DP-MINIMAL EXPANSIONS OF $(\mathbb{Z}, +)$ VIA DENSE PAIRS VIA MORDELL-LANG

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ABSTRACT. This is a contribution to the classification problem for dp-minimal expansions of $(\mathbb{Z}, +)$. Let S be a dense cyclic group order on $(\mathbb{Z}, +)$. We use results on "dense pairs" to construct uncountably many dp-minimal expansions of $(\mathbb{Z}, +, S)$. These constructions are applications of the Mordell-Lang conjecture and are the first examples of "non-modular" dp-minimal expansions of $(\mathbb{Z}, +)$. We canonically associate an o-minimal expansion $\mathcal{R}$ of $(\mathbb{R}, +, \times)$, an $\mathcal{R}$-definable circle group $\mathbb{H}$, and a character $\mathbb{Z} \to \mathbb{H}$ to a "non-modular" dp-minimal expansion of $(\mathbb{Z}, +, S)$. We also construct a "non-modular" dp-minimal expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$ from the character $\mathbb{Z} \to \mathbb{Z}_p^\times, k \mapsto \mathrm{Exp}(pk)$.

1. Introduction

We construct new dp-minimal expansions of $(\mathbb{Z}, +)$ and take some steps towards classifying dp-minimal expansions of $(\mathbb{Z}, +)$ which define either a dense cyclic group order or a p-adic valuation. (Every known proper dp-minimal expansion of $(\mathbb{Z}, +)$ defines either a dense cyclic group order, a p-adic valuation, or $<$.)

We recall the definition of dp-minimality in Section 3. Dp-minimality is a strong form of NIP which is broad enough to include many interesting structures and narrow enough to have very strong consequences. O-minimality and related notions imply dp-minimality. Johnson [21] classified dp-minimal fields. Simon [41] showed that an expansion of $(\mathbb{R}, +, <)$ is dp-minimal if and only if it is o-minimal. We summarize recent work on dp-minimal expansions of $(\mathbb{Z}, +)$ in Section 6.

It was an open question for some years whether every proper dp-minimal expansion of $(\mathbb{Z}, +)$ is interdefinable with $(\mathbb{Z}, +, <)$ [3, Question 5.32]. It turns out that this question was essentially answered before it was posed, in work on "dense pairs". We will show, applying work of Hieronymi and Günaydin [17], that if $\mathbb{S}$ is the unit circle, $t \in \mathbb{R}$ is irrational, and $\chi : \mathbb{Z} \to \mathbb{S}$ is the character $\chi(k) := e^{2\pi i t k}$ then the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ is dp-minimal.

Indeed, for every known dp-minimal expansion $\mathcal{Z}$ of $(\mathbb{Z}, +)$ there is a dp-minimal field $\mathcal{K}$, a semiabelian $\mathcal{K}$-variety V, and a character $\chi : \mathbb{Z} \to V(\mathcal{K})$ such that the structure $\mathcal{Z}_\chi$ induced on $\mathbb{Z}$ by $\mathcal{K}$ and χ is dp-minimal and $\mathcal{Z}$ is a reduct of $\mathcal{Z}_\chi$.

We now briefly describe how the known dp-minimal expansions of $(\mathbb{Z}, +)$ fall into this framework. It follows directly from the Mordell-Lang conjecture that if $\beta \in \mathbb{C}^\times$ is not a root of unity then the structure induced on $\mathbb{Z}$ by $(\mathbb{C}, +, \times)$ and the character

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$k \mapsto \beta^k$ is interdefinable with $(\mathbb{Z}, +)$. It follows from a result of Tychonievich [47, Theorem 4.1.2] that if $\beta \in \mathbb{R}^\times \setminus \{-1, 1\}$ then the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and the character $k \mapsto \beta^k$ is interdefinable with $(\mathbb{Z}, +, <)$. (It is also shown in [30] that if $\beta \in \mathbb{Q}_p^\times$ and $\mathrm{Val_p}(\beta) \neq 0$ then $(\mathbb{Z}, +, <)$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{Q}_p, +, \times)$ and $k \mapsto \beta^k$.) Below we apply work of Mariaule [30] to show that there is $\beta \in 1 + p\mathbb{Z}_p$ such that the structure induced on $\mathbb{Z}$ by $(\mathbb{Q}_p, +, \times)$ and $k \mapsto \beta^k$ is a dp-minimal expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$. The only other previously known dp-minimal expansion of $(\mathbb{Z}, +)$ is $(\mathbb{Z}, +, S)$ where S is a dense cyclic group order [46]. There is a unique $\beta \in \mathbb{S}$ such that S is the pullback of the clockwise cyclic order on $\mathbb{S}$ by $k \mapsto \beta^k$. So the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and $k \mapsto \beta^k$ is a dp-minimal expansion of $(\mathbb{Z}, +, S)$.

We produce uncountably many new dp-minimal expansions of $(\mathbb{Z}, +, S)$. Let $\mathbb{E}$ be an elliptic curve defined over $\mathbb{R}$, $\mathbb{E}^0(\mathbb{R})$ be the connected component of the identity, and $\chi : \mathbb{Z} \to \mathbb{E}^0(\mathbb{R})$ be a character such that S is the pullback by χ of the natural cyclic order on $\mathbb{E}^0(\mathbb{R})$. We apply [17] to show that the structure $\mathcal{Z}_\mathbb{E}$ induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ is a proper dp-minimal expansion of $(\mathbb{Z}, +, S)$. We also show that $\mathbb{E}^0(\mathbb{R})$ may be recovered up to semialgebraic isomorphism from $\mathcal{Z}_\mathbb{E}$. It follows that there is an uncountable family of dp-minimal expansions of $(\mathbb{Z}, +, S)$ no two of which are interdefinable.

We describe how $\mathbb{E}^0(\mathbb{R})$ may be recovered from $\mathcal{Z}_\mathbb{E}$. Let C be the usual clockwise cyclic order on $\mathbb{R}/\mathbb{Z}$. Given any dp-minimal expansion $\mathcal{Z}$ of $(\mathbb{Z}, +, S)$ we define a completion $\mathcal{Z}^\square$ of $\mathcal{Z}$, this $\mathcal{Z}^\square$ is an o-minimal expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ canonically associated to $\mathcal{Z}$. We show that $\mathcal{Z}^\square_\mathbb{E}$ is the structure induced on $\mathbb{R}/\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{E}^0(\mathbb{R})$. The recovery of $\mathbb{E}^0(\mathbb{R})$ from $\mathcal{Z}^\square_\mathbb{E}$ is a special case of a canonical correspondence between

(1) non-modular o-minimal expansions $\mathcal{C}$ of $(\mathbb{R}/\mathbb{Z}, +, C)$, and
(2) pairs $\langle \mathcal{R}, \mathbb{H} \rangle$ where $\mathcal{R}$ is an o-minimal expansion of $\mathcal{R}$ and $\mathbb{H}$ is an $\mathcal{R}$-definable circle group.

Given $\langle \mathcal{R}, \mathbb{H} \rangle$, $\mathcal{C}$ is unique up to interdefinibility. Given $\mathcal{C}$, $\mathcal{R}$ is unique up to interdefinibility and $\mathbb{H}$ is unique up to $\mathcal{R}$-definable isomorphism.

We describe $\mathcal{Z}^\square$ for a fixed dp-minimal expansion $\mathcal{Z}$ of $(\mathbb{Z}, +, S)$. Let $\psi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the unique character such that S is the pullback of C by ψ. Let $\mathcal{Z} \prec \mathcal{N}$ be highly saturated, $\mathcal{N}^{\mathrm{Sh}}$ be the Shelah completion of $\mathcal{N}$, and **Inf** be the natural subgroup of infinitesimals in $\mathcal{N}$. We identify $N/\mathbf{Inf}$ with $\mathbb{R}/\mathbb{Z}$ and identify the quotient map $N \to \mathbb{R}/\mathbb{Z}$ with the standard part map. As **Inf** is $\mathcal{N}^{\mathrm{Sh}}$-definable we regard $\mathbb{R}/\mathbb{Z}$ as an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$. A slight adaptation of [50] shows that the following structures are interdefinable:

(1) The structure on $\mathbb{R}/\mathbb{Z}$ with an n-ary relation defining the closure in $(\mathbb{R}/\mathbb{Z})^n$ of $\{(\psi(a_1), \ldots, \psi(a_n)) : (a_1, \ldots, a_n) \in X\}$ for each $\mathcal{Z}^{\mathrm{Sh}}$-definable $X \subseteq \mathbb{Z}^n$,
(2) The structure on $\mathbb{R}/\mathbb{Z}$ with an n-ary relation defining the image under the standard part map $N^n \to (\mathbb{R}/\mathbb{Z})^n$ of each $\mathcal{N}$-definable subset of N^n,
(3) The structure induced on $\mathbb{R}/\mathbb{Z}$ by $\mathcal{N}^{\mathrm{Sh}}$.

