COARSE DIMENSION AND DEFINABLE SETS IN EXPANSIONS OF THE
ORDERED REAL VECTOR SPACE

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ABSTRACT. Let $E \subseteq \mathbb{R}$. Suppose there is an $s > 0$ such that
$$|\{k \in \mathbb{Z}, -m \leq k \leq m - 1 : |k, k + 1] \cap E \neq \emptyset\}| \geq m^s$$
for all sufficiently large $m \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(E^n)$ is dense. It follows that if $E$ is in addition nowhere dense then $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

1. INTRODUCTION

Let $X \subseteq \mathbb{R}^n$ be bounded and $Z \subseteq \mathbb{R}^n$. Given a positive $\delta \in \mathbb{R}$ we let $M(\delta, X)$ be the minimum number of open $\delta$-balls required to cover $X$. Equivalently $M(\delta, X)$ is the minimal cardinality of a subset $S$ of $X$ such that every $x \in X$ lies within distance $\delta$ of some element of $S$. Let $B_n(p, r)$ be the open ball in $\mathbb{R}^n$ with center $p$ and radius $r > 0$ and let $B_n(r) = B_n(0, r)$. We define the coarse Minkowski dimension of $Z$ to be
$$\dim_{CM}(Z) := \limsup_{r \to \infty} \frac{M(1, B_n(r) \cap Z)}{\log(r)}.$$ 
It is easy to see that the coarse Minkowski dimension of $Z$ is bounded above by $n$ and the coarse Minkowski dimension of a bounded set is zero. An application of the first claim of Fact 2.1 below shows that replacing one with any fixed real number $\delta > 0$ does not change the coarse Minkowski dimension.

A simple computation shows that $\dim_{CM}(Z)$ is the infimum of the set of positive $s \in \mathbb{R}$ such that $M(1, B_n(r) \cap Z) < r^s$ for all sufficiently large $r > 0$.

We define
$$N(X) := \left| \left\{(k_1, \ldots, k_n) \in \mathbb{Z}^n : X \cap \prod_{i=1}^n [k_i, k_i + 1] \neq \emptyset \right\} \right|.$$ 
It is well-known and easy to see that there is a real number $K > 0$ depending only $n$ such that
$$K^{-1}M(1, X) \leq N(X) \leq KM(1, X).$$
So
$$\dim_{CM}(Z) = \limsup_{r \to \infty} \frac{N(B_n(r) \cap Z)}{\log(r)}.$$ 

Our main geometric result is Theorem 1.1.

**Theorem 1.1.** Suppose $E \subseteq \mathbb{R}$. If $\dim_{CM}(E) > 0$ then $T(E^n)$ is dense for some $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$. Equivalently, if there is a positive $s \in \mathbb{R}$ such that $N(B_n(r) \cap E) \geq r^s$ for all sufficiently large $r \in \mathbb{R}$ then there exist $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(E^n)$ is dense.

The converse implication to Theorem 1.1 does not hold. Let $D = \{2^n, 2^n + n : n \in \mathbb{N}\}$. A simple computation shows that $D$ has coarse Minkowski dimension zero. Let $S : \mathbb{R}^4 \to \mathbb{R}$ be given by $S(x_1, x_2, x_3, x_4) = (x_1 - x_2) + \alpha(x_3 - x_4)$ for a fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $S(D^4)$ is dense.

Theorem 1.1 is motivated by an application to logic that we now describe. Let $\mathbb{R}_{Vec}$ be the ordered vector space $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$ of real numbers. For any subset $E$ of $\mathbb{R}$ let $(\mathbb{R}_{Vec}, E)$ be the expansion of $\mathbb{R}_{Vec}$ by a unary predicate defining $E$. When we say that a subset of $\mathbb{R}^n$ is definable in a first order expansion of $(\mathbb{R}, <, +, 0)$ such as $(\mathbb{R}_{Vec}, E)$ we mean that it is first order definable possibly with parameters from $\mathbb{R}$.

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Hieronymi and Tychonievich [6] show that $(\mathbb{R}_{Vec}, Z)$ defines all bounded Borel subsets of all $\mathbb{R}^n$. In contrast, it follows from [8, 9] that every subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, <, +, 0, \mathbb{Z})$ is a finite union of locally closed sets.

