets such that $\mathcal{B}$ is a subfamily of $(B_a)_{a \in M^x}$. Let $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$. Given an $\mathcal{M}$-definable set Y we let Y' be the $\mathcal{N}$-definable set defined by the same formula as Y. We also let $(B'_a)_{a \in N^x}$ be the family of sets defined by the same formula as $(B_a)_{a \in M^x}$. Note that $\mathcal{B}' := \{U' : U \in \mathcal{B}\}$ is a subdefinable (in $\mathcal{N}$) basis for a uniform structure on X'. Let $F := \bigcap \mathcal{B}'$. Lemma 3.3 shows that F is definable in $\mathcal{N}^{\mathrm{Sh}}$. An application of saturation shows that if Y is an $\mathcal{M}$-definable subset of X^2 then $F \subseteq Y'$ if and only if Y contains some $U \in \mathcal{B}$. Let Z be the set of $a \in N^x$ such that $F \subseteq B'_a$. Then Z is definable in $\mathcal{N}^{\mathrm{Sh}}$. Facts 2.3 and 2.2 together show that $Q := Z \cap M^x$ is definable in $\mathcal{M}^{\mathrm{Sh}}$. Then Q is the set of $a \in M^x$ such that B_a

contains some $U \in \mathcal{B}$. In particular $\mathcal{B}$ is a subfamily of $(B_a)_{a\in Q}$.

Let $C_a = B_a \cap B_a^{-1}$ for all $a \in Q$. Note that each C_a is symmetric and contains some $U \in \mathcal{B}$. Note also that $\mathcal{B}$ is a subfamily of $(C_a)_{a\in Q}$. An application of Lemma 3.1 shows that $(C_a)_{a\in Q}$ is a basis for $\overline{\mathcal{B}}$. □

Corollary 3.5 follows immediately from Proposition 3.4.

Corollary 3.5. *Suppose $\mathcal{M}$ is* NIP*, X is an $\mathcal{M}$-definable set, and there is a subdefinable basis for a uniform structure on X. Let $\mathcal{X}$ be the structure induced on X by $\mathcal{M}$. Then the open core $\mathcal{X}^\circ$ of $\mathcal{X}$ is a reduct of the structure induced on X by $\mathcal{M}^{\mathrm{Sh}}$.*

3.3. **$\bigwedge$-definable equivalence relations.** We describe how uniform structures arise in model theory. **In this section $\mathcal{N}$ is an $\aleph_1$-saturated structure and X is an $\mathcal{N}$-definable set.** There is a a canonical correspondence between

- $\bigwedge$-definable equivalence relations E on X, and
- uniform structures on X which admit a basis $\mathcal{B}$ consisting of a small family of definable sets.

This correspondence is well-known to experts and used implicitly throughout the literature. If $\mathcal{B}$ is a small collection of definable subsets of X^2 forming a basis for a uniform structure on X then E is $\bigcap \mathcal{B}$. Note that E is $\bigwedge$-definable and only depends on $\overline{\mathcal{B}}$. Now suppose E is a $\bigwedge$-definable equivalence relation on X. Let $\mathcal{C}$ be a small collection of definable subsets of X such that $\bigcap \mathcal{C} = E$. After replacing $\mathcal{C}$ with

$$\{C_1 \cap \ldots \cap C_n \cap C_1^{-1} \cap \ldots \cap C_n^{-1} : C_1, \ldots, C_n \in \mathcal{C}\}$$

if necessary we may suppose that every element of $\mathcal{C}$ is symmetric and that for all $C, C' \in \mathcal{C}$ there is $B \in \mathcal{C}$ such that $B \subseteq C \cap C'$. It is now an exercise in saturation to show that $\mathcal{C}$ is a basis for a uniform structure on X and that this uniform structure does not depend on choice of $\mathcal{C}$. It is also a saturation exercise to see that the induced uniform structure on X/E is the discrete uniform structure if and only if E is definable. Finally, the quotient X/E is small if and only if the induced topology on X/E is compact. In this case the topology is known as the **logic topology** on X/E. A subset of X/E is closed in the logic topology if and only if it is the image of a $\bigwedge$-definable subset of X under the quotient map. In general we refer to the topology on X/E as the logic topology.

We include a proof of Proposition 3.6 for the sake of completeness.

Proposition 3.6. *Let E be a $\bigwedge$-definable equivalence relation on X and π be the quotient map $X \to X/E$. Then $\pi(Y)$ is closed for any definable $Y \subseteq X$.*

We omit a few details.

Proof. Suppose that $Y \subseteq X$ is definable and $p \in X$ is in the closure of $\pi(Y)$. Fix $p^* \in X$ such that $\pi(p^*) = p$. Let $\mathcal{B}$ be a small family of definable subsets of X^2 which form a basis for the uniform structure on X. Then for every $C \in \mathcal{B}$ there is $q^* \in Y$ such that $(p^*, q^*) \in C$. Applying saturation we obtain $q^* \in X$ such that $(p^*, q^*) \in C$ for every $C \in \mathcal{B}$. Then $\pi(q^*) = p$, hence $p \in \pi(Y)$. □

We also equip $(X/E)^n$ with a logic topology. Let E_n be the equivalence relation on X^n where $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are E_n-equivalent if $(a_k, b_k) \in E$ for all

$1 \leq k \leq n$. We identify $(X/E)^n$ and (X^n/E_n). Observe that E_n is a $\bigwedge$-definable equivalence relation on X^n. Observe that if $\mathcal{B}$ is a small collection of definable sets forming a basis for the uniform structure associated to E then $\mathcal{B}_n$ is a small collection definable sets forming a basis for the uniform structure associated to E_n. Thus the uniform structure on $(X/E)^n$ is simply the product uniform structure.

Let E be a $\bigwedge$-definable equivalence relation on X. A subdefinable (in $\mathcal{N}$) basis for E is a subdefinable (in $\mathcal{N}$) basis for the uniform structure on X associated to E. Lemma 3.7 is a saturation exercise which we leave to the reader.

Lemma 3.7. *Suppose $\mathcal{C}$ is a small subdefinable collection of subsets of X^2. Then $\mathcal{C}$ is a subdefinable basis for E if and only if*

(1) $\bigcap \mathcal{C} = E$,
(2) each $U \in \mathcal{C}$ is symmetric, and
(3) for all $U, U' \in \mathcal{C}$ there is $V \in \mathcal{C}$ such that $V \subseteq U \cap U'$.

If $\mathcal{C}$ satisfies (1) *and* (3) *then $\{U \cap U^{-1} : U \in \mathcal{C}\}$ is a subdefinable basis for E.*

Proposition 3.8. *Suppose there is a subdefinable (in $\mathcal{N}$) basis for E. Then $\mathcal{N}^{\mathrm{Sh}}$ defines a basis for the uniform structure on X/E.*

Proof. Lemma 3.3 shows that E is externally definable and Proposition 3.4 shows that $\mathcal{N}^{\mathrm{Sh}}$ defines a basis $\mathcal{B}$ for the uniform structure on X. Then $\pi(\mathcal{B})$ is an $\mathcal{N}^{\mathrm{Sh}}$-definable basis for the uniform structure on X/E. □

If the answer to Question 3.9 is positive then it would follow that if $\mathcal{N}$ is NIP and E is externally definable then $\mathcal{N}^{\mathrm{Sh}}$ defines a basis for the uniform structure on X/E.

Question 3.9. *If $\mathcal{N}$ is* NIP *does every equivalence relation which is both $\bigwedge$-definable and externally definable admit a subdefinable basis?*

4. Really strong Baire category theorem

Throughout this section $\mathcal{M}$ is a possibly multisorted first order structure, "definable" without modification means "$\mathcal{M}$-definable", and X is a definable set. The NIP case of Theorem 4.1 is crucial for Theorem B.

A family $(X_t : t \in \mathbb{R}_{>0})$ of sets is increasing if $X_s \subseteq X_t$ whenever $s \leq t$. The classical Baire category theorem is easily seen to be equivalent to the following: If (Z, d) is a complete metric space, and $(X_t : t \in \mathbb{R}_{>0})$ is an increasing family of nowhere dense subsets of Z, then $\bigcup_{t>0} X_t$ has empty interior. It is shown in [15, Theorem D] that if $\mathcal{R}$ is an expansion $(\mathbb{R}, <, +)$ which does not interpret the monadic second order theory of one sucessor and $(X_t : t \in \mathbb{R}_{>0})$ is an increasing definable family of nowhere dense subsets of $\mathbb{R}^n$ then $\bigcup_{t>0} X_t$ is nowhere dense. This result is known as the "strong Baire category theorem". Dolich and Goodrick [7, Proposition 2.16] show that if $\mathcal{R}$ is a strong (in particular strongly dependent) expansion of a dense ordered abelian group $(R, <, +)$, $I \subseteq R$ is an open interval, and $(X_a)_{a \in I}$ is an increasing definable family of discrete subsets of R, then $\bigcup_{a \in I} X_a$ is nowhere dense. It follows from [30, Lemma 3.5] that if $\mathcal{M}$ is a dp-minimal expansion of a divisible ordered abelian group or a non strongly minimal dp-minimal expansion of a field (equiped with the Johnson topology) and $\mathcal{A}$ is an upwards directed definable family of nowhere dense subsets of M^n then $\bigcup \mathcal{A}$ is nowhere dense.

Recall that $\mathcal{M}$ is NTP$_2$ if it is *not* TP$_2$ and $\mathcal{M}$ is TP$_2$ if there is a formula $\phi(x, y)$, with x, y possibly tuples of variables, an elementary extension $\mathcal{N}$ of $\mathcal{M}$, an array $(a_{ij} : i, j \in \mathbb{N})$ of elements of N^y, and $k \in \mathbb{N}$ such that

(1) $(\phi(x, a_{ij}) : j \in \mathbb{N})$ is k-inconsistent for every $i \in \mathbb{N}$, and
(2) $(\phi(x, a_{i,\gamma(i)}) : i \in \mathbb{N})$ is consistent for every $\gamma : \mathbb{N} \to \mathbb{N}$.

NIP implies NTP$_2$, see [27, 5.4].

Theorem 4.1 generalizes the NTP$_2$ case of the strong Baire category theorem and the latter two results described in the preceeding paragraph.

Theorem 4.1. *Suppose $\mathcal{B}$ is a subdefinable basis for a uniform structure on X, W is a nonempty open subset of X, and $\mathcal{A}$ is a definable family of subsets of X such that for every finite collection $\mathcal{U}$ of nonempty open subsets of W there is $A \in \mathcal{A}$ such that A is nowhere dense in W and intersects each $U \in \mathcal{U}$. Then $\mathcal{M}$ is* TP$_2$.

Let $[n]$ denote $\{0, \ldots, n\}$. Let $[n]^{=m}$ be the set of functions $\{0, \ldots, m\} \to \{0, \ldots, n\}$ which we identify with $\{0, \ldots, n\}$-valued sequences of length $m + 1$. Let $[n]^{\leq m}$ be the union of all $[n]^{=k}$ where $0 \leq k \leq m$. Given $\sigma \in [n]^{\leq m}$ and $\eta \in [n]^{\leq k}$ we let $|\sigma|$ be the length of σ and let $\sigma^\frown\eta \in [n]^{\leq m+k}$ be the usual concatenation of σ and η.