We refer to any of these structure as $\mathcal{Z}^\square$. It follows from (3) that $\mathcal{Z}^\square$ is dp-minimal, a slight adaptation of [41] shows that any dp-minimal expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ is o-minimal, so $\mathcal{Z}^\square$ is o-minimal. We will see that the structure induced on $\mathbb{Z}$ by $\mathcal{Z}^\square$ and ψ is a reduct of the Shelah completion of $\mathcal{Z}$. In future work we intend to show that these two are interdefinable. This will reduce the question "what are the dp-minimal expansions of $(\mathbb{Z}, +, S)$ to "for which o-minimal expansions $\mathcal{C}$ of $(\mathbb{R}/\mathbb{Z}, +, C)$ is the structure induced on $\mathbb{Z}$ by $\mathcal{C}$ and ψ dp-minimal"?

We also define an analogous completion $\mathcal{P}^\square$ of a dp-minimal expansion $\mathcal{P}$ of $(\mathbb{Z}, +, \mathrm{Val_p})$, this $\mathcal{P}^\square$ is a dp-minimal expansion of $(\mathbb{Z}_p, +, \mathrm{Val_p})$. The structure induced on $\mathbb{Z}$ by $\mathcal{P}^\square$ is reduct of $\mathcal{P}^{\mathrm{Sh}}$. We expect the induced structure to be interdefinable with $\mathcal{P}^{\mathrm{Sh}}$.

It is easy to see that $\mathcal{Z}^\square_{\mathbb{E}}$ defines an isomorphic copy of $(\mathbb{R}, +, \times)$. It follows that if $\mathcal{Z}_{\mathbb{E}} \prec \mathcal{N}_{\mathbb{E}}$ is highly saturated then the Shelah completion of $\mathcal{N}_{\mathbb{E}}$ interprets $(\mathbb{R}, +, \times)$, so $\mathcal{Z}_{\mathbb{E}}$ should be "non-modular". (One can show that $\mathcal{Z}_{\mathbb{E}}$ itself does not interpret an infinite field.) At present there is no published notion of modularity for general NIP structures, but there should be a notion of modularity for NIP (or possibly just distal) structures which satisfies the following.

- (A1) A modular structure cannot interpret an infinite field.
- (A2) Abelian groups, linearly (or cyclically) ordered abelian groups, NIP valued abelian groups, and ordered vector spaces are modular.
- (A3) If $\mathcal{M}$ is modular and the structure induced on $A \subseteq M^n$ by $\mathcal{M}$ eliminates quantifiers then the induced structure is modular. In particular the Shelah completion of a modular structure is modular. (Recall that the induced structure eliminates quantifiers if and only if every definable subset of A^m is of the form $A^m \cap X$ for $\mathcal{M}$-definable X.)
- (A4) An o-minimal structure is modular if and only if it does not define an infinite field. (This should follow from the Peterzil-Starchenko trichotomy.)

In this paper we will assume that there is a notion of modularity satisfying these conditions, but none of our results fail if this is not true. (A2) implies that all previously known dp-minimal expansions of $(\mathbb{Z}, +)$ are modular. (A1) and (A3) imply that if $\mathcal{Z}^\square$ defines $(\mathbb{R}, +, \times)$ then $\mathcal{Z}$ is non-modular. If $\mathcal{Z}^\square$ does not define $(\mathbb{R}, +, \times)$ then (A4) implies that $\mathcal{Z}^\square$ is modular. We expect that if $\mathcal{Z}^\square$ is modular then $\mathcal{Z}$ is modular.

We will see that if $\mathcal{P}$ is the structure induced on $\mathbb{Z}$ by $(\mathbb{Q}_p, +, \times)$ and the character $k \mapsto \mathrm{Exp}(pk)$ then $\mathcal{P}^\square$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Q}_p, +, \times)$ and the isomorphism $(\mathbb{Z}_p, +) \to (1 + p\mathbb{Z}_p, \times), a \mapsto \mathrm{Exp}(pa)$. It follows that the Shelah completion of a highly saturated $\mathcal{P} \prec \mathcal{N}$ interprets $(\mathbb{Q}_p, +, \times)$, so $\mathcal{P}$ is non-modular. We again expect that $\mathcal{P}$ is modular if and only if $\mathcal{P}^\square$ is modular, but we do not have a modular/non-modular dichotomy for dp-minimal expansions of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ (we lack a p-adic Peterzil-Starchenko.) It seems reasonable to conjecture that a dp-minimal expansion of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ is non-modular if and only if it defines an isomorphic copy of $(\mathbb{Q}_p, +, \times)$.

We now summarize the sections. In Section 3 we recall some background model-theoretic notions, in Section 4 we recall background on cyclically ordered abelian groups, and in Section 5 we recall some basic facts on definable groups in o-minimal expansions of $(\mathbb{R}, +, \times)$. In Section 6 we survey previous work on dp-minimal expansions of $(\mathbb{Z}, +)$. In Section 7 we construct new dp-minimal expansions of $(\mathbb{Z}, +, S)$ where S is a dense cyclic group order. In Section 8 we describe the o-minimal completion of a strongly dependent expansion of $(\mathbb{Z}, +, S)$. We also show that the Shelah completion $(\mathbb{Z}, +, S)^{\mathrm{Sh}}$ of $(\mathbb{Z}, +, S)$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z}, +, C)$ and ψ, where $\psi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the unique character such that S is the pullback of C by ψ. It follows that $(\mathbb{Z}, +, S)^{\mathrm{Sh}}$ is a reduct of each of our dp-minimal expansions of $(\mathbb{Z}, +, S)$. In Section 9 we show that two of our dp-minimal expansions of $(\mathbb{Z}, +, S)$ are interdefinable if and only if the associated semialgebraic circle groups are semialgebraically isomorphic. In Section 10 we construct a new dp-minimal expansion $\mathcal{P}$ of $(\mathbb{Z}, +, \mathrm{Val_p})$ and in Section 11 we describe the p-adic completion of a dp-minimal expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$. In Section 12 we give a conjecture which implies that one can construct uncountably many dp-minimal expansions of $(\mathbb{Z}, +, \mathrm{Val_p})$ from p-adic elliptic curves. Finally, in Section 13 we briefly discuss the question of whether our completion constructions are special cases of an abstract model-theoretic completion.

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2. Conventions, notation, and terminology

Given a tuple $x = (x_1, \dots, x_n)$ of variables we let $|x| = n$. Throughout n is a natural number, m, k, l are integers, t, r, λ, η are real numbers, and α is an element of $\mathbb{R}/\mathbb{Z}$. Suppose $\alpha \in \mathbb{R}/\mathbb{Z}$. We let ψ_α denote the character $\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $\psi_\alpha(k) = \alpha k$. We say that α is **irrational** if $\alpha = s + \mathbb{Z}$ for $s \in \mathbb{R} \setminus \mathbb{Q}$. Note that α is irrational if and only if ψ_α is injective.

All structures are first order and "definable" means "first-order definable, possibly with parameters". Suppose $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{O}$ are structures on a common domain M. Then $\mathcal{M}$ is a **reduct** of $\mathcal{O}$ (and $\mathcal{O}$ is an expansion of $\mathcal{M}$) if every $\mathcal{M}$-definable subset of every M^n is $\mathcal{O}$-definable, $\mathcal{M}$ and $\mathcal{O}$ are **interdefinable** if each is a reduct of the other, $\mathcal{M}$ is a **proper** reduct of $\mathcal{O}$ (and $\mathcal{O}$ is a proper expansion of $\mathcal{M}$) if $\mathcal{M}$ is a reduct of $\mathcal{O}$ and $\mathcal{M}$ is not interdefinable with $\mathcal{O}$, and $\mathcal{N}$ is **intermediate** between $\mathcal{M}$ and $\mathcal{O}$ if $\mathcal{M}$ is a proper reduct of $\mathcal{N}$ and $\mathcal{N}$ is a proper reduct of $\mathcal{O}$.