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The theorem of Hieronymi and Tychonievich is a special case of Theorem 1.2. Theorem 1.2 also follows from a more general theorem of Fornasiero, Hieronymi, and Walsberg [2, Theorem 7.3, Corollary 7.5]. We let $\text{Cl}(E)$ be the closure of $E \subseteq \mathbb{R}$ and $\text{Bd}(E)$ be the boundary of $E$. Recall that the boundary of a subset of $\mathbb{R}$ is always closed.

**Theorem 1.2.** Suppose that $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. Then the following are equivalent:

1. $(\mathbb{R}_{\text{vec}}, E)$ does not define every bounded Borel subset of every $\mathbb{R}^n$,
2. Every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\text{vec}}, E)$ either has interior or is nowhere dense,
3. $T(\text{Bd}(E)^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$.

The implication (3) $\Rightarrow$ (2) is a corollary of a result of Friedman and Miller [3]. The implication (1) $\Rightarrow$ (3) is a corollary of the main theorem of [6]. Note that $\text{Bd}(E)$ is nowhere dense as $E$ is not dense and co-dense in any open interval. If $E$ is bounded then (3) above is equivalent to a natural geometric condition on $E$. This equivalence, observed in [2, Theorem 7.3], is an easy consequence of the famous Marstrand projection theorem (see Mattila [7, Chapter 9]) and the classical theorem of Steinhaus that $Z - Z := \{ z - z' : z, z' \in Z \}$ has interior whenever $Z \subseteq \mathbb{R}^n$ has positive $n$-dimensional Lebesgue measure.

**Fact 1.3.** Suppose $F \subseteq \mathbb{R}$ is bounded. Then $T(F^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$ if and only if $\text{Cl}(F^n)$ has Hausdorff dimension zero for all $n \in \mathbb{N}$.

Fact 1.3 does not hold for unbounded subsets of $\mathbb{R}$. The set of integers, like any countable set, has Hausdorff dimension zero, and $T(\mathbb{Z}^2)$ is dense for any linear $T : \mathbb{R}^2 \to \mathbb{R}$ of the form $T(x, y) = x + ay$ with $a \in \mathbb{R} \setminus \mathbb{Q}$. Combining Theorem 1.1 and Theorem 1.2 we obtain the following.

**Theorem 1.4.** Suppose $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. If $\text{Bd}(E)$ has positive coarse Minkowski dimension then $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In particular if $E$ is nowhere dense and has positive coarse Minkowski dimension then $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

Note that $Z$ has coarse Minkowski dimension one so Theorem 1.4 generalizes the result of Hieronymi and Tychonievich described above. There are subsets $E$ of $\mathbb{R}$ with coarse Minkowski dimension zero such that $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$ such as $\{ 2^n, 2^n + n : n \in \mathbb{N} \}$ (see the comment after Theorem 1.1). Theorem 1.4 fails without the assumption that $E$ is not dense and co-dense in any nonempty open interval. Block-Gorman, Hieronymi, and Kaplan [4] show that every closed subset of $\mathbb{R}^n$ definable in $(\mathbb{R}_{\text{vec}}, \mathbb{Q})$ is already definable in $\mathbb{R}_{\text{vec}}$ and $\text{Bd}(\mathbb{Q}) = \mathbb{R}$ has coarse Minkowski dimension one.

The present paper is part of the broader study of the metric geometry of definable sets in first order structures expanding $(\mathbb{R}, <, +, 0)$, see [1, 2, 5]. Fornasiero, Hieronymi, and Miller [1] show that if $E \subseteq \mathbb{R}$ is nowhere dense and has positive Minkowski dimension then $(\mathbb{R}, <, +, 0, 1, E)$ defines every Borel subset of every $\mathbb{R}^n$. This statement fails over $\mathbb{R}_{\text{vec}}$, as $D = \{ \frac{1}{n} : n \in \mathbb{N}, n \geq 1 \}$ has Minkowski dimension one and Fact 1.3 and Theorem 1.2 together imply that every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\text{vec}}, D)$ either has interior or is nowhere dense. It is shown in [2] that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of $E$ is strictly less than the Hausdorff dimension of $E$ then $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

As a closed subset of $\mathbb{R}$ has topological dimension zero if it is nowhere dense and topological dimension one if it has interior, Theorem 1.4 shows that if $E \subseteq \mathbb{R}$ is closed and the topological dimension of $E$ is strictly less than the coarse Minkowski dimension of $E$ then $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. It is natural to conjecture that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of $E$ is strictly less than the coarse Minkowski dimension of $E$ then $(\mathbb{R}_{\text{vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In Theorem 4.1 we will show as a corollary to Theorem 1.4 that if $Z \subseteq \mathbb{R}^n$ is closed and has topological dimension zero and positive coarse Minkowski dimension then $(\mathbb{R}_{\text{vec}}, Z)$ defines every bounded Borel subset of $\mathbb{R}^n$.