Proof. Let R be the relation on $W \times (\mathcal{A} \times \mathcal{B} \times \mathcal{B})$ where $(a, (A, U, U')) \in R$ if and only if $a \in A[U] \setminus A[U']$. We show that R has TP$_2$. Fixing $n \geq 1$ we construct elements $(A_i : i \in [n])$ of $\mathcal{A}$, elements $(U^i_j : i, j \in [n])$ of $\mathcal{B}$, and nonempty open subsets $(W_\sigma : \sigma \in [n-1]^{\leq n})$ of W such that:

(1) $U^i_{j+1} \subsetneq U^i_j$ and $A_i[U^i_j] \setminus A_i[U^i_{j+1}]$ has interior for all $i \in [n]$ and $j \in [n-1]$,
(2) $W_{\sigma^\frown i}, W_{\sigma^\frown j}$ are pairwise disjoint subsets of W_σ for all $\sigma \in [n-1]^{<n}$ and $i, j \in [n-1]$,
(3) and if $p \in W_\sigma$ then $p \in A_i[U^i_j] \setminus A_i[U^i_{j+1}]$ if and only if $\sigma(i) = j$.

This shows that R has TP$_2$.

Let $A_0 \in \mathcal{A}$ be nowhere dense in W. Applying Lemma 3.2 inductively let $U^0_0, \ldots, U^0_n$ be elements of $\mathcal{B}$ such that $U^0_{j+1} \subsetneq U^0_j$ and $A_0[U^0_j] \setminus A_0[U^0_{j+1}]$ has interior in W for all $j \in [n-1]$. For all $j \in [n-1]$ let W_j be a nonempty open subset of W contained in $A_0[U^0_j] \setminus A_0[U^0_{j+1}]$.

Fix $1 \leq k \leq n-1$. Suppose we have constructed $(A_i : i \in [k])$, $(U^i_j : i \in [k], j \in [n])$, and $(W_\sigma : \sigma \in [n-1]^{\leq k})$. Let A_{k+1} be an element of $\mathcal{A}$ which is nowhere dense in W and intersects every element of $(W_\sigma : \sigma \in [n-1]^{=k})$. Applying Lemma 3.2 inductively let $U^{k+1}_0, \ldots, U^{k+1}_n \in \mathcal{B}$ be such that $U^{k+1}_{j+1} \subsetneq U^{k+1}_j$ and $A_{k+1}[U^{k+1}_j] \setminus A_{k+1}[U^{k+1}_{j+1}]$ has interior in each element of $(W_\sigma : \sigma \in [n-1]^{=k})$ for all $j \in [n]$. For each $\sigma \in [n-1]^{=k}$ and $j \in [n-1]$ we let $W_{\sigma^\frown j}$ be a nonempty open subset of W_σ contained in $A_{k+1}[U^i_j] \setminus A_{k+1}[U^i_{j+1}]$. □

Corollary 4.2 is often easier to apply in practice. The proof is left to the reader.

Corollary 4.2. *Suppose $\mathcal{M}$ is* NTP$_2$. *Suppose $\mathcal{B}$ is a subdefinable basis for a uniform structure on X. Let Y be a definable subset of X, $\mathcal{A}$ be a subdefinable directed family of subsets of X, and W be a nonempty open subset of Y. If each element of $\mathcal{A}$ is nowhere dense in W then $\bigcup \mathcal{A}$ is nowhere dense in W.*

5. Proof of Theorem B

Theorem 5.1. *Suppose* $\mathcal{N}$ *is* $\aleph_1$*-saturated and* NIP*. Let* E *be a* $\bigwedge$*-definable equivalence relation on* X*. Suppose one of the following:*

(1) *E is externally definable and there is a subdefinable (in $\mathcal{N}^{\mathrm{Sh}}$) basis for the uniform structure on X/E, or*
(2) *there is a subdefinable (in $\mathcal{N}$) basis for E.*

Then the structure induced on X/E by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless.

Proof. Lemma 3.3 and Proposition 3.4 shows that (2) implies (1), so we suppose (1). Let $\mathcal{B}$ be a subdefinable basis for the uniform structure on X/E. Then $\mathcal{B}_n$ is a subdefinable basis for the product uniform structure on $(X/E)^n$. Let Y, Y' be $\mathcal{N}^{\mathrm{Sh}}$-definable subsets of $(X/E)^n$. We suppose that Y is somewhere dense in Y' and show that Y has interior in Y'. Note that $\mathcal{B}_n|_{Y'}$ is an $\mathcal{N}^{\mathrm{Sh}}$-subdefinable basis for the induced uniform structure on Y'. As above, let E_n be the equivalence relation on X^n where $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are E_n-equivalent if and only if $(a_k, b_k) \in E$ for all $1 \le k \le n$. Recall that we identify $(X/E)^n$ with X^n/E_n and identify the product topology on $(X/E)^n$ with the logic topology on X^n/E_n.

Let π be the quotient map $X^n \to (X/E)^n$. Suppose towards a contradiction that W is a nonempty open subset of Y' such that Y is dense and co-dense in W. Then $Z := \pi^{-1}(Y)$ is $\mathcal{N}^{\mathrm{Sh}}$-definable and hence externally definable by Fact 2.3. Applying Fact 2.5 we obtain a definable family $(Z_b)_{b \in N^x}$ of subsets of X such that for any finite $A \subseteq Z$ there is $b \in N^x$ such that $A \subseteq Z_b \subseteq Z$. Let $Y_b := \pi(Z_b)$ for $b \in N^x$. Then $(Y_b)_{b \in N^x}$ is an $\mathcal{N}^{\mathrm{Sh}}$-definable family of sets. For any finite $A \subseteq Y$ there is $b \in N^x$ such that $A \subseteq Y_b \subseteq Z$. Suppose $\mathcal{U}$ is a finite collection of nonempty open subsets of W. As Y is dense in W there is a finite subset of Y which intersects each $U \in \mathcal{U}$. Hence there is $b \in N^x$ such that $Y_b \subseteq Y$ and Y_b intersects each element of $\mathcal{U}$. Proposition 3.6 shows that each Y_b is closed in $(X/E)^n$. If $Y_b \subseteq Y$ then Y_b is closed and co-dense in W and is therefore nowhere dense in W. As $\mathcal{N}^{\mathrm{Sh}}$ is NIP this gives a contradiction with Theorem 4.1. □

6. Locally compact Hausdorff uniform structures

In this section we prove a more general version of Theorem A. **Throughout this section, $\mathcal{M}$ is a structure, X is a definable set, $\mathcal{B}$ is a subdefinable basis for a locally compact Hausdorff uniform structure on X, and "definable" without modification means "$\mathcal{M}$-definable".**

A **subdefinable compact exhaustion** of X is a subdefinable family $\mathcal{K}$ of compact subsets of X such that every compact subset of X is contained in some $K \in \mathcal{K}$. **We also assume that X admits a subdefinable compact exhaustion $\mathcal{K}$.** If X is compact then we take $\mathcal{K} = \{X\}$ and this additional assumption is superfluous.

We equip X^n with the product uniform structure with the subdefinable basis $\{U_n : U \in \mathcal{B}\}$ defined in Section 3.1. Note that $\{K^n : K \in \mathcal{K}\}$ is a subdefinable compact exhaustion of X^n.

Let $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$. Given an $\mathcal{M}$-definable set Z we let Z' be the $\mathcal{N}$-definable set defined by any formula defining Z. Observe that $\mathcal{B}' := \{U' : U \in \mathcal{B}\}$ is a subdefinable basis for a uniform structure on X'. Let $E := \bigcap \mathcal{B}'$. Then $\mathcal{B}'$ is a subdefinable basis for E.

We define

$$O := \bigcup \{K' : K \in \mathcal{K}\}.$$

Let π be the quotient map $X' \to X'/E$. Abusing notation we also let π denote the map $(X')^n \to (X'/E)^n$ given by

$$\pi(x_1, \dots, x_n) = (\pi(x_1), \dots, \pi(x_n)).$$

Lemma 6.1 is an easy variation of Proposition 3.6. We leave the proof to the reader.

Lemma 6.1. *Suppose Y is an $\mathcal{N}$-definable subset of $(X')^n$. Then $\pi(Y \cap O^n)$ is closed in $(O/E)^n$.*

We show that O/E may be identified with X and view the quotient map $O \to O/E$ as a standard part map.

Lemma 6.2. *If $a \in O$ then the E-class of a contains exactly one element of X.*

We recall a useful definition from general topology. Suppose Z is a topological space. A *filter base* on Z is a collection $\mathcal{F}$ of nonempty subsets of Z such that for all $E, F \in \mathcal{F}$ there is $G \in \mathcal{F}$ such that $G \subseteq E \cap F$. A point $p \in Z$ is a *cluster point* of a filter base $\mathcal{F}$ if every neighbourhood of p intersects every element of $\mathcal{F}$. If Z is compact then every filter base has a cluster point.

Proof. If $x, y \in X$ are distinct then as X is Hausdorff there is $U \in \mathcal{B}$ such that $(x, y) \notin U$. Thus each E-class contains at most one element of X.

Fix $a \in O$. We show that a is E-equivalent to some element of X. Let $K \in \mathcal{K}$ be such that K' contains a. We first show that $(U'_a \cap K)_{U \in \mathcal{B}}$ is a filter base on K and then show that the cluster point is E-equivalent to a.

Fix $U \in \mathcal{B}$. As K is compact there is a finite $B \subseteq K$ such that $(U_b)_{b \in B}$ covers K. It follows that $(U'_b)_{b \in B}$ covers K'. Hence there is $b \in B$ such that $a \in U'_b$. As U' is symmetric we have $b \in U'_a$. Hence $U'_a \cap K$ is nonempty for every $U \in \mathcal{B}$. For any $U, V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W \subseteq U \cap V$ so $W'_a \subseteq U'_a \cap V'_a$ and hence

$$W'_a \cap K \subseteq (U'_a \cap K) \cap (V'_a \cap K).$$

Hence $(U'_a \cap K)_{U \in \mathcal{B}}$ is a filter base on K. As K is compact there is a cluster point $p \in K$ of $(U'_a \cap K)_{U \in \mathcal{B}}$. We show that a and p are E-equivalent. It suffices to fix $U \in \mathcal{B}$ and show that $(a, p) \in U'$. Let $V \in \mathcal{B}$ be such that $V \circ V = U$. Then V_p intersects $V'_a \cap K$. Suppose q lies in this intersection. Then $(p, q) \in V'$ and $(q, a) \in V'$, so $(p, q) \in U'$. □

Let ρ be the bijection $O/E \to X$ which maps each E-class to the unique element of X that it contains. We consider O/E as a uniform structure with basis $\pi(\mathcal{B}'|_O)$.

Lemma 6.3. *Suppose $U \in \mathcal{B}$. Then $a, b \in O/E$ satisfy $(a, b) \in \pi(U')$ if and only if $(\rho(a), \rho(b)) \in U$. Hence O/E and X are bi-uniformly equivalent and hence homeomorphic.*

Proof. Fix $a, b \in O/E$. Then $\pi(\rho(a)) = a$ and $\pi(\rho(b)) = b$. Recall that $(a, b) \in \pi(U')$ if and only if $(c, d) \in U'$ for any $c, d \in X'$ such that $\pi(c) = a$ and $\pi(d) = b$. Hence $(a, b) \in \pi(U')$ if and only if $(\rho(a), \rho(b)) \in U'$, which holds if and only if $(\rho(a), \rho(b)) \in U$. □

In light of Lemma 6.3 we identify X with O/E and identify the uniform structure on X with that on O/E. We now view $\pi : O \to X$ as a standard part map. These identifications greatly simplify notation in the proof of Theorem 6.5.

So far we have not needed to suppose that $\mathcal{B}$ is a subdefinable basis, only that each element of $\mathcal{B}$ is definable. We will now use the full assumption. As $\mathcal{B}'$ is a subdefinable basis for E, Lemma 3.3 shows that E is externally definable in $\mathcal{N}$. Lemma 6.4 shows that O is also externally definable in $\mathcal{N}$ so we regard X as an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$. Lemma 6.4 follows in the same way as Lemma 3.3 so we omit the proof.