Given a set A and an injection $f : A \to M^m$ we say that the structure induced on A by $\mathcal{M}$ and f is the structure on A with an n-ary relation defining $\{(a_1, \dots, a_n) \in A^n : ((f(a_1) \dots, f(a_n)) \in Y\}$ for every $\mathcal{M}$-definable $Y \subseteq M^{nm}$. If A is a subset of M^m and $f : A \to M^m$ is the identity we refer to this as the structure induced on A by $\mathcal{M}$.

We let $\mathrm{Cl}(X)$ denote the closure of a subset X of a topological space.

Suppose $L \subseteq L'$ are languages containing $<$, $\mathcal{R}'$ is an L'-structure expanding $(\mathbb{R}, <)$ and $\mathcal{R}$ is the L-reduct of $\mathcal{R}'$. The **open core** of $\mathcal{R}'$ is the reduct of $\mathcal{R}'$ generated by all closed $\mathcal{R}'$-definable sets. Furthermore $\mathrm{Th}(\mathcal{R})$ is **an open core** of $\mathrm{Th}(\mathcal{R}')$ if, whenever $\mathcal{R}' \prec \mathcal{N}'$ then the L'-reduct of $\mathcal{N}'$ is interdefinable with the open core of $\mathcal{N}'$. This notion clearly makes sense in much broader generality.

We use **"semialgebraic"** as a synonym of either "$(\mathbb{R}, +, \times)$-definable" or "$(\mathbb{Q}_p, +, \times)$-definable". It will be clear in context which we mean.

3. Model-theoretic preliminaries

Let $\mathcal{M}$ be a structure and $\mathcal{M} \prec \mathcal{N}$ be highly saturated.

3.1. **Dp-minimality.** Our reference is [42]. Recall that $\mathcal{M}$ is **dp-minimal** if for every small set A of parameters from $\mathcal{N}$, pair I_0, I_1 of mutually indiscernible sequences in $\mathcal{N}$ over A, and $b \in N$, I_i is indiscernible over $A \cup \{b\}$ for some $i \in \{0, 1\}$.

We now describe a second definition of dp-minimality which will be useful below. A family $(\theta_i : i \in I)$ of formulas is n-inconsistent if $\bigwedge_{i \in J} \theta_i$ is inconsistent for every $J \subseteq I$, $|J| = n$. A pair $\varphi(x; y), \phi(x; z)$ of formulas and $n \in \mathbb{N}$ *violate inp-minimality* if $|x| = 1$ and if for every $k \geq 1$ there are $a_1, \ldots, a_k \in M^{|y|}$ and $b_1, \ldots, b_k \in M^{|z|}$ such that $\varphi(x; a_1), \ldots, \varphi(x; a_k)$ and $\phi(x; b_1), \ldots, \phi(x; b_k)$ are both n-inconsistent and $\mathcal{M} \models \exists x[\varphi(x; a_i) \wedge \phi(x; b_j)]$ for any $1 \leq i, j \leq k$. We say that $\varphi(x; y)$ and $\phi(x; z)$ violate inp-minimality if there is n such that $\varphi(x; y), \phi(x; z), n$ violate inp-minimality. Then $\mathcal{M}$ is **inp-minimal** if no pair of formulas violates inp-minimality. Recall that $\mathcal{M}$ is dp-minimal if and only if $\mathcal{M}$ is inp-minimal and NIP.

Fact 3.1 is an easy application of Ramsey's theorem which we leave to the reader.

Fact 3.1. *Let $\varphi_1(x; y_1), \ldots, \varphi_m(x; y_m)$ and $\phi_1(x; z_1), \ldots, \phi_m(x; z_m)$ be formulas. If*

$$\varphi_\cup(x; y_1, \ldots, y_m) = \bigvee_{i=1}^{m} \varphi_i(x; y_i), \quad \phi_\cup(x; z_1, \ldots, z_m) := \bigvee_{i=1}^{m} \phi_i(x; z_i)$$

violate inp-minimality then $\varphi_i(x; y_i), \phi_j(x; z_j)$ violate inp-minimality for some i, j.

We also leave the proof of Fact 3.2 to the reader.

Fact 3.2. *Fix formulas $\varphi(x; y), \phi(x; y)$ with $|x| = 1$. Suppose there is n such that $\mathcal{M} \models \forall y \exists^{\leq n} x \varphi(x; y)$. Then $\varphi(x; y)$ and $\phi(x; y)$ do not violate inp-minimality.*

3.2. **External definibility.** A subset of X of M^n is **externally definable** if there is an $\mathcal{N}$-definable subset Y of N^n such that $X = M^n \cap Y$. By saturation the collection of externally definable sets does not depend on choice of $\mathcal{N}$. The **Shelah completion** $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ is the expansion by all externally definable subsets of all M^n, equivalently, the structure induced on M by $\mathcal{N}$. We will make frequent use of the following elementary observation.

Fact 3.3. *Suppose that $\mathcal{M}$ expands a linear order. Then every convex subset of M is externally definable.*

The first claim of Fact 3.4 is a theorem of Shelah [39], see also Chernikov and Simon [7]. The latter claims follow easily from the first, see for example Onshuus and Usvyatsov [34].

Fact 3.4. *If $\mathcal{M}$ is* NIP *then every $\mathcal{M}^{\mathrm{Sh}}$-definable subset of every M^n is externally definable in $\mathcal{M}$. If $\mathcal{M}$ is* NIP *then $\mathcal{M}^{\mathrm{Sh}}$ is* NIP*, if $\mathcal{M}$ is strongly dependent then $\mathcal{M}^{\mathrm{Sh}}$ is strongly dependent, and if $\mathcal{M}$ is dp-minimal then $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal.*

Fact 3.5 is a theorem of Chernikov and Simon [8, Corollary 9].

Fact 3.5. *Suppose $\mathcal{M}$ is* NIP *and X is an externally definable subset of M^n. Then there is an $\mathcal{M}$-definable family $(X_a : a \in M^m)$ of subsets of M^n such that for every finite $B \subseteq X$ there is $a \in M^m$ such that $B \subseteq X_a \subseteq X$.*

We say that $\mathcal{M}$ is **Shelah complete** if every externally definable set is definable. It follows from Fact 3.4 that $\mathcal{M}^{\mathrm{Sh}}$ is Shelah complete when $\mathcal{M}$ is NIP, so our terminology is reasonable. Fact 3.6 is the Marker-Steinhorn theorem [31].

Fact 3.6. *Any o-minimal expansion of $(\mathbb{R}, <)$ is Shelah complete.*

Fact 3.7 is a theorem of Delon [11].

Fact 3.7. *$(\mathbb{Q}_p, +, \times)$ is Shelah complete.*

3.3. **Weak minimality.** Suppose $\mathcal{O}$ expands $\mathcal{M}$. We say that $\mathcal{O}$ is **$\mathcal{M}$-minimal** if every $\mathcal{O}$-definable subset of M is definable in $\mathcal{M}$ and we say that $\mathcal{O}$ is **weakly $\mathcal{M}$-minimal** if every $\mathcal{O}$-definable subset of M is externally definable in $\mathcal{M}$.

Suppose $L \subseteq L'$ are languages, T' is a complete consistent L'-theory, and T is the L-reduct of T'. We say that T' is **T-minimal** if for every L'-formula $\varphi(x; y), |x| = 1$ there is an L-formula $\phi(x; z)$ such that for every $\mathcal{P} \models T'$ and $a \in P^{|y|}$ there is $b \in P^{|z|}$ such that $\varphi(P; a) = \phi(P; b)$. We say that T' is **weakly T-minimal** if for every L'-formula $\varphi(x; y), |x| = 1$ there is an L-formula $\phi(x; z)$ such that for every $\mathcal{P} \models T'$, highly saturated $\mathcal{P} \prec \mathcal{Q}$, and $a \in P^{|y|}$, there is $b \in Q^{|z|}$ such that $\varphi(P; a) = P \cap \phi(Q; b)$. A structure is weakly T-minimal if its theory is.