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2. Metric Notions

We recall two useful facts about $\text{M}(\delta, X)$ and $\text{N}(X)$, both of which are easy to see. One can find more information about these invariants in Yomdin and Comte [10, Chapter 2] and many other places.
Fact 2.1. Let $n \in \mathbb{N}$. There are $K, L > 0$ such that for all bounded $X, Y \subseteq \mathbb{R}^n$ and $0 < \delta < \delta'$

$$M(\delta', X) \leq M(\delta, X) \leq K \left( \frac{\delta'}{\delta} \right)^n M(\delta', X)$$

and

$$L^{-1} M(\delta, X) M(\delta, Y) \leq M(\delta, X \times Y) \leq LM(\delta, X) M(\delta, Y)$$

In particular

$$L^{-1} M(\delta, X)^2 \leq M(\delta, X^2) \leq LM(\delta, X)^2$$

for all bounded $X \subseteq \mathbb{R}^n$.

The proof of the fact below is a straightforward computation that is essentially the same as the proof of the analogous fact for Minkowski dimension. We leave the proof to the reader.

Fact 2.2. For any $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ and $k \in \mathbb{N}$ we have

$$\dim_{CM}(X \times Y) \leq \dim_{CM}(X) + \dim_{CM}(Y)$$

and

$$\dim_{CM}(X^k) = k \dim_{CM}(X).$$

Suppose that $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$, $f$ is a map $X \to Y$, and $\lambda, \delta > 0$. Then $f$ is a $(\lambda, \delta)$-quasi-isometry if

$$\frac{1}{\lambda} \|x - x'\| - \delta \leq \|f(x) - f(x')\| \leq \lambda \|x - x'\| + \delta$$

for all $x, x' \in X$, and if for every $y \in Y$ we have $\|f(x) - y\| < \delta$ for some $x \in X$. We say that $f$ is a quasi-isometry if it is a $(\lambda, \delta)$-quasi-isometry for some $\lambda, \delta > 0$. It is well-known and easy to see that if there is a quasi-isometry $X \to Y$ then there is also a quasi-isometry $Y \to X$. A map $g : X \to \mathbb{R}^n$ is a quasi-isometric embedding if it yields a quasi-isometry $X \to g(X)$.

Lemma 2.3. Suppose $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m, 0 \in X, 0 \in Y$, and $f : X \to Y$ is a quasi-isometry such that $f(0) = 0$. Then $X$ and $Y$ have the same coarse Minkowski dimension.

Lemma 2.3 holds without the assumptions that $0 \in X, 0 \in Y$, and $f(0) = 0$. We do not prove this more general result to avoid technicalities.

Proof. We show that $\dim_{CM}(Y) \leq \dim_{CM}(X)$. As there is a quasi-isometry $Y \to X$ that also maps 0 to 0 the same argument yields the other inequality. Fix $\lambda, \delta > 0$ such that $f$ is a $(\lambda, \delta)$-quasi-isometry.

Fix $r > 0$. Let $X(r) = B_n(0, r) \cap X$ and $Y(r) = B_m(0, r) \cap Y$. Let $\{B_n(p_i, 1)\}_{i=1}^k$ be a minimal covering of $X(r)$ by balls with radius 1. Then $\{f(B_n(p_i, 1))\}_{i=1}^k$ covers $f(X(r))$. Let $q_i = f(p_i)$ for all $i$. As $f$ is a $(\lambda, \delta)$-quasi-isometry we see that $f(B_n(p_i, 1))$ is contained in $B_m(q_i, \lambda + \delta)$ for all $i$. So $\{B_m(q_i, \lambda + \delta)\}_{i=1}^k$ covers $f(X(r))$.

