COARSE DIMENSION AND DEFINABLE SETS IN EXPANSIONS OF THE ORDERED REAL VECTOR SPACE

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Abstract. Let $E \subseteq \mathbb{R}$. Suppose there is an $s > 0$ such that
$$|\{k \in \mathbb{Z}, -m \leq k \leq m-1 : [k, k+1] \cap E \neq \emptyset\}| \geq m^s$$
for all sufficiently large $m \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(E^n)$ is dense. It follows that if E is in addition nowhere dense then $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

1. Introduction

Let $X \subseteq \mathbb{R}^n$ be bounded and $Z \subseteq \mathbb{R}^n$. Given a positive $\delta \in \mathbb{R}$ we let $\mathcal{M}(\delta, X)$ be the minimum number of open δ-balls required to cover X. Equivalently $\mathcal{M}(\delta, X)$ is the minimal cardinality of a subset S of X such that every $x \in X$ lies within distance δ of some element of S. Let $B_n(p, r)$ be the open ball in $\mathbb{R}^n$ with center p and radius $r > 0$ and let $B_n(r) = B_n(0, r)$. We define the **coarse Minkowski dimension** of Z to be
$$\dim_{\mathrm{CM}}(Z) := \limsup_{r \to \infty} \frac{\mathcal{M}(1, B_n(r) \cap Z)}{\log(r)}.$$
It is easy to see that the coarse Minkowski dimension of Z is bounded above by n and the coarse Minkowski dimension of a bounded set is zero. An application of the first claim of Fact 2.1 below shows that replacing one with any fixed real number $\delta > 0$ does not change the coarse Minkowski dimension. A simple computation shows that $\dim_{\mathrm{CM}}(Z)$ is the infimum of the set of positive $s \in \mathbb{R}$ such that $\mathcal{M}(1, B_n(r) \cap Z) < r^s$ for all sufficiently large $r > 0$.

We define
$$\mathcal{N}(X) := \left| \left\{ (k_1, \ldots, k_n) \in \mathbb{Z}^n : X \cap \prod_{i=1}^{n} [k_i, k_i + 1] \neq \emptyset \right\} \right|.$$
It is well-known and easy to see that there is a real number $K > 0$ depending only n such that
$$K^{-1}\mathcal{M}(1, X) \leq \mathcal{N}(X) \leq K\mathcal{M}(1, X).$$
So
$$\dim_{\mathrm{CM}}(Z) = \limsup_{r \to \infty} \frac{\mathcal{N}(B_n(r) \cap Z)}{\log(r)}.$$
Our main geometric result is Theorem 1.1.

Theorem 1.1. *Suppose $E \subseteq \mathbb{R}$. If $\dim_{\mathrm{CM}}(E) > 0$ then $T(E^n)$ is dense for some $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$. Equivalently, if there is a positive $s \in \mathbb{R}$ such that $\mathcal{N}(B_1(r) \cap E) \geq r^s$ for all sufficiently large $r \in \mathbb{R}$ then there exist $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(E^n)$ is dense.*

The converse implication to Theorem 1.1 does not hold. Let $D = \{2^n, 2^n + n : n \in \mathbb{N}\}$. A simple computation shows that D has coarse Minkowski dimension zero. Let $S : \mathbb{R}^4 \to \mathbb{R}$ be given by $S(x_1, x_2, x_3, x_4) = (x_1 - x_2) + \alpha(x_3 - x_4)$ for a fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $S(D^4)$ is dense.

Theorem 1.1 is motivated by an application to logic that we now describe. Let $\mathbb{R}_{\mathrm{Vec}}$ be the ordered vector space $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$ of real numbers. For any subset E of $\mathbb{R}$ let $(\mathbb{R}_{\mathrm{Vec}}, E)$ be the expansion of $\mathbb{R}_{\mathrm{Vec}}$ by a unary predicate defining E. When we say that a subset of $\mathbb{R}^n$ is definable in a first order expansion of $(\mathbb{R}, <, +, 0)$ such as $(\mathbb{R}_{\mathrm{Vec}}, E)$ we mean that it is first order definable possibly with parameters from $\mathbb{R}$.

Hieronymi and Tychonievich [6] show that $(\mathbb{R}_{\mathrm{Vec}}, \mathbb{Z})$ defines all bounded Borel subsets of all $\mathbb{R}^n$. In contrast, it follows from [8, 9] that every subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, <, +, 0, \mathbb{Z})$ is a finite union of locally closed sets.