Lemma 6.4. *The set O is externally definable in $\mathcal{N}$.*

Let $\mathrm{Cl}(Y)$ be the closure in X^n of $Y \subseteq X^n$. Let $\mathcal{X}$ be the structure induced on X by $\mathcal{M}$ and let $\mathcal{X}^\circ$ be the structure on X whose primitive n-ary relations are all sets of the form $\mathrm{Cl}(Y)$ for definable $Y \subseteq X^n$. If $\mathcal{B}$ is a definable basis then $\mathcal{X}^\circ$ is the open core of $\mathcal{X}$. It is not a priori clear that $\mathcal{X}^\circ$ is always a reduct of $\mathcal{X}$. Proposition 3.4 shows that if $\mathcal{M}$ is NIP then $\mathcal{X}^\circ$ is a reduct of $\mathcal{X}^{\mathrm{Sh}}$.

Theorem 6.5. *Suppose $\mathcal{M}$ is* NIP*. Then $\mathcal{X}^\circ$ is strongly noiseless.*

Proof. Theorem 5.1 shows that the structure induced on X'/E by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless. As we identify X with $O/E \subseteq X'/E$ it follows that the structure induced on X by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless. It now suffices to show that $\mathcal{X}^\circ$ is a reduct of the structure induced on X by $\mathcal{N}^{\mathrm{Sh}}$.

We fix an $\mathcal{M}$-definable subset Y of X^n and show that $\mathrm{Cl}(Y)$ is definable in $\mathcal{N}^{\mathrm{Sh}}$. We show that $\mathrm{Cl}(Y) = \pi(Y' \cap O^n)$. As π is the identity on X^n we have $Y \subseteq \pi(Y' \cap O^n)$. Lemma 6.1 shows that $\pi(Y' \cap O^n)$ is closed so $\mathrm{Cl}(Y) \subseteq \pi(Y' \cap O^n)$. We prove the other inclusion. Fix $p \in \pi(Y' \cap O^n)$. We show that $U_p \cap Y \neq \emptyset$ for all $U \in \mathcal{B}_n$. Fix $U \in \mathcal{B}_n$. Let $q \in Y' \cap O^n$ satisfy $\pi(q) = p$. Then $q \in U'_p$, so $Y' \cap U'_p$ is nonempty. Then $Y \cap U_p$ is nonempty as $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$. □

As an application we show that $\mathcal{X}^\circ$-definable functions are generically continuous.

Proposition 6.6. *Suppose $\mathcal{M}$ is* NIP*, Y is a $\mathcal{X}^\circ$-definable subset of X^m, and $f : Y \to X^n$ is definable in $\mathcal{X}^\circ$. Then there is a dense definable open subset of Y on which f is continuous.*

Proof. Let
$$f(a) = (f_1(a), \dots, f_n(a)) \quad \text{for all} \quad a \in Y.$$
Suppose that for all $1 \le k \le n$ there is a dense definable open subset U_k of Y on which f_k is continuous. Then $\bigcap_{k=1}^n U_k$ is a dense definable open subset of Y on which f is continuous. Hence we assume $n = 1$.

After applying Proposition 3.4 and replacing $\mathcal{M}$ by $\mathcal{M}^{\mathrm{Sh}}$ we may suppose that $\mathcal{B}$ is a definable basis.

For each $U \in \mathcal{B}$ we let D_U be the set of $a \in Y$ such that for all $V \in \mathcal{B}$ there is $b \in Y$ such that $(a,b) \in V$ and $(f(a), f(b)) \notin U$. Note that $(D_U)_{U \in \mathcal{B}}$ is a definable family and $\bigcup_{U \in \mathcal{B}} D_U$ is the set of points at which f is discontinuous. We suppose towards a contradiction that $\bigcup_{U \in \mathcal{B}} D_U$ is somewhere dense in Y. Observe that if $U, V \in \mathcal{B}$ satisfy $U \subseteq V$ then $D_V \subseteq D_U$. For any $U, U' \in \mathcal{B}$ there is $V \in \mathcal{B}$ such that $V \subseteq U \cap U'$, hence $D_U, D_{U'} \subseteq D_V$, and $(D_U)_{U \in \mathcal{B}}$ is directed. We apply Corollary 4.2 to obtain $U \in \mathcal{B}$ such that D_U is somewhere dense in Y. It follows by strong noiselessness that D_U has nonempty interior in Y. Let W be a nonempty open subset of Y contained in D_U. As $\mathcal{B}$ is a definable basis we may suppose W is definable.

For each $K \in \mathcal{K}$ let F_K be $f^{-1}(K) \cap D_U$. Then $(F_K)_{K \in \mathcal{K}}$ is subdefinable. For any $K, L \in \mathcal{K}$ there is $P \in \mathcal{K}$ such that $K, L \subseteq P$, hence $F_K, F_L \subseteq F_P$. Therefore $(F_K)_{K \in \mathcal{K}}$ is directed. As $\bigcup_{K \in \mathcal{K}} F_K = D_U$ Corollary 4.2 and strong noiselessness yield a $K \in \mathcal{K}$ such that F_K has interior in W. After replacing W with a smaller nonempty definable open set if necessary we suppose that W is contained in F_K. Then $f(W) \subseteq K$.

Fix $V \in \mathcal{B}$ such that $V \circ V \subseteq U$. As K is compact there is a finite $A \subseteq K$ such that $(V_a)_{a \in A}$ covers K. Then $(f^{-1}(V_a))_{a \in A}$ covers F_K and in particular covers W. Thus there is $b \in A$ such that $f^{-1}(V_b)$ is somewhere dense in W. Then $f^{-1}(V_b)$ has interior in W by strong noiselessness. Let Z be the interior of $f^{-1}(V_b)$ in W, note Z is definable as $\mathcal{B}$ is a definable basis. As $p \in D_U$ there is $q \in Z$ such that $(f(p), f(q)) \notin U$. However as $p, q \in f^{-1}(V_b)$ we have $(f(p), b) \in V$ and $(b, f(q)) \in V$, so $(f(p), f(q)) \in U$ by choice of V. Contradiction. $\square$

6.1. **Expansions of locally compact groups.** Let G be a group. Recall that if $\mathcal{C}$ is a subdefinable basis for a group topology on G then the collection $\{(g, g') \in G^2 : g^{-1}g' \in U\}$, $U \in \mathcal{C}$ is a subdefinable basis for a uniform structure on G inducing the group topology. Theorem 6.7 is a special case of Theorem 6.5.

Theorem 6.7. *Suppose that G is a group, $\mathcal{G}$ expands G, and $\mathcal{G}$ admits a subdefinable neighbourhood basis at the identity for a locally compact Hausdorff group topology and a subdefinable compact exhaustion. If $\mathcal{G}$ is* NIP *then $\mathcal{G}^\circ$ is strongly noiseless.*

Note that if G is compact then the assumption of a subdefinable compact exhaustion is superfluous.

Theorem 6.7 shows that any $\mathcal{G}^\circ$-definable subset of G^n with empty interior is topologically small. Under the additional assumption that G is second countable we show that any $\mathcal{G}^\circ$-definable subset of G^n with empty interior is measure-theoretically small. The key tool is a theorem of Simon [29]. Proposition 6.8 is already known for expansions of $(\mathbb{R}, <, +)$, see [15, Theorem D].

Proposition 6.8. *Assume the assumptions of Theorem 6.7. Suppose G is second countable, $\mathcal{G}$ is* NIP *and X is a $\mathcal{G}^\circ$-definable subset of G^n. Then any X has empty interior if and only if X is Haar null*

The Haar measure is unique up to rescaling, so the collection of Haar null sets does not depend on the choice of a Haar measure.

Proof. As G is second countable any open subset of G^n has non-zero Haar measure. Suppose that X has empty interior and let $\mathrm{Cl}(X)$ be the closure of X in G^n. Then X is nowhere dense, so $\mathrm{Cl}(X)$ is nowhere dense. Corollary 3.5 shows that $\mathrm{Cl}(X)$ is $\mathcal{G}^{\mathrm{Sh}}$-definable. As $\mathcal{G}^{\mathrm{Sh}}$ is NIP [29, Theorem 3.6] shows that $\mathrm{Cl}(X)$ is Haar null. □

7. Three counterexamples

It is natural to ask if Theorem 6.5 holds when we only have a definable basis for a locally compact Hausdorff topology on X. We show that this is not the case. Each example below is dp-minimal and the second is definable in $(\mathbb{R}, <)$. Let I be $[0, 1]$.

7.1. **The Cantor function.** Let K be the middle-thirds Cantor set and $f : I \to I$ be the Cantor function. If $t \in K$ and $t = \sum_{i=1}^{\infty} a_i/3^i$ where $a_i \in \{0, 2\}$ for all i, then $f(t) = (1/2) \sum_{i=1}^{\infty} a_i/2^i$ and if $t \notin K$ then $f(t) = \sup\{f(s) : s \in K, s < t\}$. See [10] for background. Continuity of f implies $(\mathbb{R}, <, f)$ and $(\mathbb{R}, <, f)^\circ$ are interdefinable.

We first show that $(\mathbb{R}, <, f)$ is NIP (in fact dp-minimal). A subset X of $\mathbb{R}^2$ is monotone if whenever $(s, s') \in X$ and $t \leq s, s' \leq t'$ then $(t, t') \in X$. It is a special case of [28, Proposition 4.2] that the expansion of $(\mathbb{R}, <)$ by all monotone subsets of $\mathbb{R}^2$ is dp-minimal. Suppose $h : \mathbb{R} \to \mathbb{R}$ is increasing. Then $G_h := \{(s, s') \in \mathbb{R}^2 : g(s) \leq s'\}$ is monotone. It is easy to see that h is definable in $(\mathbb{R}, <, G_h)$. It follows that the expansion of $(\mathbb{R}, <)$ by all increasing functions $\mathbb{R} \to \mathbb{R}$ is dp-minimal. As $f(0) = 0$ and $f(1) = 1$, we let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(t) = 0$ when $t < 0$, $h(t) = f(t)$ when $t \in I$, and $h(t) = 1$ when $t > 1$. Then h is increasing and $(\mathbb{R}, <, h)$ is interdefinable with $(\mathbb{R}, <, f)$. Hence we see that $(\mathbb{R}, <, f)$ is dp-minimal.

It follows from the definition of f that K is the set of $t \in [0, 1]$ at which f is not locally constant. Hence K is definable in $(\mathbb{R}, <, f)$. Note that each connected component of $I \setminus K$ is an open interval, let L be the set of endpoints of connected components of $I \setminus K$, and observe that L is definable in $(\mathbb{R}, <, f)$. As $f(K) = I$, L is dense in K, and f is continuous, we see that $f(L)$ is dense in I. As L is countable $f(L)$ is co-dense in I. Hence $(\mathbb{R}, <, f)$ is noisy.

7.2. **The double arrow space.** Let X be $I \times \{0, 1\}$, $\lhd$ be the lexicographic order on X, and equip X with the associated order topology. This is known as the double arrow space or the split interval. This topology is compact Hausdorff as $\lhd$ is complete and has a maximum and minimum. Any linear order is dp-minimal [28, Proposition 4.2], so $(X, \lhd)$ is dp-minimal.

Note that the collection of $\lhd$-open intervals is a $(X, \lhd)$-definable basis. Note that $\lhd$ is open in the product topology on X^2, so $(X, \lhd)$ and $(X, \lhd)^\circ$ are interdefinable. Now $I \times \{0\}$ is the set of $a \in X$ such that $\{b \in X : a \lhd b\}$ has a minimum. (If $a = (t, 0)$ then this minimum is $(t, 1)$.) Hence $I \times \{0\}$ and $I \times \{1\}$ are both definable in $(X, \lhd)$. Finally $I \times \{0\}$ and $I \times \{1\}$ are both dense in X, so $(X, \lhd)$ is noisy.

7.3. **The Cantor set.** Let K be any Cantor (nonempty, nowhere dense, perfect) subset of $\mathbb{R}$. A similar argument shows that $(K, <)$ is dp-minimal, interdefinable with $(K, <)^\circ$, and noisy.