Weak minimality was introduced in [43]. If T is a complete theory of dense linear orders then T' is T-minimal if and only if T' is o-minimal and T' is weakly T-minimal if and only if T' is weakly o-minimal.

Suppose $\bigstar$ is an NIP-theoretic property such that T has $\bigstar$ if and only if every T-model omits a certain configuration involving only unary definable sets. It is then easy to see that if T is $\bigstar$ and T' is weakly T-minimal then T' is $\bigstar$.

Fact 3.8. *Suppose T' is weakly T-minimal. If T satisfies any one of the following properties, then so does T'.*

1. *stability,*
2. NIP,
3. *strong dependence,*
4. *dp-minimality.*

4. Cyclically ordered abelian groups

We give basic definitions and results concerning cyclically ordered groups. We also set notation to be used throughout. See [46] for more information and references.

A **cyclic order** S on a set G is a ternary relation such that for all $a, b, c \in G$,

(1) if $S(a, b, c)$, then $S(b, c, a)$,
(2) if $S(a, b, c)$, then $\neg S(c, b, a)$,
(3) if $S(a, b, c)$ and $S(a, c, d)$ then $S(a, b, d)$,
(4) if a, b, c are distinct, then either $S(a, b, c)$ or $S(c, b, a)$.

An open S-interval is a set of the form $\{b \in G : S(a, b, c)\}$ for some $a, c \in G$, likewise define closed and half open intervals. A subset of G is S-convex if it is the union of a nested family of intervals. We drop the "S" when it is clear from context.

If $(G, +)$ is an abelian group then a cyclic group order on $(G, +)$ is a $+$-invariant cyclic order. Suppose S is a cyclic group order on $(G, +)$. A subset of G is an **S tmc set** if it is of the form $a + mJ$ for S-convex $J \subseteq G$ and $a \in G$. We drop the "S" when it is clear from context.

Note that $\{(a, b, c) \in G^2 : S(c, b, a)\}$ is a cyclic group order which we refer to as the **opposite** of S. (If $<$ is a linear group order on $(G, +)$ then $\{(a, b) \in G^2 : b < a\}$ is also a linear group order which we refer to as the opposite of $<$.)

Throughout C is the cyclic group order on $(\mathbb{R}/\mathbb{Z}, +)$ such that whenever $t, t', t'' \in \mathbb{R}$ and $0 \leq t, t', t'' < 1$ then $C(t + \mathbb{Z}, t' + \mathbb{Z}, t'' + \mathbb{Z})$ holds if and only if either $t < t' < t''$, $t' < t'' < t$, or $t'' < t < t'$. Given irrational $\alpha \in \mathbb{R}/\mathbb{Z}$ we let $\boldsymbol{C_\alpha}$ be the cyclic group order on $(\mathbb{Z}, +)$ where $C_\alpha(k, k', k'')$ if and only if $C(\alpha k, \alpha k', \alpha k'')$, so C_α is the pullback of C by ψ_α. Every dense cyclic group order on $(\mathbb{Z}, +)$ is of this form for unique $\alpha \in \mathbb{R}/\mathbb{Z}$.

Let $<$ be a linear group order on $(G, +)$. There are two associated cyclic orders:

$$S_< := \{(a, b, c) \in G^3 : (a < b < c) \vee (b < c < a) \vee (c < a < b)\},$$

and

$$S_> := \{(a, b, c) \in G^3 : (c < b < a) \vee (b < a < c) \vee (a < c < b)\}.$$

Note that $S_<$ is the opposite of $S_>$. See for example [46] for a proof of Fact 4.1.

Fact 4.1. *Every cyclic group order on $(\mathbb{Z}, +)$ is either C_α for some irrational $\alpha \in \mathbb{R}/\mathbb{Z}$ or $S_<$ or $S_>$ for the usual order $<$.*

We will frequently apply Fact 4.2, which is elementary and left to the reader.

Fact 4.2. *Suppose $\mathbb{H}$ is a topological group and γ is an isomorphism $\mathbb{H} \to \mathbb{R}/\mathbb{Z}$ of topological groups. Then γ is unique up to sign, i.e. if $\xi : \mathbb{H} \to \mathbb{R}/\mathbb{Z}$ is a topological group isomorphism then either $\xi = \gamma$ or $\xi = -\gamma$.*

4.1. **The universal cover.** We describe the universal cover of $(G, +, S)$. A universal cover of $(G, +, S)$ is an ordered abelian group $(H, +, <)$, a distinguished positive $u \in H$ such that $u\mathbb{Z}$ is cofinal in H, and a surjective group homomorphism $\pi : H \to G$ with kernel $u\mathbb{Z}$ such that if $a, b, c \in H$ and $0 \leq a, b, c < u$ then $S(\pi(a), \pi(b), \pi(c))$ if and only if we either have $a < b < c$, $b < c < a$, or $c < a < b$.

The universal cover $(H, +, <, u, \pi)$ is unique up to unique isomorphism and every cyclically ordered abelian group has a universal cover.

So $(\mathbb{R}, +, <, 1, \pi)$ is a universal cover of $(\mathbb{R}/\mathbb{Z}, +, C)$, where $\pi(t) = t + \mathbb{Z}$ for all $t \in \mathbb{R}$ and $(\mathbb{Z} + s\mathbb{Z}, +, <, 1, \pi)$ is a universal cover of $(\mathbb{Z}, +, C_\alpha)$ when $\alpha = s + \mathbb{Z}$.

5. Definable groups

We recall some basic facts from the extensive theory of definable groups in o-minimal structures. **Throughout this section $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, \times)$, $\mathbb{H}$ is an $\mathcal{R}$-definable group, and "dimension" without modification is the o-minimal dimension.**

Fact 5.1 follows from work of Pillay [37] and [48, 10.1.8].

Fact 5.1. *There is an $\mathcal{R}$-definable group $\mathbb{G}$ with underlying set $G \subseteq \mathbb{R}^m$ such that $\mathbb{G}$ is a topological group with respect to the topology induced by $\mathbb{R}^m$ and an $\mathcal{R}$-definable group isomorphism $\xi : \mathbb{H} \to \mathbb{G}$. If $\mathbb{G}'$ is an $\mathcal{R}$-definable group with underlying set $G' \subseteq \mathbb{R}^n$, $\mathbb{G}'$ is a topological group with respect to the topology induced by $\mathbb{R}^n$, and $\xi' : \mathbb{H} \to \mathbb{G}'$ is an $\mathcal{R}$-definable group isomorphism, then $\xi' \circ \xi^{-1}$ is a topological group isomorphism $\mathbb{G} \to \mathbb{G}'$.*

We let $\mathcal{T}_{\mathbb{H}}$ be the canonical group topology on $\mathbb{H}$ and consider $\mathbb{H}$ as a topological group. Recall that any connected topological group of topological dimension one is isomorphic (as a topological group) to either $(\mathbb{R}, +)$ or $(\mathbb{R}/\mathbb{Z}, +)$. It follows that if $\mathbb{H}$ is one-dimensional and connected then $\mathbb{H}$ is isomorphic as a topological group to either $(\mathbb{R}, +)$ or $(\mathbb{R}/\mathbb{Z}, +)$. In the first case we say that $\mathbb{H}$ is a **line group**, in the second case $\mathbb{H}$ is a **circle group**.

Suppose X is an $\mathcal{R}$-definable subset of $\mathbb{R}^m$. An easy application of the good directions lemma [48, Theorem 4.2] shows that if X is homeomorphic to $\mathbb{R}$ then there is an $\mathcal{R}$-definable homeomorphism $X \to \mathbb{R}$ and if X is homeomorphic to $\mathbb{R}/\mathbb{Z}$ then there is an $\mathcal{R}$-definable homeomorphism from X to the unit circle. (The analogous fact fails in higher dimensions, there are homeomorphic semialgebraic sets X, X' for which there is no homeomorphism $X \to X'$ definable in an o-minimal expansion of $(\mathbb{R}, +, \times)$, this is a consequence of Shiota's o-minimal Hauptvermutung [40] together with the failure of the classical Hauptvermutung.) Fact 5.2 easily follows.