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The theorem of Hieronymi and Tychonievich is a special case of Theorem 1.2. Theorem 1.2 also follows from a more general theorem of Fornasiero, Hieronymi, and Walsberg [2, Theorem 7.3, Corollary 7.5]. We let $\mathrm{Cl}(E)$ be the closure of $E \subseteq \mathbb{R}$ and $\mathrm{Bd}(E)$ be the boundary of E. Recall that the boundary of a subset of $\mathbb{R}$ is always closed.

Theorem 1.2. *Suppose that $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. Then the following are equivalent:*

(1) $(\mathbb{R}_{\mathrm{Vec}}, E)$ does not define every bounded Borel subset of every $\mathbb{R}^n$,
(2) Every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\mathrm{Vec}}, E)$ either has interior or is nowhere dense,
(3) $T(\mathrm{Bd}(E)^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$.

The implication (3) ⇒ (2) is a corollary of a result of Friedman and Miller [3]. The implication (1) ⇒ (3) is a corollary of the main theorem of [6]. Note that $\mathrm{Bd}(E)$ is nowhere dense as E is not dense and co-dense in any open interval. If E is bounded then (3) above is equivalent to a natural geometric condition on E. This equivalence, observed in [2, Theorem 7.3], is an easy consequence of the famous Marstrand projection theorem (see Mattila [7, Chapter 9]) and the classical theorem of Steinhaus that $Z - Z := \{z - z' : z, z' \in Z\}$ has interior whenever $Z \subseteq \mathbb{R}^n$ has positive n-dimensional Lebesgue measure.

Fact 1.3. *Suppose $F \subseteq \mathbb{R}$ is bounded. Then $T(F^n)$ is nowhere dense for every linear $T : \mathbb{R}^n \to \mathbb{R}$ if and only if $\mathrm{Cl}(F)^n$ has Hausdorff dimension zero for all $n \in \mathbb{N}$.*

Fact 1.3 does not hold for unbounded subsets of $\mathbb{R}$. The set of integers, like any countable set, has Hausdorff dimension zero, and $T(\mathbb{Z}^2)$ is dense for any linear $T : \mathbb{R}^2 \to \mathbb{R}$ of the form $T(x, y) = x + \alpha y$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Combining Theorem 1.1 and Theorem 1.2 we obtain the following.

Theorem 1.4. *Suppose $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. If $\mathrm{Bd}(E)$ has positive coarse Minkowski dimension then $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In particular if E is nowhere dense and has positive coarse Minkowski dimension then $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.*

Note that $\mathbb{Z}$ has coarse Minkowski dimension one so Theorem 1.4 generalizes the result of Hieronymi and Tychonievich described above. There are subsets E of $\mathbb{R}$ with coarse Minkowski dimension zero such that $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$ such as $\{2^n, 2^n + n : n \in \mathbb{N}\}$ (see the comment after Theorem 1.1). Theorem 1.4 fails without the assumption that E is not dense and co-dense in any nonempty open interval. Block-Gorman, Hieronymi, and Kaplan [4] show that every closed subset of $\mathbb{R}^n$ definable in $(\mathbb{R}_{\mathrm{Vec}}, \mathbb{Q})$ is already definable in $\mathbb{R}_{\mathrm{Vec}}$ and $\mathrm{Bd}(\mathbb{Q}) = \mathbb{R}$ has coarse Minkowski dimension one.

The present paper is part of the broader study of the metric geometry of definable sets in first order structures expanding $(\mathbb{R}, <, +, 0)$, see [1, 2, 5]. Fornsiero, Hieronymi, and Miller [1] show that if $E \subseteq \mathbb{R}$ is nowhere dense and has positive Minkowski dimension then $(\mathbb{R}, <, +, \cdot, 0, 1, E)$ defines every Borel subset of every $\mathbb{R}^n$. This statement fails over $\mathbb{R}_{\mathrm{Vec}}$, as $D = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$ has Minkowski dimension one and Fact 1.3 and Theorem 1.2 together imply that every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\mathrm{Vec}}, D)$ either has interior or is nowhere dense. It is shown in [2] that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of E is strictly less than the Hausdorff dimension of E then $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

As a closed subset of $\mathbb{R}$ has topological dimension zero if it is nowhere dense and topological dimension one if it has interior, Theorem 1.4 shows that if $E \subseteq \mathbb{R}$ is closed and the topological dimension of E is strictly less than the coarse Minkowski dimension of E then $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. It is natural to conjecture that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of E is strictly less then the coarse Minkowski dimension of E then $(\mathbb{R}_{\mathrm{Vec}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In Theorem 4.1 we will show as a corollary to Theorem 1.4 that if $Z \subseteq \mathbb{R}^n$ is closed and has topological dimension zero and positive coarse Minkowksi dimension then $(\mathbb{R}_{\mathrm{Vec}}, Z)$ defines every bounded Borel subset of $\mathbb{R}^n$.