8. NIP expansions of $(\mathbb{R},<,+)$

In this section we prove Theorem C. We first prove a general fact on expansions of $(\mathbb{R},<,+)$. This fact is known to experts but has not appeared in the literature.

Theorem 8.1. *The following are equivalent:*

1. *$\mathcal{R}$ is generically locally o-minimal,*
2. *every definable subset of $\mathbb{R}$ either has interior or contains an isolated point,*
3. *$\mathcal{R}$ is strongly noiseless,*
4. *the structure induced on $[0,1]$ by $\mathcal{R}$ is strongly noiseless,*
5. *$\mathcal{R}$ is strongly noiseless in one variable,*
6. *$\mathcal{R}$ is noiseless in one variable and does not define a Cantor subset of $\mathbb{R}$.*

We need a result of Miller. Fact 8.2 follows from [24, Proposition 3.4].

Fact 8.2. *Suppose every $\mathcal{R}$-definable subset of $\mathbb{R}$ either has interior or contains an isolated point. Let X be a nonempty $\mathcal{R}$-definable subset of $\mathbb{R}^n$. Then there is a dense open subset V of X such that for every $p \in V$ there is $d, n \in \mathbb{N}$ with $d \leq n$, a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$, and an open neighbourhood W of p such that $\pi(X \cap W)$ is open and π induces a homeomorphism $X \cap W \to \pi(X \cap W)$.*

We now prove Theorem 8.1.

Proof. We first show that (1) and (2) are equivalent. Let $X \subseteq \mathbb{R}$ be definable. Suppose $\mathcal{R}$ is generically locally o-minimal. Then there is a $p \in X$ and an open interval I containing p such that $I \cap X$ is $(\mathbb{R},<)$ definable. Then $I \cap X$ either has interior or is finite and hence contains an isolated point. Suppose (2) holds. Let U be the interior of X and D be the set of isolated points of X. Note that $U \cup D$ is open in X. It suffices to show that $U \cap D$ is dense in X. Suppose otherwise. Then there is $p \in X$ and an open interval I containing p such that I is disjoint from $U \cap D$. Then $I \cap X$ does not have interior and does not contain an isolated point, contradiction.

(2) $\Rightarrow$ (3): Note that (2) implies that $\mathcal{R}$ is noiseless in one variable, so $\mathcal{R}$ is noiseless by Fact 9.4. Let X, Y be definable subsets of $\mathbb{R}^n$. Suppose that X is somewhere dense in Y. Fix a nonempty definable open subset W of Y such that X is dense in $W \cap Y$. After replacing Y with $W \cap Y$ if necessary we suppose X is dense in Y. Applying Fact 8.2 we obtain $d, n \in \mathbb{N}$ with $d \leq n$, definable open $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$, and a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that π restricts to a homeomorphism $U \cap Y \to V$. Then $\pi(X \cap U)$ is dense in V, so $\pi(X \cap U)$ has interior in V as $\mathcal{R}$ is noiseless. It follows that X has interior in $U \cap Y$ and thus has interior in Y.

It is clear that (3) implies (4). We show that (4) implies (5). Suppose $\mathcal{R}$ is not strongly noiseless in one variable. Let X, Y be definable subsets of $\mathbb{R}$ such that X is somewhere dense in Y and has empty interior in Y. Let p be an element of X and I be an open interval containing p such that $I \cap X$ is dense in $I \cap Y$. Then $(X - p + \frac{1}{2}) \cap [0,1]$ is somewhere dense in $(Y - p + \frac{1}{2}) \cap [0,1]$ but $(X - p + \frac{1}{2}) \cap [0,1]$ has empty interior in $(Y - p + \frac{1}{2}) \cap [0,1]$. Thus the structure induced on $[0,1]$ by $\mathcal{R}$ is not strongly noiseless.

(5) $\Rightarrow$ (6) : It suffices to suppose there is a definable Cantor set and show that (5) fails. Let Y be a definable Cantor set. Observe that each connected component of

$\mathbb{R}\setminus Y$ is an interval. Let X be the set of endpoints of connected components of $\mathbb{R}\setminus Y$. Note X is a definable subset of Y. It is easy to see that X is dense and co-dense in Y.

(6) $\Rightarrow$ (2) : Let X be a definable subset of $\mathbb{R}$. We show that X either has interior or contains an isolated point. If X is somewhere dense then X has interior. Suppose X is nowhere dense. It suffices to show that $\mathrm{Cl}(X)$ has an isolated point. After replacing X by its closure we suppose X is closed nowhere dense. As X is totally disconnected there is a bounded open interval I such that $I \cap X \neq \emptyset$ and $\mathrm{Bd}(I) \cap X = \emptyset$. Thus $I \cap X$ is nonempty, closed, bounded, and nowhere dense. As $I \cap X$ is not a Cantor set, it has an isolated point. □

Theorem 8.3 follows from Theorem 6.7 and Theorem 8.1.

Theorem 8.3. *If $\mathcal{R}$ is* NIP *then $\mathcal{R}^\circ$ is generically locally o-minimal.*

9. Strongly Dependent Expansions of $(\mathbb{R}, <, +)$

We continue to suppose that $\mathcal{R}$ is an expansion of $(\mathbb{R}, <, +)$. In this section we analyse strongly dependent expansions of $(\mathbb{R}, <, +)$ by closed sets and in particular prove Theorem D. In Section 9.1 we prove a general result about expansions of $(\mathbb{R}, <, +)$. In Section 9.2 we discuss $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimality. We then show that a strongly dependent expansion of $(\mathbb{R}, <, +)$ by closed sets is either o-minimal or $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal for some $\alpha > 0$. (This α is unique up to rational multiples.) We make crucial use of work of Dolich and Goodrick on strongly dependent expansions of ordered abelian groups [7] and work of Kawakami, Takeuchi, Tanaka, and Tsuboi [18] on locally o-minimal structures.

9.1. **Local o-minimality.**

We will need two results about locally o-minimal expansions of $(\mathbb{R}, <, +)$. The first gives a number equivalent conditions to local o-minimality. The equivalence of (2) and (3) below was proven in [31].

Theorem 9.1. *The following are equivalent:*

1. *$\mathcal{R}$ is locally o-minimal,*
2. *for every definable $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ there is an open interval I containing a such that $I \cap X$ is $(\mathbb{R}, <)$-definable,*
3. *the structure induced by $\mathcal{R}$ on any bounded interval I is o-minimal,*
4. *the expansion of $(\mathbb{R}, <, +)$ by all bounded $\mathcal{R}$-definable sets is o-minimal,*
5. *$\mathcal{R}$ is noiseless and does not define a bounded discrete subset of D of $\mathbb{R}_{>0}$ such that $\mathrm{Cl}(D) = D \cup \{0\}$,*
6. *$\mathcal{R}$ is noiseless and every nowhere dense definable subset of $\mathbb{R}$ is closed and discrete.*

We will need four results for the proof of Theorem 9.1. The first is an easy fact about subsets of $\mathbb{R}$ whose verification we leave to the reader.

Fact 9.2. *Suppose I is an open interval and X is a subset of I. Then X is a finite union of open intervals and singletons if and only if $\mathrm{Bd}(X) \cap I$ is finite.*

We say that $\mathcal{R}$ is **o-minimal at infinity** if for every definable $X \subseteq \mathbb{R}$ there is $t > 0$ such that (t, ∞) is either contained in or disjoint from X. Fact 9.3 is a special case of a theorem of Belegradek, Verbovskiy, and Wagner [1, Theorem 19].

Fact 9.3. *The expansion of $(\mathbb{R}, <, +)$ by all bounded subsets of all $\mathbb{R}^n$ is o-minimal at infinity.*

Fact 9.4 is due to Miller [24, Theorem 3.2].

Fact 9.4. *$\mathcal{R}$ is noiseless if and only if it is noiseless is one variable.*

We now prove Theorem 9.1.

Proof. (2) $\Rightarrow$ (3) : Let I be a bounded interval. After replacing I with its closure if necessary we suppose I is closed. Fix a definable subset X of I. We show that X is definable in $(\mathbb{R},<)$. For every $a \in I$ let J_a be an open interval containing a such that $J_a \cap X$ is definable in $(\mathbb{R},<)$. As I is compact there is a finite $A \subseteq I$ such that $(J_a)_{a\in A}$ covers I. Then

$$X = \bigcup_{a\in A} J_a \cap X$$

so X is definable in $(\mathbb{R},<)$.

(3) $\Rightarrow$ (4) : Let $\mathcal{B}$ be the collection of all bounded definable sets. Suppose $X \subseteq \mathbb{R}$ is definable in $(\mathbb{R},<,+,\mathcal{B})$. Fact 9.3 yields a $t > 0$ such that $X \setminus [-t,t]$ is definable in $(\mathbb{R},<)$. The induced structure on $[-t,t]$ is o-minimal, so $X \cap [-t,t]$ is definable in $(\mathbb{R},<)$, hence X is $(\mathbb{R},<)$-definable.

(4) $\Rightarrow$ (5) : Suppose that $\mathcal{R}$ is noisy. Let X be a definable subset of $\mathbb{R}^n$ which is dense and co-dense in a nonempty definable open subset U of $\mathbb{R}^n$. We may suppose that U is bounded. Then $U \cap X$ is bounded and definable and $(\mathbb{R},<,+,U\cap X)$ is not o-minimal, contradiction. Suppose that D is a bounded discrete subset of $\mathbb{R}_{>0}$ such that $\mathrm{Cl}(D) = D \cup \{0\}$. Then $\mathrm{Bd}(D) = D \cup \{0\}$ is infinite, an application of Fact 9.2 shows that $(\mathbb{R},<,+,D)$ is not o-minimal.

(5) $\Rightarrow$ (6) : Suppose that X is a nowhere dense definable subset of $\mathbb{R}$ which is not closed and discrete. Fix a bounded open interval I such that $I \cap X$ is infinite. Note that each connected component of $I \setminus \mathrm{Cl}(X)$ is a nonempty open interval. We let D be the set of lengths of connected components of $I \setminus \mathrm{Cl}(X)$. Then D is infinite as $\mathrm{Cl}(X)$ is infinite and bounded as I is bounded. As I is bounded there are only finitely many connected components of length $> t$ for any $t > 0$. Thus $D \setminus (0,t)$ is finite for any $t > 0$. It follows that D is discrete and $\mathrm{Cl}(D) = D \cup \{0\}$.

(6) $\Rightarrow$ (2) : Fix a definable subset X of $\mathbb{R}$ and $a \in X$. Then $\mathrm{Bd}(X)$ is nowhere dense as X is nowhere dense and co-dense, so $\mathrm{Bd}(X)$ is closed and discrete. Let I be a bounded open interval containing a such that $I \cap \mathrm{Bd}(X)$ is finite. Fact 9.2 shows that $I \cap X$ is definable in $(\mathbb{R},<)$.

It is immediate that (2) implies (1). We show that (1) implies (6). Let $X \subseteq \mathbb{R}$ be definable and somewhere dense and I be a nonempty open interval in which X is dense. Fix $p \in I \cap X$ and a subinterval J of I such that $J \cap X$ is $(\mathbb{R},<)$-definable. As X is dense in J, $J \cap X$ is cofinite, so X has interior. Now suppose that $Y \subseteq \mathbb{R}$ is nowhere dense and definable. It suffices to show that every $p \in \mathrm{Cl}(X)$ is isolated. Let I be a open interval containing p such that $I \cap \mathrm{Cl}(X)$ is $(\mathbb{R},<)$-definable. As $\mathrm{Cl}(X)$ is nowhere dense, $I \cap \mathrm{Cl}(X)$ is finite hence p is isolated in $I \cap \mathrm{Cl}(X)$. ☐

We now recall two results of Kawakami, Takeuchi, Tanaka, and Tsuboi [18]. Fact 9.5 is [18, Theorem 18]. Let $+_1 : [0,1)^2 \to [0,1)$ be given by $t +_1 t' = t + t'$ when $t + t' < 1$ and $t +_1 t' = t + t' - 1$ otherwise.