Fact 5.2. *Suppose $\mathbb{H}$ is one-dimensional, connected, and has underlying set H and group operation $\oplus$. Then there is a unique up to opposite $\mathcal{R}$-definable cyclic group order S on $\mathbb{H}$. If $\mathbb{H}$ is a line group then $(H, +, S)$ is isomorphic to $(\mathbb{R}, +, S_<)$. If $\mathbb{H}$ is a circle group the $(H, +, S)$ is isomorphic to $(\mathbb{R}/\mathbb{Z}, +, C)$.*

So if $\mathbb{H}$ is one-dimensional and connected and A is a subgroup of $\mathbb{H}$ then we may speak without ambiguity of a tmc subset of A.

Finally we recall the interpretation-rigidity theorem for o-minimal expansions of $(\mathbb{R}, +, \times)$. Fact 5.3 is due to Otero, Peterzil, and Pillay [35].

Fact 5.3. *Let $\mathbb{F}$ be an infinite field interpretable in $\mathcal{R}$. Then there is either an $\mathcal{R}$-definable field isomorphism $\mathbb{F} \to (\mathbb{R}, +, \times)$ or $\mathbb{F} \to (\mathbb{C}, +, \times)$. It follows that if an expansion $\mathcal{S}$ of $(\mathbb{R}, +, \times)$ is interpretable in $\mathcal{R}$ then $\mathcal{S}$ is isomorphic to a reduct of $\mathcal{R}$,* *and if a structure $\mathcal{M}$ is mutually interpretable with $\mathcal{R}$ then $\mathcal{R}$ is (up to interdefinibility) the unique expansion of* $(\mathbb{R}, +, \times)$ *mutually interpretable with* $\mathcal{M}$.

6. What we know about dp-minimal expansions of $(\mathbb{Z}, +)$

We survey what is known about dp-minimal expansions of $(\mathbb{Z}, +)$.

The first result on dp-minimal expansions of $(\mathbb{Z}, +)$ is Fact 6.1, proven in [4, Proposition 6.6]. Fact 6.1 follows easily from two results, the Michaux-Villemaire theorem [33] that there are no proper $(\mathbb{Z}, +, <)$-minimal expansions of $(\mathbb{Z}, +, <)$, and Simon's theorem [41, Lemma 2.9] that a definable family of unary sets in a dp-minimal expansion of a linear order has only finitely many germs at infinity.

Fact 6.1. *There are no proper dp-minimal expansions of* $(\mathbb{Z}, +, <)$*. Equivalently: there are no proper dp-minimal expansions of* $(\mathbb{N}, +)$*.*

The authors of [4] raised the question of whether there is a dp-minimal expansion of $(\mathbb{Z}, +)$ which is not a reduct of $(\mathbb{Z}, +, <)$. Conant and Pillay [10] proved Fact 6.2. Their proof relies on earlier work of Palacín and Sklinos [36], who apply the Buechler dichotomy theorem and other sophisticated tools of stability theory.

Fact 6.2. *There are no proper stable dp-minimal expansions of* $(\mathbb{Z}, +)$*.*

Conant [9] proved Fact 6.3 via a geometric analysis of $(\mathbb{Z}, +, <)$-definable sets. Facts 6.2 and 3.8 show that a proper dp-minimal expansion of $(\mathbb{Z}, +)$ is not $\mathrm{Th}(\mathbb{Z}, +)$-minimal. Alouf and d'Elbée [2] used this to give a quicker proof of Fact 6.3.

Fact 6.3. *There are no intermediate structures between* $(\mathbb{Z}, +)$ *and* $(\mathbb{Z}, +, <)$*.*

Alouf and d'Elbée [2] proved Fact 6.4. Given a prime p we let $\mathrm{Val_p}$ be the p-adic valuation on $(\mathbb{Z}, +)$ and $\prec_p$ be the partial order on $\mathbb{Z}$ where $m \prec_p n$ if and only if $\mathrm{Val_p}(m) < \mathrm{Val_p}(n)$. We can view $(\mathbb{Z}, +, \mathrm{Val_p})$ as either $(\mathbb{Z}, +, \prec_p)$ or as the two sorted structure with disjoint sorts $\mathbb{Z}$ and $\mathbb{N} \cup \{\infty\}$, addition on $\mathbb{Z}$, and $\mathrm{Val_p} : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$. It makes no difference which of these two options we take.

Fact 6.4. *Let p be a prime. Then* $(\mathbb{Z}, +, \mathrm{Val_p})$ *is dp-minimal and* $(\mathbb{Z}, +)$*-minimal, and there are no structures intermediate between* $(\mathbb{Z}, +)$ *and* $(\mathbb{Z}, +, \mathrm{Val_p})$*.*

Alouf and d'Elbée also show that $(\mathbb{Z}, +, (\mathrm{Val_p})_{p \in I})$ has dp-rank $|I|$ for any nonempty set I of primes. So if $p \neq q$ are primes then $(\mathbb{Z}, +, \mathrm{Val_p})$ and $(\mathbb{Z}, +, \mathrm{Val}_q)$ do not have a common dp-minimal expansion.

So far we have described countably many dp-minimal expansions of $(\mathbb{Z}, +)$. Fact 6.5, proven by Tran and Walsberg [46], shows that there is an uncountable collection of dp-minimal expansions of $(\mathbb{Z}, +)$, no two of which are interdefinable.

Fact 6.5. *Suppose $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ are irrational. Then* $(\mathbb{Z}, +, C_\alpha)$ *is dp-minimal. Furthermore* $(\mathbb{Z}, +, C_\alpha)$ *and* $(\mathbb{Z}, +, C_\beta)$ *are interdefinable if and only if α and β are $\mathbb{Z}$-linearly dependent.*

Fact 6.5, Fact 4.1, and dp-minimality of $(\mathbb{Z}, +, <)$ together show that any expansion of $(\mathbb{Z}, +)$ by a cyclic group order is dp-minimal.

It is shown in [46] that every unary definable set in every elementary extension of $(\mathbb{Z}, +, C_\alpha)$ is a finite union of tmc sets. It follows by Fact 3.8 that if $\mathcal{Z}$ expands

$(\mathbb{Z},+,C_\alpha)$ and every unary definable set in every elementary extension of $\mathcal{Z}$ is a finite union of tmc sets, then $\mathcal{Z}$ is dp-minimal. A converse is proven in [43].

Fact 6.6. *Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Suppose $\mathcal{Z}$ is a dp-minimal expansion of $(\mathbb{Z},+,C_\alpha)$. Then $\mathcal{Z}$ is weakly* $\mathrm{Th}(\mathbb{Z},+,C_\alpha)$*-minimal (equivalently: every unary definable set in every elementary extension of $\mathcal{Z}$ is a finite union of tmc sets).*

In particular a dp-minimal expansion of $(\mathbb{Z},+,C_\alpha)^{\mathrm{Sh}}$ cannot add new unary sets.

Suppose $\alpha,\beta \in \mathbb{R}/\mathbb{Z}$ are irrational and $\mathbb{Z}$-linearly independent. An easy application of Kronecker density shows that if I is an infinite and co-infinite C_α-convex set then I is not a finite union of C_β-tmc sets, see [46]. Fact 6.7 follows.

Fact 6.7. *Suppose $\alpha,\beta \in \mathbb{R}/\mathbb{Z}$ are irrational and $\mathbb{Z}$-linearly independent. Suppose $\mathcal{Z}_\alpha$ is a dp-minimal expansion of $(\mathbb{Z},+,C_\alpha)$ and $\mathcal{Z}_\beta$ is a dp-minimal expansion of $(\mathbb{Z},+,C_\beta)$. If I is an infinite and co-infinite C_α-interval then I is not $\mathcal{Z}_\beta$-definable, and vice versa. So $\mathcal{Z}_\alpha$ defines a subset of $\mathbb{Z}$ which is not $\mathcal{Z}_\beta$-definable, and vice versa. In particular $\mathcal{Z}_\alpha$ and $\mathcal{Z}_\beta$ do not have a common dp-minimal expansion.*

We now describe a striking recent result of Alouf [1]. We first recall Fact 6.8, a special case of [20, Lemma 3.1].