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2. Metric Notions

We recall two useful facts about $\mathcal{M}(\delta, X)$ and $\mathcal{N}(X)$, both of which are easy to see. One can find more information about these invariants in Yomdin and Comte [10, Chapter 2] and many other places.

Fact 2.1. *Let $n \in \mathbb{N}$. There are $K, L > 0$ such that for all bounded $X, Y \subseteq \mathbb{R}^n$ and $0 < \delta < \delta'$*

$$\mathcal{M}(\delta', X) \leq \mathcal{M}(\delta, X) \leq K\left(\frac{\delta'}{\delta}\right)^n \mathcal{M}(\delta', X)$$

and

$$L^{-1}\mathcal{M}(\delta, X)\mathcal{M}(\delta, Y) \leq \mathcal{M}(\delta, X \times Y) \leq L\mathcal{M}(\delta, X)\mathcal{M}(\delta, Y)$$

In particular

$$L^{-1}\mathcal{M}(\delta, X)^2 \leq \mathcal{M}(\delta, X^2) \leq L\mathcal{M}(\delta, X)^2$$

for all bounded $X \subseteq \mathbb{R}^n$.

The proof of the fact below is a straightforward computation that is essentially the same as the proof of the analogous fact for Minkowski dimension. We leave the proof to the reader.

Fact 2.2. *For any $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ and $k \in \mathbb{N}$ we have*

$$\dim_{\text{CM}}(X \times Y) \leq \dim_{\text{CM}}(X) + \dim_{\text{CM}}(Y)$$

and

$$\dim_{\text{CM}}(X^k) = k \dim_{\text{CM}}(X).$$

Suppose that $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, f is a map $X \to Y$, and $\lambda, \delta > 0$. Then f is a (λ, δ)-quasi-isometry if

$$\frac{1}{\lambda}\|x - x'\| - \delta \leq \|f(x) - f(x')\| \leq \lambda\|x - x'\| + \delta \quad \text{for all} \quad x, x' \in X,$$

and if for every $y \in Y$ we have $\|f(x) - y\| < \delta$ for some $x \in X$. We say that f is a quasi-isometry if it is a (λ, δ)-quasi-isometry for some $\lambda, \delta > 0$. It is well-known and easy to see that if there is a quasi-isometry $X \to Y$ then there is also a quasi-isometry $Y \to X$. A map $g : X \to \mathbb{R}^n$ is a quasi-isometric embedding if it yields a quasi-isometry $X \to g(X)$.

Lemma 2.3. *Suppose $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m, 0 \in X, 0 \in Y$, and $f : X \to Y$ is a quasi-isometry such that $f(0) = 0$. Then X and Y have the same coarse Minkowski dimension.*

Lemma 2.3 holds without the assumptions that $0 \in X, 0 \in Y$, and $f(0) = 0$. We do not prove this more general result to avoid technicalities.

Proof. We show that $\dim_{\text{CM}}(Y) \leq \dim_{\text{CM}}(X)$. As there is a quasi-isometry $Y \to X$ that also maps 0 to 0 the same argument yields the other inequality. Fix $\lambda, \delta > 0$ such that f is a (λ, δ)-quasi-isometry.

Fix $r > 0$. Let $X(r) = B_n(0, r) \cap X$ and $Y(r) = B_m(0, r) \cap Y$. Let $\{B_n(p_i, 1)\}_{i=1}^k$ be a minimal covering of $X(r)$ by balls with radius 1. Then $\{f(B_n(p_i, 1))\}_{i=1}^k$ covers $f(X(r))$. Let $q_i = f(p_i)$ for all i. As f is a (λ, δ)-quasi-isometry we see that $f(B_n(p_i, 1))$ is contained in $B_m(q_i, \lambda + \delta)$ for all i. So $\{B_m(q_i, \lambda + \delta)\}_{i=1}^k$ covers $f(X(r))$.