We now show that every point in $Y(r\lambda^{-1} - 2\delta)$ lies within distance $\delta$ of $f(X(r))$. Fix $y \in Y(r\lambda^{-1} - 2\delta)$. As $f$ is a $(\lambda, \delta)$-quasi-isometry there is $x \in X$ such that $\|f(x) - y\| < \delta$. Suppose $\|x\| > r$. Then as $f(0) = 0$ we have

$$\|f(x)\| \geq \frac{1}{\lambda} \|x\| - \delta > r\lambda^{-1} - \delta.$$$$

As $\|f(x) - y\| < \delta$ the triangle inequality yields $\|y\| > r\lambda^{-1} - 2\delta$. Contradiction.

Combining the previous paragraphs we see that $\{B_m(q_i, \lambda + 2\delta)\}_{i=1}^k$ covers $Y(r\lambda^{-1} - 2\delta)$. Thus

$$M(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta)) \leq M(1, X(r))$$

for all $r > 0$.

Applying the first claim of Fact 2.1 we obtain a constant $L > 0$ depending only on $m$ such that

$$LM(1, Y(r\lambda^{-1} - 2\delta)) \leq M(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta))$$

hence

$$LM(1, Y(r\lambda^{-1} - 2\delta)) \leq M(1, X(r)).$$

Taking logarithms of both sides of the expression above, dividing both sides by $\log(r)$, and taking the limit as $r \to \infty$ we see that $\dim_{CM}(Y) \leq \dim_{CM}(X)$. \qed
3. Proof of Theorem 1.1

Let $S$ be the unit circle in $\mathbb{R}^2$. Given $u \in S$ we let $T_u : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection parallel to $u$, i.e., $T_u$ is the orthogonal projection such that $T_u(x) = T_u(y)$ if and only if $x - y = tu$ for some $t \in \mathbb{R}$. For our purposes a double wedge around $u \in S$ is a subset of $\mathbb{R}^2$ of the form

$$C_{s, \varepsilon}^u := \{tv : t \in \mathbb{R}, |t| > s, v \in S, \|v - u\| < \varepsilon\}$$

for some $s, \varepsilon > 0$.

**Lemma 3.1.** Let $F$ be a nonempty subset of $\mathbb{R}^2$ and $u \in S$. If $F \setminus F = \{x - y : x, y \in F\}$ is disjoint from some double wedge around $u$ then the restriction of $T_u$ to $F$ is a quasi-isometric embedding $F \to \mathbb{R}$.

Lemma 3.1 is a quasi-isometric version of a well-known fact from geometric measure theory: if $F$ is a nonempty subset of $\mathbb{R}^2$ such that $F \setminus F$ is disjoint from a double wedge of the form $C_{s, \varepsilon}^u$ then the restriction of $T_u$ to $F$ is a bilipschitz embedding $F \to \mathbb{R}$. This fact is applied in [1, 5].

**Proof.** Suppose that $F \setminus F$ is disjoint from $C_{s, \varepsilon}^u$. As $T_u$ is an orthogonal projection we have $\|T_u(x) - T_u(x')\| \leq \|x - x'\|$ for all $x, x' \in \mathbb{R}^2$, so it suffices to obtain a lower bound on $\|T_u(x) - T_u(x')\|$ of the appropriate form.

After making a change of coordinates if necessary we suppose $u = (0, 1)$ so that $T_u(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Then we have

$$C_{s, \varepsilon}^u = \{(x, y) \in \mathbb{R}^2 : |y| > \lambda|x| \quad \text{and} \quad \|(x, y)\| > s\}$$

for some $\lambda > 0$ depending only on $\varepsilon$. Thus, if $(x, y) \in F \setminus F$ then either $\|(x, y)\| < s$ or $|y| \leq \lambda|x|$. Equivalently, for all $(x, y), (x', y') \in F$ we either have

$$\|(x, y) - (x', y')\| < s \quad \text{or} \quad |y - y'| \leq \lambda|x - x'|.$$  

In the latter case we have

$$\|(x, y) - (x', y')\| \leq |x - x'| + |y - y'| \leq (1 + \lambda)|x - x'|$$

hence

$$\frac{1}{1 + \lambda} \|(x, y) - (x', y')\| \leq |x - x'|.$$  

In the first case we have

$$\|(x, y) - (x', y')\| - s < |x - x'|.$$  

So for all $(x, y), (x', y') \in F$ we have

$$\frac{1}{1 + \lambda} \|(x, x') - (y, y')\| - s < |x - x'|.$$  

So the restriction of $T_u$ to $F$ is a quasi-isometric embedding $F \to \mathbb{R}^2$. $\square$

We let $\mathbb{H}$ be the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and let $S^+ = S \cap \mathbb{H}$. A wedge in $\mathbb{H}$ around $u \in S^+$ is a set of the form

$$C_{s, \varepsilon}^{u,+} := \{tv : t \in \mathbb{R}, t > s, v \in S, \|v - u\| < \varepsilon\}$$

such that $C_{s, \varepsilon}^{u,+} \subseteq \mathbb{H}$.