Fact 6.8. *Suppose $\mathcal{G}$ is a dp-minimal expansion of a group G which defines a non-discrete Hausdorff group topology on G. Then $\mathcal{G}$ eliminates $\exists^\infty$.*

Fact 6.8 shows that any dp-minimal expansion of $(\mathbb{Z},+,\mathrm{Val_p})$ or $(\mathbb{Z},+,C_\alpha)$ eliminates $\exists^\infty$. Fact 6.9 is proven in [1].

Fact 6.9. *Suppose $\mathcal{Z}$ is a dp-minimal expansion of $(\mathbb{Z},+)$ which either*

(1) does not eliminate $\exists^\infty$,

(2) or defines an infinite subset of $\mathbb{N}$.

Then $\mathcal{Z}$ defines $<$.

So $(\mathbb{Z},+,<)$ is, up to interdefinibility, the only dp-minimal expansion of $(\mathbb{Z},+)$ which does not eliminate $\exists^\infty$. Conjecture 1 is now natural.

Conjecture 1. *Any proper dp-minimal expansion of $(\mathbb{Z},+)$ which eliminates $\exists^\infty$ defines a non-discrete group topology on $(\mathbb{Z},+)$.*

Johnson [22] shows that a dp-minimal expansion of a field which is not strongly minimal admits a definable non-discrete field topology. His proof makes crucial use of the fact that any dp-minimal expansion of a field eliminates $\exists^\infty$.

6.1. **Interpretations.** We describe what we know about interpretations between dp-minimal expansions of $(\mathbb{Z},+)$. We suspect that bi-interpretable dp-minimal expansions of $(\mathbb{Z},+)$ are interdefinable.

Proposition 6.10. *Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Suppose $\mathcal{Z}$ is a dp-minimal expansion of $(\mathbb{Z},+,C_\alpha)$. Then $\mathcal{Z}^{\mathrm{eq}}$ eliminates $\exists^\infty$, so $\mathcal{Z}$ does not interpret $(\mathbb{Z},+,<)$ or $(\mathbb{Z},+,\mathrm{Val_p})$ for any prime p.*

Note that $(\mathbb{Z},+,\mathrm{Val_p})^{\mathrm{eq}}$ does not eliminate $\exists^\infty$ as $(\mathbb{Z},+,\mathrm{Val_p})$ interprets $(\mathbb{N},<)$.

Given a structure $\mathcal{M}$ we say that $\mathcal{M}^{\mathrm{eq}}$ eliminates $\exists^\infty$ in one variable if for every definable family $(E_a : a \in M^k)$ of equivalence relations on M there is n such that for all $a \in M^k$ we either have $|M/E_a| < n$ or $|M/E_a| \geq \aleph_0$. Proposition 6.10 requires Fact 6.11, which is routine and left to the reader.

Fact 6.11. *Let $\mathcal{M} \prec \mathcal{N}$ be highly saturated. Suppose that $\mathcal{N}$ eliminates $\exists^\infty$ and there is no $\mathcal{N}$-definable equivalence relation on N with infinitely many infinite classes. Then $\mathcal{M}^{\mathrm{eq}}$ eliminates $\exists^\infty$.*

We now prove Proposition 6.10. We use the notation and results of [43], so the reader will need to have a copy of that paper at hand.

Proof. Let $(H, +, <, u, \pi)$ be a universal cover of $(\mathbb{Z}, +, C_\alpha)$, $I := (-u, u)$. So let $\mathcal{I}$ be the structure induced on I by $\mathcal{Z}$ and π. It is shown in [43] that $\mathcal{I}$ and $\mathcal{Z}$ define isomorphic copies of each other, so it suffices to show that $\mathcal{I}^{\mathrm{eq}}$ eliminates $\exists^\infty$. Let $\mathcal{I} \prec \mathcal{J}$ be highly saturated. The proof of Fact 6.8 shows that $\mathcal{J}$ eliminates $\exists^\infty$. We show that every $\mathcal{J}$-definable equivalence relation on J has only finitely many infinite classes and apply Fact 6.11.

Suppose E is a $\mathcal{J}$-definable equivalence relation on J with infinitely many infinite classes. By [43, Lemma 8.7] there is a finite partition $\mathcal{A}$ of J into $\mathcal{J}$-definable sets such that every E-class is a finite union of sets of the form $K \cap A$ for convex K and $A \in \mathcal{A}$. Fix $A \in \mathcal{A}$ which intersects infinitely many E-classes. Note that the intersection of each E-class with A is a finite union of convex sets. Let F be the equivalence relation on J where $a < b$ are F-equivalent if and only if there are $a' < a < b < b'$ such that $a', b' \in A$, a' and b' are E-equivalent, and a', b' lie in the same convex component of $E_{a'} \cap A$. It is easy to see that every F-class is convex and there are infinitely many F-classes. However, it is shown in the proof of [43, Lemma 8.7] that any definable equivalence relation on J with convex equivalence classes has only finitely many infinite classes. $\square$

Fact 6.12 is proven in [50, Proposition 5.6].

Fact 6.12. *Suppose $\mathcal{Z}$ is an* NTP$_2$ *expansion of $(\mathbb{Z}, <)$ and $\mathcal{G}$ is an expansion of a group G which defines a non-discrete Hausdorff group topology on G. Then $\mathcal{Z}$ does not interpret $\mathcal{G}$. So in particular an* NTP$_2$ *expansion of $(\mathbb{Z}, +, <)$ does not interpret $(\mathbb{Z}, +, C_\alpha)$ for any irrational $\alpha \in \mathbb{R}/\mathbb{Z}$ or $(\mathbb{Z}, +, \mathrm{Val_p})$ for any prime p.*

In Section 10 we construct a dp-minimal expansion $\mathcal{P}$ of $(\mathbb{Z}, +, \mathrm{Val_p})$ which defines addition on the value set, so in particular $\mathcal{P}$ interprets $(\mathbb{Z}, +, <)$.

7. New dp-minimal expansions of $(\mathbb{Z}, +, C_\alpha)$

We describe new dp-minimal expansions of $(\mathbb{Z}, +, C_\alpha)$.

7.1. **Dense pairs.** We first recall Hieronymi and Günaydin [17]. Let $\mathbb{H}$ be an abelian semialgebraic group with underlying set $H \subseteq \mathbb{R}^m$ and group operation $\oplus$, and A be a subgroup of $\mathbb{H}$. Then A has the **Mordell-Lang property** if for every $f \in \mathbb{R}[x_1, \ldots, x_{nm}]$ the set $\{a \in A^n : f(a) = 0\}$ is a finite union of sets of the form

$$\{(a_1, \ldots, a_n) \in A^n : k_1 a_1 \oplus \ldots \oplus k_n a_n = b\} \quad \text{for some } k_1, \ldots, k_n \in \mathbb{Z}, b \in A.$$

We say that $\mathbb{H}$ is a **Mordell-Lang** group if every finite rank subgroup of $\mathbb{H}$ has the Mordell-Lang property. Fact 7.1 is essentially in [17], but see the comments below.

Fact 7.1. *Suppose* $\mathbb{H}$ *is a one-dimensional connected Mordell-Lang group. Let* A *be a dense finite rank subgroup of* $\mathbb{H}$*. Then* $(\mathbb{R}, +, \times, A)$ *is* NIP*,* $\mathrm{Th}(\mathbb{R}, +, \times)$ *is an open core of* $\mathrm{Th}(\mathbb{R}, +, \times, A)$*, and every subset of* A^k *definable in* $(\mathbb{R}, +, \times, A)$ *is a finite union of sets of the form* $b \oplus n(X \cap A^k)$ *for semialgebraic* X *and* $b \in A^k$*.*

Note that the last claim of Fact 7.1 shows that structure induced on A by $(\mathbb{R}, +, \times)$ is interdefinable with the structure induced by $(\mathbb{R}, +, \times, A)$ are interdefinable.