We now show that every point in $Y(r\lambda^{-1} - 2\delta)$ lies within distance δ of $f(X(r))$. Fix $y \in Y(r\lambda^{-1} - 2\delta)$. As f is a (λ, δ)-quasi-isometry there is $x \in X$ such that $\|f(x) - y\| < \delta$. Suppose $\|x\| > r$. Then as $f(0) = 0$ we have

$$\|f(x)\| \geq \frac{1}{\lambda}\|x\| - \delta > r\lambda^{-1} - \delta.$$

As $\|f(x) - y\| < \delta$ the triangle inequality yields $\|y\| > r\lambda^{-1} - 2\delta$. Contradiction.

Combining the previous paragraphs we see that $\{B_m(q_i, \lambda + 2\delta)\}_{i=1}^k$ covers $Y(r\lambda^{-1} - 2\delta)$. Thus

$$\mathcal{M}(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta)) \leq \mathcal{M}(1, X(r)) \quad \text{for all} \quad r > 0.$$

Applying the first claim of Fact 2.1 we obtain a constant $L > 0$ depending only on m such that

$$L\mathcal{M}(1, Y(r\lambda^{-1} - 2\delta)) \leq \mathcal{M}(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta))$$

hence

$$L\mathcal{M}(1, Y(r\lambda^{-1} - 2\delta)) \leq \mathcal{M}(1, X(r)).$$

Taking logarithms of of both sides of the expression above, dividing both sides by $\log(r)$, and taking the limit as $r \to \infty$ we see that $\dim_{\text{CM}}(Y) \leq \dim_{\text{CM}}(X)$. □

3. Proof of Theorem 1.1

Let $\mathbb{S}$ be the unit circle in $\mathbb{R}^2$. Given $u \in \mathbb{S}$ we let $T_u : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection parallel to u, i.e., T_u is the orthogonal projection such that $T_u(x) = T_u(y)$ if and only if $x - y = tu$ for some $t \in \mathbb{R}$. For our purposes a **double wedge** around $u \in \mathbb{S}$ is a subset of $\mathbb{R}^2$ of the form

$$C^u_{s,\varepsilon} := \{tv : t \in \mathbb{R}, |t| > s, v \in \mathbb{S}, \|v - u\| < \varepsilon\}$$

for some $s, \varepsilon > 0$.

Lemma 3.1. *Let F be a nonempty subset of $\mathbb{R}^2$ and $u \in \mathbb{S}$. If $F - F = \{x - y : x, y \in F\}$ is disjoint from some double wedge around u then the restriction of T_u to F is a quasi-isometric embedding $F \to \mathbb{R}$.*

Lemma 3.1 is a quasi-isometric version of a well-known fact from geometric measure theory: if F is a nonempty subset of $\mathbb{R}^2$ such that $F - F$ is disjoint from a double wedge of the form $C^u_{\varepsilon,0}$ then the restriction of T_u to F is a bilipschitz embedding $F \to \mathbb{R}$. This fact is applied in [1, 5].

Proof. Suppose that $F - F$ is disjoint from $C^u_{s,\varepsilon}$. As T_u is an othogonal projection we have $\|T_u(x) - T_u(x')\| \le \|x - x'\|$ for all $x, x' \in \mathbb{R}^2$, so it suffices to obtain a lower bound on $\|T_u(X) - T_u(x')\|$ of the appropriate form.

After making a change of coordinates if necessary we suppose $u = (0, 1)$ so that $T_u(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Then we have

$$C^u_{s,\varepsilon} = \{(x, y) \in \mathbb{R}^2 : |y| > \lambda|x| \quad \text{and} \quad \|(x, y)\| > s\}$$

for some $\lambda > 0$ depending only on ε. Thus, if $(x, y) \in F - F$ then either $\|(x, y)\| < s$ or $|y| \le \lambda|x|$. Equivalently, for all $(x, y), (x', y') \in F$ we either have

$$\|(x, y) - (x', y')\| < s \quad \text{or} \quad |y - y'| \le \lambda|x - x'|.$$