Fact 9.5. *Suppose $\mathcal{I}$ is an o-minimal expansion of $([0,1),<,+_1)$ and $\mathcal{D}$ is an arbitrary first order expansion of $(\mathbb{Z},<,+)$. Then there is a first order expansion $\mathcal{S}$ of $(\mathbb{R},<,+)$ such that a subset of $\mathbb{R}^n$ is $\mathcal{S}$-definable if and only if it is a finite union of sets of the form*

$$\bigcup_{b\in B} b+A.$$

for $\mathcal{I}$-definable $A\subseteq[0,1)^n$ and $\mathcal{D}$-definable $B\subseteq\mathbb{Z}^n$. This $\mathcal{S}$ is locally o-minimal and is bi-interpretable with the disjoint union of $\mathcal{I}$ and $\mathcal{D}$.

If $\mathcal{I}$ is $([0,1),<,+_1)$ and $\mathcal{D}$ is $(\mathbb{Z},<,+)$ then $\mathcal{S}$ is interdefinable with $(\mathbb{R},<,+,\mathbb{Z})$. It follows that any $(\mathbb{R},<,+,\mathbb{Z})$-definable subset of $\mathbb{R}^n$ is a finite union of sets of the form $\bigcup_{b\in B} b+A$ for $(\mathbb{R},<,+)$-definable $A\subseteq[0,1)^n$ and $(\mathbb{Z},<,+)$-definable $B\subseteq\mathbb{Z}^n$. This description of definable sets previously appeared in computer science [3, 4].

For each n let ι_n be the bijection $[0,1)^n\times\mathbb{Z}^n\to\mathbb{R}^n$ given by $\iota_n(a,d)=a+d$. Thus for any $A\subseteq[0,1)^n$ and $B\subseteq\mathbb{Z}^n$ we have

$$\iota_n(A\times B)=\bigcup_{b\in B} A+b.$$

Fact 9.6 is [18, Lemma 23, Theorem 24]. Fact 9.6 shows that any locally o-minimal expansion of $(\mathbb{R},<,+)$ which defines $\mathbb{Z}$ is bi-interpretable with the disjoint union of an o-minimal expansion of $([0,1),<,+_1)$ and an expansion of $(\mathbb{Z},<,+)$. Fact 9.6 is a converse to Fact 9.5.

Fact 9.6. *Suppose $\mathcal{R}$ is locally o-minimal and defines $\mathbb{Z}$. Every definable subset $[0,1)^n\times\mathbb{Z}^n$ is a finite union of sets of the form $A\times B$ for definable $A\subseteq[0,1)^n$ and $B\subseteq\mathbb{Z}^n$, hence every definable subset of $\mathbb{R}^n$ is a finite union of sets of the form*

$$\iota_n(A\times B)=\bigcup_{b\in B} A+b$$

for definable $A\subseteq[0,1)^n$ and $B\subseteq\mathbb{Z}^n$. Therefore $\mathcal{R}$ is bi-interpretable with the disjoint union of $\mathcal{I}$ and $\mathcal{D}$ where $\mathcal{I}$ is the structure induced on $[0,1)$ by $\mathcal{R}$ and $\mathcal{D}$ is the structure induced on $\mathbb{Z}$ by $\mathcal{R}$. Note that $\mathcal{I}$ is o-minimal as $\mathcal{R}$ is locally o-minimal.

9.2. **$(\mathbb{R},<,+,\mathbb{Z})$-minimality.** Recall that $\mathcal{R}$ is $(\mathbb{R},<,+,\mathbb{Z})$-minimal if $\mathcal{R}$ expands $(\mathbb{R},<,+,\mathbb{Z})$ and every $\mathcal{R}$-definable subset of $\mathbb{R}$ is $(\mathbb{R},<,+,\mathbb{Z})$-definable. Equivalently: $\mathcal{R}$ is $(\mathbb{R},<,+,\mathbb{Z})$-minimal if the $\mathcal{R}$-definable subsets of $\mathbb{R}$ are all finite unions of sets of the form $A+I$ where $A=s+t\mathbb{N}$ for some $s\in\mathbb{R},t\in\mathbb{Q}$ and I is an interval. Note that $(\mathbb{R},<,+,\mathbb{Z})$-minimality implies local o-minimality.

We will apply Fact 9.7, a theorem of Michaux and Villemaire [23].

Fact 9.7. *There are no proper $(\mathbb{Z},+,<)$-minimal expansions of $(\mathbb{Z},+,<)$.*

We can now prove Proposition 9.8.

Proposition 9.8. *Suppose $\mathcal{R}$ is $(\mathbb{R},<,+,\mathbb{Z})$-minimal. Then every $\mathcal{R}$-definable subset of $\mathbb{R}^n$ is a finite union of sets of the form $\bigcup_{b\in B} b+A$ where A is an $\mathcal{R}$-definable subset of $[0,1)^n$ and B is a $(\mathbb{Z},<,+)$-definable subset of $\mathbb{Z}^n$. In particular any subset of $\mathbb{Z}^n$ definable in $\mathcal{R}$ is definable in $(\mathbb{Z},<,+)$. It follows that $\mathcal{R}$ is bi-interpretable with the disjoint union of the induced structure on $[0,1)$ and $(\mathbb{Z},<,+)$.*

Strong dependence is preserved under disjoint unions and bi-interpretations, so Proposition 9.8 shows that $(\mathbb{R}, <, +, \mathbb{Z})$-minimality implies strong dependence.

Proof. By Fact 9.6 it suffices to show that the structure induced on $\mathbb{Z}$ by $\mathcal{R}$ is interdefinable with $(\mathbb{Z}, <, +)$. It follows from the description of $(\mathbb{R}, <, +, \mathbb{Z})$-definable sets that every $\mathcal{R}$-definable subset of $\mathbb{Z}$ is $(\mathbb{Z}, <, +)$-definable. Apply Fact 9.7. □

We now classify $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal structures. Suppose $\mathcal{S}$ is an o-minimal expansion of $(\mathbb{R}, <, +)$. A **pole** is an $\mathcal{S}$-definable surjection from a bounded interval to an unbounded interval. We say that $\mathcal{S}$ has **rational scalars** if the function $\mathbb{R} \to \mathbb{R}$ given by $t \mapsto \lambda t$ is only definable when $\lambda \in \mathbb{Q}$.

Theorem 9.9. *Fix $\alpha > 0$. The following are equivalent:*

(1) $\mathcal{R}$ is $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal,
(2) There is a collection $\mathcal{B}$ of bounded subsets of Euclidean space such that $(\mathbb{R}, <, +, \mathcal{B})$ is o-minimal and $\mathcal{R}$ is interdefinable with $(\mathbb{R}, <, +, \mathcal{B}, \alpha\mathbb{Z})$,
(3) There is an o-minimal expansion $\mathcal{S}$ of $(\mathbb{R}, <, +)$ which has no poles and has rational scalars such that $\mathcal{R}$ is interdefinable with $(\mathcal{S}, \alpha\mathbb{Z})$.
(4) There is an o-minimal expansion $\mathcal{S}$ of $(\mathbb{R}, <, +)$ such that $(\mathcal{S}, \alpha\mathbb{Z})$ is locally o-minimal and $\mathcal{R}$ is interdefinable with $(\mathcal{S}, \alpha\mathbb{Z})$.

We apply Fact 9.10, a special case of a theorem of Edmundo [11].

Fact 9.10. *The following are equivalent for an o-minimal expansion $\mathcal{S}$ of $(\mathbb{R}, <, +)$,*

- *$\mathcal{S}$ has no poles and has rational scalars,*
- *$\mathcal{S}$ is interdefinable with $(\mathbb{R}, <, +, \mathcal{B})$ for some collection $\mathcal{B}$ of bounded subsets of Euclidean space.*

We now prove Theorem 9.9.

Proof. Note that $t \mapsto \alpha^{-1}t$ gives an isomorphism $(\mathbb{R}, <, +, \alpha\mathbb{Z}) \to (\mathbb{R}, <, +, \mathbb{Z})$. Applying this isomorphism reduces to the case when $\alpha = 1$. We first show (1) and (2) are equivalent.

Suppose that $\mathcal{R}$ is $(\mathbb{R}, <, +, \mathbb{Z})$-minimal. Let $\mathcal{B}$ be the collection of all $\mathcal{R}$-definable subsets of all $[0, 1)^n$. As $\mathcal{R}$ is locally o-minimal Theorem 9.1 shows that $(\mathbb{R}, <, +, \mathcal{B})$ is o-minimal. It is immediate that $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$ is a reduct of $\mathcal{R}$. Proposition 9.8 shows that $\mathcal{R}$ is a reduct of $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$.

Now suppose (2). Rescaling and translating reduces to the case when every element of $\mathcal{B}$ is a subset of some $[0, 1)^n$. Let $\mathcal{I}$ be the structure induced on $[0, 1)$ by $(\mathbb{R}, <, +, \mathcal{B})$. Note that $\mathcal{I}$ is an o-minimal expansion of $([0, 1), <, +_1)$. Let $\mathcal{S}$ be the expansion of $(\mathbb{R}, <, +)$ constructed from $\mathcal{I}$ and $(\mathbb{Z}, <, +)$ as in Fact 9.5. The description of $\mathcal{S}$-definable sets in Fact 9.5 shows that $\mathcal{S}$ is $(\mathbb{R}, <, +, \mathbb{Z})$-minimal. It is immediate that $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$ is a reduct of $\mathcal{S}$ and the description of $\mathcal{S}$-definable sets in Fact 9.5 shows that $\mathcal{S}$ is a reduct of $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$. Therefore $\mathcal{S}$ and $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$ are interdefinable. It follows that $(\mathbb{R}, <, +, \mathcal{B}, \mathbb{Z})$ is $(\mathbb{R}, <, +, \mathbb{Z})$-minimal.

Fact 9.10 shows that (2) and (3) are equivalent. So (1) and (3) are equivalent, it follows that (3) implies (4) as $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimality implies local o-minimality. We show that (4) implies (3) by showing that if $\mathcal{S}$ is an o-minimal expansion of

$(\mathbb{R}, <, +)$ and $(\mathcal{S}, \mathbb{Z})$ is locally o-minimal then $\mathcal{S}$ has no poles and has rational scalars.

Suppose $\tau : I \to J$ is a pole. Applying the monotonicity theorem for o-minimal structures, reflecting, and translating, we suppose that J contains a final segment of $\mathbb{R}$, τ is strictly increasing, and τ is continuous. Then $\tau^{-1}(\mathbb{N})$ is an infinite discrete subset of a bounded interval so $(\mathcal{S}, \mathbb{Z})$ is not locally o-minimal. Suppose $\lambda \in \mathbb{R}$ is irrational and the map $\mathbb{R} \to \mathbb{R}$, $t \mapsto \lambda t$ is $\mathcal{S}$-definable. Then $\mathbb{Z} + \lambda\mathbb{Z}$ is dense and co-dense in $\mathbb{R}$ so $(\mathcal{S}, \mathbb{Z})$ is not locally o-minimal. □

Corollaries 9.11 and 9.12 will be used below. Both are easy consequences of Proposition 9.8 and standard facts from o-minimality, we leave the details to the reader. Here we let $\dim X$ be the topological dimension[1] of a subset X of $\mathbb{R}^n$.

Corollary 9.11. *Suppose that $\mathcal{R}$ is $(\mathbb{R}, <, +, \mathbb{Z})$-minimal and X, Y are $\mathcal{R}$-definable subsets of $\mathbb{R}^n$. If Y is a nowhere dense subset of X then* $\dim Y < \dim X$.