**Lemma 3.2.** Suppose $F \subseteq \mathbb{H}$ intersects every wedge in $\mathbb{H}$. Then there is a $u \in S^+$ such that $T_u(F)$ is dense.

The reader may find that drawing a few pictures greatly assists in comprehending the proof of Lemma 3.2. We let $p = (-1, 0)$ and $o = (0, 0)$. Note that if $z \in \mathbb{H}$, $q$ is a positive real number, and $u \in S^+$, then $T_u(z) = q$ if and only if $\angle pou = \angle pqz$.

**Proof.** We show that the set of $u \in S^+$ such that $T_u(F)$ is dense in $\mathbb{R}$ is comeager in $S^+$. It suffices to show that

$$\{u \in S^+ : T_u(F) \cap I \neq \emptyset\}$$

is open and dense in $S^+$ for every nonempty open interval $I$ with rational endpoints. Fix a nonempty open interval $I = (q_1, q_2)$ with rational endpoints. We suppose that $q_1, q_2 > 0$ for the sake of simplicity, the more general case follows by trivial modifications of our argument. The map $T : S^+ \times \mathbb{R}^2 \to \mathbb{R}$ given by $T(u, x) = T_u(x)$ is continuous. Thus if $T_u(x) \in I$ then $T_u(x) \in I$ for all $v \in S^+$ sufficiently close to $u$. It follows that the set of $u$ such that $T_u(F) \cap I \neq \emptyset$ is open in $S^+$.

It now suffices to show that the set of $w \in S^+$ such that $T_w(F) \cap I \neq \emptyset$ is dense in $S^+$. Fix $u, v \in S^+$ such that $\angle pou < \angle pov$ and let $J$ be the set of $w \in S^+$ such that $\angle pou < \angle pov < \angle pov$. We show there
is a \( w \in J \) such that \( T_w(F) \cap I \neq \emptyset \). Let \( r_1, r_2 \in \mathbb{H} \) be such that \( \angle pq_1 r_1 = \angle pou \) and \( \angle pq_2 r_2 = \angle pov \).

Let \( D \) be the set of points in \( \mathbb{H} \) that lie in between the rays \( q_1 r_1 \) and \( q_2 r_2 \). It is easy to see that

\[
D = \bigcup_{q \in I} \{ r \in \mathbb{H} : \angle pou < \angle pqr < \angle pov \} = \bigcup_{q \in I} \bigcup_{w \in J} T_w^{-1}(\{ q \}) = \bigcup_{w \in J} T_w^{-1}(I).
\]

It therefore suffices to show that \( D \) intersects \( F \). Let \( z_1, z_2 \in S^+ \) be such that

\[
\angle pou < \angle poz_1 < \angle poz_2 < \angle pov.
\]

As \( \angle pq_1 r_1 < \angle poz_1 < \angle poz_2 < \angle pq_2 r_2 \), we see that every element of \( \overline{o z_1} \) or \( \overline{o z_2} \) sufficiently far from the origin lies in \( D \). It follows that there is a \( t > 0 \) such that

\[
W := \{ z \in \mathbb{H} : \| z \| \geq t, \angle poz_1 < \angle poz < \angle poz_2 \} \subseteq D.
\]

Then \( W \) is a wedge in \( \mathbb{H} \) and so contains an element of \( F \). Thus \( D \) contains an element of \( F \). \( \square \)

**Lemma 3.3.** Suppose \( E \subseteq \mathbb{R} \). Then one of the following holds:

1. There is a \( u \in \mathbb{S} \) such that the restriction of \( T_u \) to \( E^2 \) is a quasi-isometric embedding \( E^2 \rightarrow \mathbb{R} \).
2. There is a linear \( S : \mathbb{R}^4 \rightarrow \mathbb{R} \) such that \( S(E^4) \) is dense.