The reader will not find the exact statement of the last claim of Fact 7.1 in [17]. It is incorrectly claimed in [17, Proposition 3.10] that every subset of A^k definable in $(\mathbb{R}, +, \times, A)$ is a finite union of sets of the form $X \cap (b \oplus nA^k)$ where $X \subseteq \mathbb{H}$ is semialgebraic. This is true when $\mathbb{H}$ is a line group, but fails when $\mathbb{H}$ is a circle group. If $\mathbb{H}$ is a circle group and I is an infinite and co-infinite open interval in $\mathbb{H}$ then $2I$ is not of this form. A slightly corrected version of the proof of [17, Proposition 3.10] yields the last statement of Fact 7.1[1].

Proposition 7.2 is proven in [43].

Proposition 7.2. *Suppose* $(G, +, S)$ *is a cyclically order abelian group and* $\mathcal{G}$ *expands* $(G, +, S)$*. Suppose* $|G/nG| < \aleph_0$ *for all* n*. Then* $\mathcal{G}$ *is dp-minimal if and only if every unary definable set in every elementary extension of* $\mathcal{G}$ *is a finite union of tmc sets. So* $\mathcal{G}$ *is dp-minimal if and only if* $\mathrm{Th}(\mathcal{G})$ *is weakly* $\mathrm{Th}(G, +, S)$*-minimal.*

Let $\mathcal{A}$ be the structure induced on A by $(\mathbb{R}, +, \times)$. Fact 7.1 shows that every $\mathcal{A}$-definable unary set is a finite union of tmc sets, and that the same claim holds in every elementary extension of $\mathcal{A}$. Proposition 7.3 follows.

Proposition 7.3. *If* $\mathbb{H}$ *is a one-dimensional connected Mordell-Lang group and* A *is a dense finite rank subgroup of* $\mathbb{H}$*, then the structure induced on* A *by* $(\mathbb{R}, +, \times)$ *is dp-minimal. So if* $\mathbb{H}$ *is a Mordell-Lang circle group and* $\chi : \mathbb{Z} \to \mathbb{H}$ *is an injective character then the structure induced on* $\mathbb{Z}$ *by* $(\mathbb{R}, +, \times)$ *and* χ *is dp-minimal.*

Of course Proposition 7.3 is only relevant because there are semialgebraic Mordell-Lang circle groups by the general Mordell-Lang conjecture. This is a theorem of Faltings, Vojta, McQuillan and others, see [32] for a survey.

Fact 7.4. *If* W *is a semiabelian variety defined over* $\mathbb{C}$*,* V *is a subvariety of* W*, and* Γ *is a finite rank subgroup of* $W(\mathbb{C})$*, then* $\Gamma \cap V(\mathbb{C})$ *is a finite union of cosets of subgroups of* Γ*. So* $(\mathbb{R}_{>0}, \times)$*, the unit circle equipped with complex multiplication, and the real points of an elliptic curve defined over* $\mathbb{R}$ *are all Mordell-Lang groups.*

7.2. **Specific examples.** Suppose $\mathbb{H}$ is a semialgebraic group equipped with $\mathcal{T}_{\mathbb{H}}$. By [19] there is an open neighbourhood $U \subseteq \mathbb{H}$ of the identity, an algebraic group W defined over $\mathbb{R}$, a neighbourhood $V \subseteq W(\mathbb{R})$ of the identity, and a semialgebraic local group isomorphism $U \to V$. We say that $\mathbb{H}$ is semiabelian when W is semiabelian. Suppose $\mathbb{H}$ is one-dimensional. Then W is one dimensional, so we may take $W(\mathbb{R})$ to be either $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \times)$, the unit circle, or the real points of an elliptic curve. In the latter three cases $\mathbb{H}$ is semiabelian.

[1]Thanks to Philipp Hieronymi for discussions on this point.

One-dimensional semialgebraic groups were classified up to semialgebraic isomorphism by Madden and Stanton [28]. There are three families of semiabelian semialgebraic circle groups.

We describe the first family. Given $\lambda > 1$ we let $\mathbb{G}_\lambda := ([1,\lambda), \otimes_\lambda)$ where $t \otimes_\lambda t' = tt'$ when $tt' < \lambda$ and $t \otimes_\lambda t' = tt'\lambda^{-1}$ otherwise. Let $\lambda, \eta > 1$. The unique (up to sign) topological group isomorphism $\mathbb{G}_\lambda \to \mathbb{G}_\eta$ is $t \mapsto t^{\log_\lambda \eta}$. So $\mathbb{G}_\lambda$ and $\mathbb{G}_\eta$ are semialgebraically isomorphic if and only if $\log_\lambda \eta \in \mathbb{Q}$.

Lemma 7.5. *Fix $\lambda > 1$. Suppose A is a finite rank subgroup of $\mathbb{G}_\lambda$. Then $(\mathbb{R}, +, \times, A)$ is* NIP*,* $\mathrm{Th}(\mathbb{R}, +, \times)$ *is an open core of* $\mathrm{Th}(\mathbb{R}, +, \times, A)$*, and the structure induced on A by $(\mathbb{R}, +, \times, A)$ is dp-minimal.*

We let S be the cyclic order on $[1, \lambda)$ where $S(t, t', t'')$ if and only if either $t < t' < t''$, $t' < t'' < t$, or $t'' < t < t'$. So S is the unique (up to opposite) semialgebraic cyclic group order on $\mathbb{G}_\lambda$.

Proof. Identify $\mathbb{G}_\lambda$ with $(\mathbb{R}_{>0}/\lambda^{\mathbb{Z}}, \times)$ and let ρ be the quotient map $\mathbb{R}_{>0} \to \mathbb{G}_\lambda$. So $(\mathbb{R}_{>0}, \times, <, \lambda, \rho)$ is a universal cover of $(\mathbb{G}_\lambda, S)$. Let $H := \rho^{-1}(A)$. So H is finite rank and $(H, \times, <, \lambda, \rho)$ is a universal cover of $(A, \otimes_\lambda, S)$. As $(\mathbb{R}_{>0}, \times)$ is a Mordell-Lang group and H is dense in $\mathbb{R}_{>0}$, $(\mathbb{R}, +, \times, H)$ is NIP, $\mathrm{Th}(\mathbb{R}, +, \times)$ is an open core of $\mathrm{Th}(\mathbb{R}, +, \times, H)$, and the structure induced on H by $(\mathbb{R}, +, \times, H)$ is dp-minimal. Observe that A is definable in $(\mathbb{R}, +, \times, H)$. So $(\mathbb{R}, +, \times, A)$ is NIP and $\mathrm{Th}(\mathbb{R}, +, \times)$ is an open core of $\mathrm{Th}(\mathbb{R}, +, \times, A)$. Finally the structure induced on A by $(\mathbb{R}, +, \times)$ is interdefinable with the structure induced on $H \cap [0, \lambda)$ by $(\mathbb{R}, +, \times)$. So the structure induced on A by $(\mathbb{R}, +, \times)$ is dp-minimal. □

The unique (up to sign) topological group isomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{G}_\lambda$ is $\gamma(t + \mathbb{Z}) = \lambda^{t - \lfloor t \rfloor}$. Given irrational $\alpha = s + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ we let $\chi_\alpha : \mathbb{Z} \to \mathbb{G}_\lambda$ be

$$\chi_\alpha(k) := \gamma(\alpha k) = \lambda^{sk - \lfloor sk \rfloor}$$

and let $\mathcal{G}_{\alpha,\lambda}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ_α.

Proposition 7.6. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\lambda > 1$. Then $\mathcal{G}_{\alpha,\lambda}$ is a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)$.*

Let $\mathbb{S}$ be the unit circle equipped with complex multiplication. The second family of consists of $\mathbb{S}$ and other circle groups constructed from $\mathbb{S}$ in roughly the same way that $\mathbb{G}_\lambda$ is constructed from $(\mathbb{R}_{>0}, \times)$. We only discuss $\mathbb{S}$. The unique (up to sign) topological group isomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{S}$ is given by $\gamma(t + \mathbb{Z}) = e^{2\pi i t}$. Given irrational $\alpha = s + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ we let $\chi_\alpha : \mathbb{Z} \to \mathbb{S}$ be

$$\chi_\alpha(k) := \gamma(\alpha k) = e^{2\pi i s k}.$$

and let $\mathcal{S}_\alpha$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ_α.