Corollary 9.12. *If $\mathcal{R}$ is $(\mathbb{R}, <, +, \mathbb{Z})$-minimal then every $\mathcal{R}$-definable subset of $\mathbb{R}^n$ is a boolean combination of closed $\mathcal{R}$-definable sets.*

Corollaries 9.11 and 9.12 hold more generally for any locally o-minimal expansion of $(\mathbb{R}, <, +)$, we will not need this.

9.3. **Strong dependence.** We now show that strongly dependent expansions of $(\mathbb{R}, <, +)$ by closed sets are either o-minimal or $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal for some $\alpha > 0$ which is unique up to rational multiples. Strong dependence should be seen as "finite dimensional NIP". We refer to [27, Chapter 4] for an account of strong dependence. We need (special cases of) several results of Dolich and Goodrick. The first claim of Fact 9.13 is a special case of [7, Corollary 2.13]. The second claim is essentially a case of [7, Theorem 2.18].

Fact 9.13. *Suppose that $\mathcal{R}$ is strongly dependent and $E \subseteq \mathbb{R}$ is definable and discrete. Then E has no accumulation points and there are $s_1, \ldots, s_n, t_1, \ldots, t_n$ so that $E = (s_1 + t_1\mathbb{N}) \cup \ldots \cup (s_n + t_n\mathbb{N})$ and $t_i/t_j \in \mathbb{Q}$ for all i, j such that $t_j \neq 0$.*

We say that the second claim is "essentially" a special case of [7, Theorem 2.18] because that theorem is slightly incorrect as stated. We explain. Dolich and Goodrick assert, under the assumptions of Fact 9.13, that E is a union of a finite set together with *commensurable* progressions $s_1 + t_1\mathbb{N}, \ldots, s_n + t_n\mathbb{N}$. (Recall $s_1 + t_1\mathbb{N}$ and $s_2 + t_2\mathbb{N}$ are commensurable if they both lie in $s + t\mathbb{Z}$ for some s, t.) This need not be the case, for example $\mathbb{N}$ and $\alpha + \mathbb{N}$ are not commensurable for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The mistake lies in [7, Corollary 2.30]. In the proof of that corollary it is asserted that if $s_1 + t_1\mathbb{N}, s_2 + t_2\mathbb{N}$ are both infinite and contained in $\mathbb{R}_{>0}$ and

$$\{t - \min\{r \in s_1 + t_1\mathbb{N}, s_2 + t_2\mathbb{N} : r > t\} : t \in s_1 + t_1\mathbb{N}, s_2 + t_2\mathbb{N}\}$$

is disjoint from $[0, \varepsilon)$ for some $\varepsilon > 0$ then $s_1 + t_1\mathbb{N}$ and $s_2 + t_2\mathbb{N}$ are commensurable. This is incorrect, but it does imply that $t_1/t_2 \in \mathbb{Q}$. The rest of the proof is correct[2].

Fact 9.14 is a theorem of Bès and Choffrut [2]. We only need the case $n = 1$. The case $n = 1$ was also proven independently by Miller and Speissegger [25].

[1]Topological dimension here refers to either small inductive dimension, large inductive dimension, or Lebesgue covering dimension. On subsets of $\mathbb{R}^n$ these three dimensions coincide (see Engelking [13] for details and definitions).

[2]Thanks to John Goodrick for discussions on this point.

Fact 9.14. *Suppose X is a subset of $\mathbb{R}^n$ which is definable in $(\mathbb{R},<,+,\mathbb{Z})$ and not definable in $(\mathbb{R},<,+)$. Then $(\mathbb{R},<,+,X)$ defines $\mathbb{Z}$.*

We can now describe strongly dependent expansions of $(\mathbb{R},<,+)$ by closed sets. Recall our standing assumption that $\mathcal{R}$ is an expansion of $(\mathbb{R},<,+)$.

Theorem 9.15. *The following are equivalent:*

(1) $\mathcal{R}$ is a strongly dependent expansion by closed sets.
(2) $\mathcal{R}$ is strongly dependent and noiseless.
(3) $\mathcal{R}$ is either o-minimal or $(\mathbb{R},<,+,\alpha\mathbb{Z})$-minimal for some $\alpha>0$.
(4) $\mathcal{R}$ is either o-minimal or interdefinable with $(\mathbb{R},<,+,\mathcal{B},\alpha\mathbb{Z})$ for some $\alpha>0$ and collection $\mathcal{B}$ of bounded subsets of Euclidean space such that $(\mathbb{R},<,+,\mathcal{B})$ is o-minimal.
(5) $\mathcal{R}$ is either o-minimal or locally o-minimal and interdefinable with $(\mathcal{S},\alpha\mathbb{Z})$ for some $\alpha>0$ and o-minimal expansion $\mathcal{S}$ of $(\mathbb{R},<,+)$.

Proof. Theorem 8.3 shows that (1) implies (2). We show that (2) implies (3). Suppose that $\mathcal{R}$ is strongly dependent, noiseless, and not o-minimal. The first claim of Fact 9.13 and Theorem 9.1 together show that $\mathcal{R}$ is locally o-minimal. We show that $\mathcal{R}$ defines $\alpha\mathbb{Z}$ for some $\alpha>0$. As $\mathcal{R}$ is locally o-minimal and not o-minimal there is an infinite definable discrete subset E of $\mathbb{R}$. Let $s_1,\dots,s_n,t_1,\dots,t_n$ be as in Fact 9.13. After possibly removing a finite subset of E we suppose that $t_i\neq 0$ for all i. Let $\alpha:=|t_1|$. As $t_i/\alpha\in\mathbb{Q}$ for all $2\le i\le n$ each $s_i+t_i\mathbb{N}$ is $(\mathbb{R},<,+,\alpha\mathbb{Z})$-definable. Thus E is $(\mathbb{R},<,+,\alpha\mathbb{Z})$-definable. As E is not $(\mathbb{R},<,+)$-definable, rescaling and applying Fact 9.14 shows that $(\mathbb{R},<,+,E)$ defines $\alpha\mathbb{Z}$, hence $\mathcal{R}$ defines $\alpha\mathbb{Z}$.

We may now apply Fact 9.6. We get that any $\mathcal{R}$-definable subset of $\mathbb{R}$ is a finite union of sets of the form $\bigcup_{b\in B} b+A$ where $A\subseteq[0,\alpha)$ is a finite union of intervals and singletons and $B\subseteq\alpha\mathbb{Z}$ is $\mathcal{R}$-definable. It therefore suffices to show that an $\mathcal{R}$-definable subset B of $\alpha\mathbb{Z}$ is $(\alpha\mathbb{Z},<,+)$-definable. As B is discrete this follows from Fact 9.13.

The equivalence of (3), (4), and (5) is Theorem 9.9. We show that (3) implies (1). Suppose $\mathcal{R}$ is $(\mathbb{R},<,+,\alpha\mathbb{Z})$-minimal for fixed $\alpha>0$. By Corollary 9.12 $\mathcal{R}$ is an expansion by closed sets. The remarks after Proposition 9.8 show that $\mathcal{R}$ is strongly dependent. □

10. Archimedean quotients

Proposition 10.1 is used in the proof of Theorem E below.

Through this section $\mathcal{N}$ is an $\aleph_1$-saturated expansion of a dense ordered abelian group $(N,<,+)$. Given a positive element a of N we let $\mathbf{Fin}_a$ be the convex hull of $a\mathbb{Z}$ and $\mathbf{Inf}_a$ be the set of $b\in N$ such that $|nb|<a$ for all n. Then $\mathbf{Fin}_a$ and $\mathbf{Inf}_a$ are convex subgroups of $(N,<,+)$. We order $\mathbf{Fin}_a/\mathbf{Inf}_a$ by declaring $a+\mathbf{Inf}_a<b+\mathbf{Inf}_a$ if every element of $a+\mathbf{Inf}_a$ is strictly less then every element of $b+\mathbf{Inf}_a$. Then $\mathbf{Fin}_a/\mathbf{Inf}_a$ is an ordered abelian group. As $\mathbf{Fin}_a$ and $\mathbf{Inf}_a$ are convex Fact 2.1 shows that they are externally definable. We consider $\mathbf{Fin}_a/\mathbf{Inf}_a$ to be an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$.

Let $\mathrm{st}_a : \mathbf{Fin}_a \to \mathbb{R}$ be given by $\mathrm{st}_a(b) := \sup\{\frac{m}{n} \in \mathbb{Q} : ma \le nb\}$. Then st_a is an ordered group homomorphism with kernal $\mathbf{Inf}_a$. Note that $\mathrm{st}_a(\alpha) = 1$. Density of $(N, <, +)$ and saturation together imply that st_a is surjective, hence $\mathbf{Fin}_a/\mathbf{Inf}_a$ is isomorphic to $(\mathbb{R}, <, +)$.

Proposition 10.1. *Suppose $\mathcal{N}$ is NIP, fix a positive $a \in N$, and let $\mathcal{R}$ be the structure induced on $\mathbf{Fin}_a/\mathbf{Inf}_a$ by $\mathcal{N}^{\mathrm{Sh}}$. Then $\mathcal{R}$ is isomorphic to a generically locally o-minimal expansion of $(\mathbb{R}, <, +)$. If $\mathcal{N}$ is strongly dependent then $\mathcal{R}$ is either isomorphic to an o-minimal expansion of $(\mathbb{R}, <, +)$ or an $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal expansion of $(\mathbb{R}, <, +, \alpha\mathbb{Z})$ for some $\alpha > 0$.*

Proof. For the sake of simplicity we identify $\mathbf{Fin}_a/\mathbf{Inf}_a$ with $\mathbb{R}$. We show that the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless. By Theorem 8.1 it suffices to show that the structure induced on $[0, 1]$ by $\mathcal{N}^{\mathrm{Sh}}$ is strongly noiseless. Let $I = [0, a]$. Note that $\mathrm{st}_a(I) = [0, 1]$. Let E be the equivalence relation of equality modulo $\mathbf{Inf}_a$ on I. Applying density we select for each $n \ge 1$ an $a_n \in N$ satisfying $(n - 1)a_n < a \le na_n$. Let F_n be the set of $(a, b) \in I^2$ such that $|a - b| < a_n$. Then each F_n is $\mathcal{N}$-definable, $(F_n)_{n\in\mathbb{N}}$ is nested, and $\bigcap_{n\ge1} F_n = E$. Then E is $\bigwedge$-definable and Lemma 3.7 shows that $(F_n)_{n\ge1}$ is a subdefinable basis for F.

By Theorem 8.1 is suffices to show that the logic topology on $[0, 1]$ agrees with the usual order topology. As both topologies are compact Hausdorff it suffices to show that any subset of $[0, 1]$ which is closed in the order topology is also closed in the logic topology. For this purpose it is enough to fix elements $t < t'$ of $[0, 1]$ and show that $[t, t']$ is closed in the logic topology. For each m, n, m', n' such that $\frac{m}{n} \le t, t' \le \frac{m'}{n'}$ we let $X^{m,n}_{m',n'}$ be the set of $a \in I$ such that $m \le na$ and $n'a \le m'$. Hence each $X^{m,n}_{m',n'}$ is $\mathcal{N}$-definable. Then $\mathrm{st}^{-1}([a, b]) \cap I$ is the intersection of all $X^{m,n}_{m',n'}$ and is hence $\bigwedge$-definable, so $[a, b]$ is closed in the logic topology. The proposition now follows by Theorems 5.1, 8.1, and 9.15. □

11. Archimedean structures

In this section we prove Theorem E, which is broken up into several theorems. **We suppose throughout this section that $(R, <, +)$ is an archimededan dense ordered abelian group, which we take to be a substructure of $(\mathbb{R}, <, +)$, and that $\mathcal{R}$ is a NIP structure expanding $(R, <, +)$.** By the Hahn embedding theorem any archimedean ordered abelian group has a unique up to rescaling embedding into $(\mathbb{R}, <, +)$ so our assumption that $(R, <, +)$ is a substructure of $(\mathbb{R}, <, +)$ is mild.