**Proof.** Consider \( E^2 - E^2 \subseteq \mathbb{R}^2 \). If \( E^2 - E^2 \) is disjoint from a double wedge in \( \mathbb{R}^2 \) then Lemma 3.2 shows that some \( T_u \) quasi-isometrically embeds \( E^2 \) into \( \mathbb{R} \).

Suppose \( E^2 - E^2 \) intersects every double wedge in \( \mathbb{R}^2 \). Note that if \( (x, y) \in E^2 - E^2 \) then \( (-x, -y) \) is also an element of \( E^2 - E^2 \). It is easy to see that this implies that \( E^2 - E^2 \) intersects every wedge in \( \mathbb{H} \). Applying Lemma 3.3 we fix a \( u \in \mathbb{S} \) such that \( T_u(E^2 - E^2) \) is dense. Let \( S : \mathbb{R}^4 \rightarrow \mathbb{R} \) be the linear function given by

\[
S(x, y, x', y') = T_u(x - x', y - y') \quad \text{for all} \quad x, y, x', y' \in \mathbb{R}.
\]

Then \( S(E^4) \) is dense. \( \square \)

We now prove Theorem 1.1.

**Proof.** Suppose towards a contradiction that \( E \subseteq \mathbb{R} \) has positive coarse Minkowski dimension and \( T(E^n) \) is not dense for every \( n \in \mathbb{N} \) and linear \( T : \mathbb{R}^n \rightarrow \mathbb{R} \). We may suppose that \( 0 \in E \). Let \( S \) be the collection of sets of the form \( T(E^n) \) for linear \( T : \mathbb{R}^n \rightarrow \mathbb{R} \). It is easy to see that if \( F \in \mathbb{S} \) and \( T : \mathbb{R}^n \rightarrow \mathbb{R} \) is linear then \( T(F^n) \) is also in \( \mathbb{S} \). Let \( s \) be the supremum of the coarse Minkowski dimensions of members of \( \mathbb{S} \). Every element of \( \mathbb{S} \) has coarse Minkowski dimension \( \leq 1 \), so \( s \) exists and \( s \leq 1 \). As \( \dim_{CM}(E) > 0 \) we have \( s > 0 \). Let \( F \in \mathbb{S} \) be such that \( \dim_{CM}(F) > \frac{1}{8} s \). An application of Lemma 3.3 yields a linear \( T : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that the restriction of \( T \) to \( F^2 \) is a quasi-isometric embedding \( F^2 \rightarrow \mathbb{R} \). Lemma 2.3 and Fact 2.2 together show that

\[
\dim_{CM} T(F^2) = \dim_{CM}(F^2) = 2 \dim_{CM}(F) > s.
\]

But \( T(F^2) \in \mathbb{S} \), contradiction. \( \square \)

**4. A COROLLARY IN \( \mathbb{R}^n \)**

We prove a higher dimensional version of the second claim of Theorem 1.4. (Recall that a closed subset of \( \mathbb{R}^n \) has topological dimension zero if and only if it is nowhere dense.)

**Theorem 4.1.** Suppose \( Z \) is a closed subset of \( \mathbb{R}^n \) with topological dimension zero. If \( Z \) has positive coarse Minkowski dimension then \( (\mathbb{R}_{vec}, Z) \) defines all bounded Borel subsets of all \( \mathbb{R}^n \).

**Proof.** We suppose that \( (\mathbb{R}_{vec}, Z) \) does not define all bounded Borel subsets of all \( \mathbb{R}^n \) and show that \( \dim_{CM}(Z) = 0 \). Given \( 1 \leq k \leq n \) we let \( \pi_k : \mathbb{R}^n \rightarrow \mathbb{R} \) be given by

\[
\pi_k(x_1, \ldots, x_n) = x_k \quad \text{for all} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

An application of [2, Theorem D, Theorem E] shows that \( \pi_k(Z) \) is nowhere dense for all \( 1 \leq k \leq n \). Theorem 1.4 shows that \( \dim_{CM} \pi_k(Z) = 0 \) for all \( 1 \leq k \leq n \). As \( Z \) is a subset of \( \pi_1(Z) \times \ldots \times \pi_n(Z) \) repeated application of Fact 2.2 shows that \( \dim_{CM}(Z) = 0 \). \( \square \)

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