In the latter case we have

$$\|(x, y) - (x', y')\| \le |x - x'| + |y - y'| \le (1 + \lambda)|x - x'|$$

hence

$$\frac{1}{1 + \lambda}\|(x, y) - (x', y')\| \le |x - x'|.$$

In the first case we have

$$\|(x, y) - (x', y')\| - s < |x - x'|.$$

So for all $(x, y), (x', x') \in F$ we have

$$\frac{1}{1 + \lambda}\|(x, x') - (y, y')\| - s \le |x - x'|.$$

So the restriction of T_u to F is a quasi-isometric embedding $F \to \mathbb{R}^2$. $\square$

We let $\mathbb{H}$ be the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and let $\mathbb{S}^+ = \mathbb{S} \cap \mathbb{H}$. A wedge in $\mathbb{H}$ around $u \in \mathbb{S}^+$ is a set of the form

$$C^{u,+}_{s,\varepsilon} := \{tv : t \in \mathbb{R}, t > s, v \in \mathbb{S}, \|v - u\| < \varepsilon\}$$

such that $C^{u,+}_{s,\varepsilon} \subseteq \mathbb{H}$.

Lemma 3.2. *Suppose $F \subseteq \mathbb{H}$ intersects every wedge in $\mathbb{H}$. Then there is a $u \in \mathbb{S}^+$ such that $T_u(F)$ is dense.*

The reader may find that drawing a few pictures greatly assists in comprehending the proof of Lemma 3.2. We let $p = (-1, 0)$ and $o = (0, 0)$. Note that if $z \in \mathbb{H}$, q is a positive real number, and $u \in \mathbb{S}^+$, then $T_u(z) = q$ if and only if $\angle pou = \angle pqz$.

Proof. We show that the set of $u \in \mathbb{S}^+$ such that $T_u(F)$ is dense in $\mathbb{R}$ is comeager in $\mathbb{S}^+$. It suffices to show that

$$\{u \in \mathbb{S}^+ : T_u(F) \cap I \neq \emptyset\}$$

is open and dense in $\mathbb{S}^+$ for every nonempty open interval I with rational endpoints. Fix a nonempty open interval $I = (q_1, q_2)$ with rational endpoints. We suppose that $q_1, q_2 > 0$ for the sake of simplicity, the more general case follows by trivial modifications of our argument. The map $T : \mathbb{S}^+ \times \mathbb{R}^2 \to \mathbb{R}$ given by $T(u, x) = T_u(x)$ is continuous. Thus if $T_u(x) \in I$ then $T_v(x) \in I$ for all $v \in \mathbb{S}^+$ sufficiently close to u. It follows that the set of u such that $T_u(F) \cap I \neq \emptyset$ is open in $\mathbb{S}^+$.

It now suffices to show that the set of $w \in \mathbb{S}^+$ such that $T_w(F) \cap I \neq \emptyset$ is dense in $\mathbb{S}^+$. Fix $u, v \in \mathbb{S}^+$ such that $\angle pou < \angle pov$ and let J be the set of $w \in \mathbb{S}^+$ such that $\angle pou < \angle pow < \angle pov$. We show there is a $w \in J$ such that $T_w(F) \cap I \neq \emptyset$. Let $r_1, r_2 \in \mathbb{H}$ be such that $\angle pq_1r_1 = \angle pou$ and $\angle pq_2r_2 = \angle pov$. Let D be the set of points in $\mathbb{H}$ that lie in between the rays $\overrightarrow{q_1r_1}$ and $\overrightarrow{q_2r_2}$. It is easy to see that

$$D = \bigcup_{q\in I} \{r \in \mathbb{H} : \angle pou < \angle pqr < \angle pov\} = \bigcup_{q\in I} \bigcup_{w\in J} T_w^{-1}(\{q\}) = \bigcup_{w\in J} T_w^{-1}(I).$$

It therefore suffices to show that D intersects F. Let $z_1, z_2 \in \mathbb{S}^+$ be such that

$$\angle pou < \angle poz_1 < \angle poz_2 < \angle pov.$$

As $\angle pq_1r_1 < \angle poz_1 < \angle poz_2 < \angle pq_2r_2$, we see that every element of $\overrightarrow{oz_1}$ or $\overrightarrow{oz_2}$ sufficiently far from the origin lies in D. It follows that there is a $t > 0$ such that

$$W := \{z \in \mathbb{H} : \|z\| \geq t, \angle poz_1 < \angle poz < \angle poz_2\} \subseteq D.$$

Then W is a wedge in $\mathbb{H}$ and so contains an element of F. Thus D contains an element of F. $\square$

Lemma 3.3. *Suppose $E \subseteq \mathbb{R}$. Then one of the following holds:*

(1) there is a $u \in \mathbb{S}$ such that the restriction of T_u to E^2 is a quasi-isometric embedding $E^2 \to \mathbb{R}$,
(2) there is a linear $S : \mathbb{R}^4 \to \mathbb{R}$ such that $S(E^4)$ is dense.