It is natural to try to extend Theorem 8.3 to expansions of archimedean ordered abelian groups. Fix $n \ge 2$ and suppose that $(R, <, +)$ is not n-divisible. Observe that nR is dense and co-dense, hence $(R, <, +)$ is generated by closed sets and is noisy. The correct generalization of Theorem 8.3 concerns the canonical completion of a NIP expansion of an archimedean ordered abelian group.

11.1. The completion of $\mathcal{R}$.

We define the **completion** $\mathcal{R}^{\square}$ of $\mathcal{R}$ to be the structure on $\mathbb{R}$ whose primitive n-ary relations are all sets of the form $\mathrm{Cl}(X)$ for $\mathcal{R}^{\mathrm{Sh}}$-definable $X \subseteq R^n$. Note that $\mathcal{R}^{\square}$ expands $(\mathbb{R}, <, +)$.

Proposition 11.1 follows as $\mathrm{Cl}(X) \cap R^n = X$ for any $X \subseteq R^n$ which is closed in R^n.

Proposition 11.1. *Suppose $\mathcal{R}$ is an expansion of $(R,<,+)$ by closed sets. Then $\mathcal{R}$ is a reduct of the structure induced on R by $\mathcal{R}^{\square}$.*

Let $\mathcal{N}$ be an $|\mathbb{R}|^+$-saturated elementary extension of $\mathcal{R}$. Let **Fin** be the convex hull of R in N and let **Inf** be the set of $a \in N$ such that $|a| < b$ for all positive $b \in R$. Following Section 10 we identify **Fin**/**Inf** with $\mathbb{R}$ and consider $\mathbb{R}$ to be an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$. We let $\mathrm{st} : \mathbf{Fin} \to \mathbb{R}$ be the quotient map and, abusing notation, let $\mathrm{st} : \mathbf{Fin}^n \to \mathbb{R}^n$ be given by

$$\mathrm{st}(a_1,\ldots,a_n) = (\mathrm{st}(a_1),\ldots,\mathrm{st}(a_n)).$$

Given an $\mathcal{R}$-definable subset Y of R^n we let Y' be the subset of N^n defined by any formula which defines Y.

Proposition 11.2. *The following are interdefinable*

(1) $\mathcal{R}^{\square}$,
(2) The expansion of $(\mathbb{R},<,+)$ by all $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ for $\mathcal{N}$-definable $Y \subseteq N^n$,
(3) The open core of the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$.

In particular $\mathcal{R}^{\square}$ is a reduct of the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$. Thus $\mathcal{R}^{\square}$ is NIP *and generically locally o-minimal and if $\mathcal{R}$ is strongly dependent then $\mathcal{R}^{\square}$ is either o-minimal or $(\mathbb{R},<,+,\alpha\mathbb{Z})$-minimal for some $\alpha > 0$.*

We do not know if $\mathcal{R}^{\square}$ is always interdefinable with the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$. We need the next two propositions to prove the first claim of Proposition 11.2.

Lemma 11.3. *Suppose $X \subseteq R^n$ is externally definable in $\mathcal{R}$ and $Y \subseteq N^n$ is an honest definition of X. Then $\mathrm{Cl}(X) = \mathrm{st}(Y \cap \mathbf{Fin}^n)$.*

Proof. As $X \subseteq Y \cap \mathbf{Fin}^n$ and st is the identity on R^n we see that X is contained in $\mathrm{st}(Y \cap \mathbf{Fin}^n)$. An easy saturation argument shows that $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is closed so $\mathrm{Cl}(X)$ is contained in $\mathrm{st}(Y \cap \mathbf{Fin}^n)$. We prove the other inclusion. Let $I_1,\ldots,I_n$ be nonempty open intervals in $\mathbb{R}$ with endpoints in R. Let U be $I_1 \times \ldots \times I_n$. Note that the collection of such open boxes forms a basis for $\mathbb{R}^n$. It suffices to suppose $\mathrm{Cl}(X)$ is disjoint from U and show that $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is disjoint from U. It is enough to show that Y is disjoint from $\mathrm{st}^{-1}(U)$. Note that $V := R^n \cap U$ is definable in $\mathcal{R}$. Then X is disjoint from V, so Y is disjoint from V' by honestness. It is easy to see that $\mathrm{st}^{-1}(U)$ is a subset of V', so Y is disjoint from $\mathrm{st}^{-1}(U)$. ☐

Lemma 11.4. *Any closed $\mathcal{N}^{\mathrm{Sh}}$-definable subset of $\mathbb{R}^n$ is $\mathcal{R}^{\square}$-definable.*

Given $a = (a_1,\ldots,a_m) \in N^m$ we let $\|a\| = \max\{|a_1|,\ldots,|a_m|\}$. We will apply the fact that if $p,p' \in N^k$ and $q,q' \in N^l$ then $\|(p,q)-(p',q')\| = \max\{\|p-p'\|,\|q-q'\|\}$.

Proof. Let $X \subseteq \mathbb{R}^n$ be closed and $\mathcal{N}^{\mathrm{Sh}}$-definable. Let $Y := \mathrm{st}^{-1}(X)$, so Y is $\mathcal{N}^{\mathrm{Sh}}$-definable. Then

$$\{(\delta,a) \in N_{>0} \times N^n : \|a-a'\| < \delta \quad \text{for some } a' \in Y\}$$

is $\mathcal{N}^{\mathrm{Sh}}$-definable, hence externally definable in $\mathcal{N}$. Applying Fact 2.2 we see that

$$W := \{(\delta,a) \in R_{>0} \times R^n : \|a-a'\| < \delta \quad \text{for some } a' \in Y\}$$

is $\mathcal{R}^{\mathrm{Sh}}$-definable. Let $Z := \mathrm{Cl}(W) \cap (\mathbb{R}_{>0} \times \mathbb{R}^n)$. Then Z is $\mathcal{R}^{\square}$-definable. We show that $X = \bigcap_{t>0} Z_t$.

To prove the left to right inclusion we fix $p \in X$ and $t \in \mathbb{R}_{>0}$ and show that $p \in Z_t$. As Z_t is closed it suffices to fix $\varepsilon \in \mathbb{R}_{>0}$ and produce $p' \in Z_t$ such that $\|p - p'\| < \varepsilon$. We may suppose $\varepsilon < t$. Fix $q \in Y$ such that $\mathrm{st}(q) = p$ and $p' \in R^n$ such that $\|p - p'\| < \varepsilon$. As $\mathrm{st}\,\|p - q\| = 0$ we have $\|p' - q\| < \varepsilon < t$. Hence $p' \in Z_t$.

We prove the other inclusion. Suppose $p \in Z_t$ for all $t \in \mathbb{R}_{>0}$. We show $p \in X$. As X is closed it suffices to fix $\varepsilon \in \mathbb{R}_{>0}$ and produce $p' \in X$ such that $\|p - p'\| < \varepsilon$. As $p \in Z_{\frac{1}{4}\varepsilon}$ we fix $(t, p'') \in W$ such that $\|(\frac{1}{4}\varepsilon, p) - (t, p'')\| < \frac{1}{4}\varepsilon$. Then $|t - \frac{1}{4}\varepsilon| < \frac{1}{4}\varepsilon$ and $\|p - p''\| < \frac{1}{4}\varepsilon$. As $(t, p'') \in W$ there is $q \in Y$ such that $\|p'' - q\| < t < \frac{1}{2}\varepsilon$. Set $p' := \mathrm{st}(q)$, so $p' \in X$. As $\mathrm{st}\,\|q - p'\| = 0$ we have $\|p'' - p'\| < \frac{1}{2}\varepsilon$, so $\|p - p'\| < \varepsilon$. □

We now prove Proposition 11.2.

Proof. The second claim of Proposition 11.2 follows immediately from the first claim and the third claim follows from the second claim and Proposition 10.1. We prove the first claim. Fact 2.6 and Lemma 11.3 together show that (1) is a reduct of (2). A saturation argument shows that if $Y \subseteq N^n$ is $\mathcal{N}$-definable then $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is closed. Thus (2) is a reduct of (3). By Lemma 11.4 (3) is a reduct of (1). □

We expect that if $\mathcal{R}$ is a naturally arising expansion by closed sets then $\mathcal{R}^{\mathrm{Sh}}$ is interdefinable with the structure induced on R by $\mathcal{R}^{\square}$.

Proposition 11.5. *The structure induced on R by $\mathcal{R}^{\square}$ is a reduct of $\mathcal{R}^{\mathrm{Sh}}$.*

Propositions 11.1 and 11.5 together show that if $\mathcal{R}$ is an NIP expansion by closed sets then the structure induced on R by $\mathcal{R}^{\square}$ is between $\mathcal{R}$ and $\mathcal{R}^{\mathrm{Sh}}$.

Proof. Suppose X is a subset of $\mathbb{R}^n$ definable in $\mathcal{R}^{\square}$. By Proposition 11.2 $\mathrm{st}^{-1}(X)$ is definable in $\mathcal{N}^{\mathrm{Sh}}$, hence externally definable in $\mathcal{N}$. As st is the identity on R^n we have $X \cap R^n = \mathrm{st}^{-1}(X) \cap R^n$. By Fact 2.2 $X \cap R^n$ is externally definable in $\mathcal{R}$. □

11.2. **The strongly dependent case.** We obtain stronger results when $\mathcal{R}$ is strongly dependent. We do not know if these results generalize to the NIP case.

Corollary 11.6. *Suppose $\mathcal{R}$ is strongly dependent. Then $\mathcal{R}^{\square}$ is interdefinable with the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$.*

Proof. Let $\mathcal{S}$ be the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$. By Proposition 10.1 $\mathcal{S}$ is either o-minimal or $(\mathbb{R}, <, +, \alpha\mathbb{Z})$-minimal for some α. Corollary 9.12 shows that $\mathcal{S}$ is interdefinable with $\mathcal{S}^{\circ}$. Now apply Proposition 11.2. □

Proposition 11.7. *Suppose $\mathcal{R}$ is noiseless and strongly dependent. For every $\mathcal{R}^{\mathrm{Sh}}$-definable $X \subseteq R^n$ there is an $\mathcal{R}^{\square}$-definable $Y \subseteq \mathbb{R}^n$ such that $X = Y \cap R^n$. Equivalently: the structure induced on R by $\mathcal{R}^{\square}$ eliminates quantifiers and is interdefinable with $\mathcal{R}^{\mathrm{Sh}}$.*

The proof of Proposition 11.7 requires Lemma 11.8 and Proposition 11.9.

Lemma 11.8. *If $\mathcal{R}$ is* NIP *then $\mathcal{R}$ is noiseless if and only if $\mathcal{R}$ is strongly noiseless.*

Proof. Suppose $\mathcal{R}$ is NIP and not strongly noiseless. Let X, Y be $\mathcal{R}$-definable subsets of R^n such that X is somewhere dense in Y and has empty interior in Y. Let U be a definable open subset of Y such that X is dense in U. After replacing Y with U if necessary we suppose X is dense in Y. As $\mathcal{R}^{\square}$ is generically locally o-minimal an application of Fact 8.2 yields $d, n \in \mathbb{N}$ with $d \leq n$, a point $p \in Y$, an open neighbourhood V of p, and a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that π gives a homeomorphism $\mathrm{Cl}(Y) \cap U \to \pi(\mathrm{Cl}(Y) \cap U)$ and $\pi(\mathrm{Cl}(Y) \cap U)$ is an open subset of $\mathbb{R}^d$. Then $\pi(X \cap U)$ is dense and co-dense in an open subset of R^d, so $\mathcal{R}$ is noisy. □

We also need Proposition 11.9 which we prove under general assumptions.

Proposition 11.9. *Suppose $\mathcal{M}$ is* NIP *and one-sorted. Suppose there is a subdefinable basis for a uniform structure on M. If $\mathcal{M}$ is (strongly) noiseless then $\mathcal{M}^{\mathrm{Sh}}$ is (strongly) noiseless.*

The proof of Proposition 11.9 is very similar to that of Theorem B.

Proof. We suppose that $\mathcal{M}$ is strongly noiseless and show that $\mathcal{M}^{\mathrm{Sh}}$ is strongly noiseless, the same argument shows that if $\mathcal{M}$ is noiseless then $\mathcal{M}^{\mathrm{Sh}}$ is noiseless. Applying Proposition 3.4 we suppose that $\mathcal{B}$ is an $\mathcal{M}^{\mathrm{Sh}}$-definable basis for the uniform structure on M. Suppose X, Y are $\mathcal{M}^{\mathrm{Sh}}$-definable subsets of M^n such that X is somewhere dense in Y and X has empty interior in Y. Let U be a nonempty open subset of Y in which X is dense. As $\mathcal{B}$ is an $\mathcal{M}$-definable basis we may suppose U is $\mathcal{M}^{\mathrm{Sh}}$-definable. After replacing Y with $U \cap Y$ we suppose X is dense in Y. We apply Theorem 4.1 to Y and $\mathcal{B}^n|_Y$. Applying Facts 2.3 and 2.5 we let $(X_a)_{a \in M^m}$ be an $\mathcal{M}$-definable family of sets such that for every finite $A \subseteq X$ there is $a \in M^m$ such that $A \subseteq X_a \subseteq X$. As X has empty interior in Y and $\mathcal{M}$ is strongly noiseless X_a is nowhere dense in Y when $X_a \subseteq X$. Let $\mathcal{U}$ be a finite collection of nonempty open subsets of Y. As X is dense in Y there is a finite subset A of X which intersects each $U \in \mathcal{U}$. There is $a \in M^m$ such that $A \subseteq X_a \subseteq X$, this X_a intersects each $U \in \mathcal{U}$ and is nowhere dense in Y. This contradicts Theorem 4.1 □

We now prove Proposition 11.7. We let $\mathrm{Cl}_R(X)$ be the closure in R^n of $X \subseteq R^n$ and let $\dim X$ be the topological dimension of a subset X of $\mathbb{R}^n$.

Proof. The second claim follows from the first claim by Proposition 11.5, so we only prove the first claim. Lemma 11.8 shows that $\mathcal{R}$ is strongly noiseless. Proposition 11.9 shows that $\mathcal{R}^{\mathrm{Sh}}$ is strongly noiseless.

Let X be an $\mathcal{R}^{\mathrm{Sh}}$-definable subset of R^n. We apply induction on $\dim \mathrm{Cl}(X)$. If $\dim \mathrm{Cl}(X) = -1$ then X is empty, so we take $Y = \emptyset$. Suppose $\dim \mathrm{Cl}(X) \geq 0$. Let U be the interior of X in $\mathrm{Cl}_R(X)$ and $Z := \mathrm{Cl}_R(X) \setminus U$. Then U, Z are $\mathcal{R}^{\mathrm{Sh}}$-definable and Z is closed in R^n. By strong noiselessness U is dense in $\mathrm{Cl}_R(X)$ so Z is nowhere dense in $\mathrm{Cl}_R(X)$ hence $\mathrm{Cl}(Z)$ is nowhere dense in $\mathrm{Cl}(X)$. By Corollary 9.11 we have $\dim \mathrm{Cl}(Z) < \dim \mathrm{Cl}(X)$. Proposition 11.5 shows that $Z \cap X$ is definable in $\mathcal{R}^{\mathrm{Sh}}$ so an application of induction shows that $Z \cap X = Y' \cap R^n$ for some $\mathcal{R}^{\square}$-definable $Y' \subseteq \mathbb{R}^n$. We have $X \setminus Z = U$ and $U = [\mathrm{Cl}(X) \setminus \mathrm{Cl}(Z)] \cap R^n$. Then

$$X = [Y' \cup (\mathrm{Cl}(X) \setminus \mathrm{Cl}(Z))] \cap R^n$$

and

$$Y := Y' \cup (\mathrm{Cl}(X) \setminus \mathrm{Cl}(Z))$$

is $\mathcal{R}^{\square}$-definable. □

11.3. **Examples of completions.** We discuss some examples of completions.

11.3.1. *Ordered groups.* The NIP-theoretic completion of $(R, <, +)$ agrees with the usual completion.

Proposition 11.10. *$(R, +, <)^\square$ is interdefinable with $(\mathbb{R}, +, <)$.*

Proposition 11.10 requires the quantifier elimination for archimedean ordered abelian groups. See Weispfennig [32] for a proof.

Fact 11.11. *The structure $(R, +, <)$ admits quantifier elimination after a unary relation for every nR is added to the language.*

We now prove Proposition 11.10. If $T : R^n \to R$ is a $\mathbb{Z}$-linear function given by $T(a_1, \ldots, a_n) = k_1 a_1 + \ldots + k_n a_n$ for integers $k_1, \ldots, k_n$ then we also let T denote the function $\mathbb{R}^n \to \mathbb{R}$ given by $(t_1, \ldots, t_n) \mapsto k_1 t_1 + \ldots + k_n t_n$.

Proof. Let $(R+ <) \prec (N, +, <)$ be $|\mathbb{R}|^+$-saturated and let $\mathbf{Fin}, \mathrm{st} : \mathbf{Fin}^n \to \mathbb{R}^n$ be as above. As $(R, +, <)$ is NIP, it suffices by Proposition 11.2 to suppose that $Y \subseteq N^n$ is $\mathcal{N}$-definable and show that $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is $(\mathbb{R}, +, <)$-definable. If Z is the closure of Y in N^n then $\mathrm{st}(Z \cap \mathbf{Fin}^n) = \mathrm{st}(Y \cap \mathbf{Fin}^n)$. We therefore suppose that Y is closed. A straightforward application of Fact 11.11 shows that Y is a finite union of sets of the form

$$\{a \in N^n : T_1(a) \leq s_1, \ldots, T_k(a) \leq s_k\}$$

for $\mathbb{Z}$-linear $T_1, \ldots, T_k : N^n \to N$ and $s_1, \ldots, s_k \in N$. Thus we may suppose that Y is of this form. If $s_i > \mathbf{Fin}$ then $\mathbf{Fin}^n$ is contained in $\{a \in N^n : T_i(a) \leq s_i\}$ and if $s_i < \mathbf{Fin}$ then $\{a \in N^n : T_i(a) \leq s_i\}$ is disjoint from $\mathbf{Fin}^n$, so we suppose $s_1, \ldots, s_k \in \mathbf{Fin}$. It is now easy to see that

$$\mathrm{st}(Y \cap \mathbf{Fin}^n) = \{a \in \mathbb{R}^n : T_1(a) \leq \mathrm{st}(s_1), \ldots, T_k(a) \leq \mathrm{st}(s_k)\}.$$

Hence $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is $(\mathbb{R}, +, <)$-definable. $\square$

11.3.2. *Ordered fields.* Suppose R is an NIP subfield of $(\mathbb{R}, <, +, \times)$. One hopes that $(R, <, +, \times)^\square$ is interdefinable with $(\mathbb{R}, <, +, \times)$. We show below that this follows from the main conjecture on NIP fields. Let R be real closed. Then $(R, <, +, \times)$ is o-minimal and $(\mathbb{R}, <, +, \times)$ is an elementary extension of $(R, <, +, \times)$, so it follows from Section 11.3.3 below that $(R, <, +, \times)^\square$ is interdefinable with $(\mathbb{R}, <, +, \times)$ and $(R, <, +, \times)^{\mathrm{Sh}}$ is interdefinable with the structure induced on R by $(\mathbb{R}, <, +, \times)$. It is a well-known conjecture that an infinite NIP field is either separably closed, real closed, or admits a non-trivial Henselian valuation (see for example [16]). An ordered field cannot be separably closed, a Henselian valuation on an ordered field has a convex valuation ring [14, Lemma 4.3.6], and an archimedean ordered field does not admit a non-trivial convex subring. Thus the conjecture implies that an NIP archimedean ordered field is real closed.

11.3.3. *O-minimal structures.* Laskowski and Steinhorn [20] show that if $\mathcal{R}$ is o-minimal then there is a unique o-minimal expansion $\mathcal{S}$ of $(\mathbb{R}, <, +)$ such that $\mathcal{R} \prec \mathcal{S}$.

Proposition 11.12. *Let $\mathcal{R}$ be o-minimal and $\mathcal{S}$ be the unique elementary extension of $\mathcal{R}$ which expands $(\mathbb{R}, <, +)$. Then $\mathcal{S}$ is interdefinable with $\mathcal{R}^\square$.*

Proof. We first show that $\mathcal{S}$ is a reduct of $\mathcal{R}^\square$. It suffices to suppose that $X \subseteq \mathbb{R}^n$ is 0-definable in $\mathcal{S}$ and show that X is $\mathcal{R}^\square$-definable. By o-minimal cell decomposition X is a boolean combination of closed subsets of $\mathbb{R}^n$ which are 0-definable in $\mathcal{S}$, so we suppose that X is closed. Let X' be the $\mathcal{R}$-definable set defined by any parameter-free formula defining X. Observe that X is the closure of X' in $\mathbb{R}^n$.

We now show that $\mathcal{R}^\square$ is a reduct of $\mathcal{S}$. By the Marker-Steinhorn theorem [22] $\mathcal{S}$ is Shelah complete, so by Lemma 2.4 any $\mathcal{R}^{\mathrm{Sh}}$-definable subset of R^n is of the form $Y \cap R^n$ for some $\mathcal{S}$-definable $Y \subseteq \mathbb{R}^n$. Thus the structure induced on R by $\mathcal{S}$ is interdefinable with $\mathcal{R}^{\mathrm{Sh}}$. Work of Dolich, Miller, and Steinhorn [8, Section 5] shows that $(\mathcal{S}, R)^\circ$ is interdefinable with $\mathcal{S}$. Thus if $X \subseteq R^n$ is definable in the structure induced on R by $\mathcal{S}$ then $\mathrm{Cl}(X)$ is definable in $\mathcal{S}$. □

11.3.4. *Weakly o-minimal structures.* A weakly o-minimal structure may not have weakly o-minimal theory. However, a result of Marker, Macpherson, and Steinhorn [21, Theorem 6.7] shows that a weakly o-minimal expansion of an archimedean ordered abelian group has weakly o-minimal theory, so we ignore this distinction.

Suppose $\mathcal{R}$ is weakly o-minimal. We describe the o-minimal completion of $\mathcal{R}$. This completion was first constructed by Wencel [33, 34]. The construction we use is due to Keren [19], see also [12]. Let $\mathcal{C}(\mathcal{R})$ be set of all $t \in \mathbb{R}$ such that $(-\infty, t)$ is definable in $\mathcal{R}$. Note that R is a subset of $\mathcal{C}(\mathcal{R})$ and $\mathcal{C}(\mathcal{R})$ is a subgroup of $(\mathbb{R}, +)$. The completion $\overline{\mathcal{R}}$ of $\mathcal{R}$ is the expansion of $(\mathcal{C}(\mathcal{R}), <, +)$ by all closures in $\mathcal{C}(\mathcal{R})^n$ of $\mathcal{R}$-definable subsets of R^n. Then $\overline{\mathcal{R}}$ is o-minimal and the structure induced on R by $\overline{\mathcal{R}}$ is interdefinable with $\mathcal{R}$. Proposition 11.13 is now clear.

Proposition 11.13. *If $\mathcal{R}$ is weakly o-minimal then $\mathcal{R}^\square$ is interdefinable with $\overline{\mathcal{R}^{\mathrm{Sh}}}$.*

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