Proof. Consider $E^2 - E^2 \subseteq \mathbb{R}^2$. If $E^2 - E^2$ is disjoint from a double wedge in $\mathbb{R}^2$ then Lemma 3.2 shows that some T_u quasi-isometrically embeds E^2 into $\mathbb{R}$.

Suppose $E^2 - E^2$ intersects every double wedge in $\mathbb{R}^2$. Note that if $(x, y) \in E^2 - E^2$ then $(-x, -y)$ is also an element of $E^2 - E^2$. It is easy to see that this implies that $E^2 - E^2$ intersects every wedge in $\mathbb{H}$. Applying Lemma 3.3 we fix a $u \in \mathbb{S}$ such that $T_u(E^2 - E^2)$ is dense. Let $S : \mathbb{R}^4 \to \mathbb{R}$ be the linear function given by

$$S(x, y, x', y') = T_u(x - x', y - y') \quad \text{for all} \quad x, y, x', y' \in \mathbb{R}.$$

Then $S(E^4)$ is dense. $\square$

We now prove Theorem 1.1.

Proof. Suppose towards a contradiction that $E \subseteq \mathbb{R}$ has positive coarse Minkowski dimension and $T(E^n)$ is not dense for every $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$. We may suppose that $0 \in E$. Let $\mathcal{S}$ be the collection of sets of the form $T(E^n)$ for linear $T : \mathbb{R}^n \to \mathbb{R}$. It is easy to see that if $F \in \mathcal{S}$ and $T : \mathbb{R}^n \to \mathbb{R}$ is linear then $T(F^n)$ is also in $\mathcal{S}$. We let s be the supremum of the coarse Minkowski dimensions of members of $\mathcal{S}$. Every element of $\mathcal{S}$ has coarse Minkowski dimension ≤ 1, so s exists and $s \leq 1$. As $\dim_{\mathrm{CM}}(E) > 0$ we have $s > 0$. Let $F \in \mathcal{S}$ be such that $\dim_{\mathrm{CM}}(F) > \frac{1}{2}s$. An application of Lemma 3.3 yields a linear $T : \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of T to F^2 is a quasi-isometric embedding $F^2 \to \mathbb{R}$. Lemma 2.3 and Fact 2.2 together show that

$$\dim_{\mathrm{CM}} T(F^2) = \dim_{\mathrm{CM}}(F^2) = 2\dim_{\mathrm{CM}}(F) > s.$$

But $T(F^2) \in \mathcal{S}$, contradiction. $\square$

4. A corollary in $\mathbb{R}^n$

We prove a higher dimensional version of the second claim of Theorem 1.4. (Recall that a closed subset of $\mathbb{R}^n$ has topological dimension zero if and only if it is nowhere dense.)

Theorem 4.1. *Suppose Z is a closed subset of $\mathbb{R}^n$ with topological dimension zero. If Z has positive coarse Minkowski dimension then $(\mathbb{R}_{\mathrm{Vec}}, Z)$ defines all bounded Borel subsets of all $\mathbb{R}^n$.*

Proof. We suppose that $(\mathbb{R}_{\mathrm{Vec}}, Z)$ does not define all bounded Borel subsets of all $\mathbb{R}^n$ and show that $\dim_{\mathrm{CM}}(Z) = 0$. Given $1 \leq k \leq n$ we let $\pi_k : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\pi_k(x_1, \ldots, x_n) = x_k \quad \text{for all} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

An application of [2, Theorem D, Theorem E] shows that $\pi_k(Z)$ is nowhere dense for all $1 \leq k \leq n$. Theorem 1.4 shows that $\dim_{\mathrm{CM}} \pi_k(Z) = 0$ for all $1 \leq k \leq n$. As Z is a subset of $\pi_1(Z) \times \ldots \times \pi_n(Z)$ repeated application of Fact 2.2 shows that $\dim_{\mathrm{CM}}(Z) = 0$. $\square$

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