Proposition 7.7. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational. Then $\mathcal{S}_\alpha$ is a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)$.*

The third family comes from elliptic curves. Given an elliptic curve $\mathbb{E}$ defined over $\mathbb{R}$ we let $\mathbb{E}(\mathbb{R})$ be the real points of $\mathbb{E}$. We consider $\mathbb{E}$ as a subvariety of $\mathbb{P}^2$ via the Weierstrass embedding. We let $\mathbb{E}^0(\mathbb{R})$ be the connected component of the identity of $\mathbb{E}(\mathbb{R})$, so $\mathbb{E}^0(\mathbb{R})$ is a semialgebraic circle group. The fourth family of semialgebraic circle groups consists of such $\mathbb{E}^0(\mathbb{R})$ and circle groups constructed from $\mathbb{E}^0(\mathbb{R})$ in roughly the same way as $\mathbb{G}_\lambda$ is constructed from $(\mathbb{R}_{>0}, \times)$. We only discuss $\mathbb{E}_0(\mathbb{R})$.

Fix $\lambda > 0$ and let Λ be the lattice $\mathbb{Z} + i\lambda\mathbb{Z}$. Let $\mathbb{E}_\lambda$ be the elliptic curve associated to Λ, recall that $\mathbb{E}_\lambda$ is defined over $\mathbb{R}$ and any elliptic curve defined over $\mathbb{R}$ is isomorphic to some $\mathbb{E}_\lambda$. Given $\eta > 0$ there is a semialgebraic group isomorphism $\mathbb{E}^0_\lambda(\mathbb{R}) \to \mathbb{E}^0_\eta(\mathbb{R})$ if and only if $\lambda/\eta \in \mathbb{Q}$, see [28].

Let $\wp_\lambda$ be the Weierstrass elliptic function associated to Λ and $\mathfrak{p}_\lambda : \mathbb{R} \to \mathbb{E}^0_\lambda(\mathbb{R})$ be given by $\mathfrak{p}_\lambda(t) = [\wp_\lambda(t) : \wp'_\lambda(t) : 1]$. The unique (up to sign) topological group isomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{E}^0_\lambda(\mathbb{R})$ is $\gamma(t + \mathbb{Z}) = \mathfrak{p}_\lambda(t)$. Fix irrational $\alpha = s + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ and let $\chi_\alpha : \mathbb{Z} \to \mathbb{E}^0_\lambda(\mathbb{R})$ be the character

$$\chi_\alpha(k) := \gamma(\alpha k) = \mathfrak{p}_\lambda(sk) = [\wp_\lambda(sk) : \wp'_\lambda(sk) : 1].$$

Let $\mathcal{E}_{\alpha,\lambda}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ_α.

Proposition 7.8. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\lambda > 0$. Then $\mathcal{E}_{\alpha,\lambda}$ is a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)$.*

7.3. **Another possible family of expansions.** We describe an approach to constructing uncountably many dp-minimal expansions of each example described above. Let I be a closed bounded interval with interior. Let $C^\infty(I)$ be the topological vector space of smooth functions $I \to \mathbb{R}$ where the topology is that induced by the seminorms $f \mapsto \max\{|f^{(n)}(t)| : t \in I\}$. So $C^\infty(I)$ is a Polish space. Le Gal has shown that the set of $f \in C^\infty(I)$ such that $(\mathbb{R}, +, \times, f)$ is o-minimal is comeager [26].

Conjecture 2. *Let $\mathbb{H}$ be a semialgebraic Mordell-Lang circle group, γ be the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{H}$, $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational, $\chi : \mathbb{Z} \to \mathbb{H}$ be given by $\chi(k) = \gamma(\alpha k)$, and $A := \chi(\mathbb{Z})$. There is a comeager subset Λ of $C^\infty(I)$ (possibly depending on α) such that if $f \in \Lambda$ then*

1. *$(\mathbb{R}, +, \times, f)$ is o-minimal,*
2. *if $f \neq g$ are in Λ then $(\mathbb{R}, +, \times, f)$ and $(\mathbb{R}, +, \times, g)$ are not interdefinable.*
3. *Every $(\mathbb{R}, +, \times, f)$-definable group is definably isomorphic to a semialgebraic group and any $(\mathbb{R}, +, \times, f)$-definable homomorphism between semialgebraic groups is semialgebraic.*
4. *$(\mathbb{R}, +, \times, A)$ is* NIP *and* $\mathrm{Th}(\mathbb{R}, +, \times)$ *is an open core of* $\mathrm{Th}(\mathbb{R}, +, \times, A)$.
5. *Every $(\mathbb{R}, +, \times, A)$-definable subset of A^k is a finite union of sets of the form $b \oplus n(X \cap A^k)$ for semialgebraic X and $b \in A^k$. So in particular the structure induced on A by $(\mathbb{R}, +, \times)$ is a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)$.*

Gorman, Hieronymi, and Kaplan generalized the Mordell-Lang property to an abstract model theoretic setting [15]. Item (4) of Conjecture 2 should follow by verifying that the conditions in their paper are satisfied.

Suppose Conjecture 2 holds. Let $\mathcal{H}_\alpha$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ and for each $f \in \Lambda$ let $\mathcal{H}_{\alpha,f}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times, f)$ and χ. So each $\mathcal{H}_{\alpha,f}$ is a dp-minimal expansion of $\mathcal{H}_\alpha$.

It is easy to see that our expansions of $(\mathbb{Z}, +, C_\alpha)$ define the same subsets of $\mathbb{Z}$ as $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$, so is $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ a reduct of these expansions? It is intuitively obvious that these expansions defines subsets of $\mathbb{Z}^2$ which are not definable in $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$, but how do we show this? When are two of the expansions described above interdefinable? We now develop tools to answer these questions.

8. The o-minimal completion

We associate an o-minimal expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ to a strongly dependent expansion of $(\mathbb{Z}, +, C_\alpha)$. We will show that $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z}, +, C)$ and ψ_α. It will follow that each of the dp-minimal expansions of $(\mathbb{Z}, +, C_\alpha)$ describe above in fact expands $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$.

We first recall the completion of an NIP expansion of a dense archimedean ordered abelian group defined in [50].

8.1. **The linearly ordered case.** Suppose that $(H, +, <)$ is a dense subgroup of $(\mathbb{R}, +, <)$, $\mathcal{H}$ is an expansion of $(H, +, <)$, and $\mathcal{H} \prec \mathcal{N}$ is highly saturated. Let $\mathbf{Fin}$ be the convex hull of H in N and $\mathbf{Inf}$ be the set of $a \in N$ such that $|a| < b$ for all positive $b \in H$. We identify $\mathbf{Fin}/\mathbf{Inf}$ with $\mathbb{R}$ so the quotient map $\mathrm{st} : \mathbf{Fin} \to \mathbb{R}$ is the usual standard part map. Note that $\mathbf{Fin}$ and $\mathbf{Inf}$ are both $\mathcal{H}^{\mathrm{Sh}}$-definable so we regard $\mathbb{R}$ as an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$. We let $\mathrm{st} : \mathbf{Fin}^n \to \mathbb{R}^n$ be given by $\mathrm{st}(a_1, \ldots, a_n) = (\mathrm{st}(a_1), \ldots, \mathrm{st}(a_n))$. Fact 8.1 is [50, Theorem F].

Fact 8.1. *Suppose $\mathcal{H}$ is* NIP*. Then the following structures are interdefinable.*

1. *The structure $\mathcal{H}^{\square}$ on $\mathbb{R}$ with an n-ary relation symbol defining the closure in $\mathbb{R}^n$ of every subset of H^n which is externally definable in $\mathcal{H}$.*
2. *The structure on $\mathbb{R}$ with an n-ary relation symbol defining, for each $\mathcal{N}$-definable subset X of N^n, the image of $\mathbf{Fin}^n \cap X$ under the standard part map $\mathbf{Fin}^n \to \mathbb{R}^n$.*
3. *The open core of the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$.*

Furthermore the structure induced on H by $\mathcal{H}^{\square}$ is a reduct of $\mathcal{H}^{\mathrm{Sh}}$. If $\mathcal{H}$ is strongly dependent then $\mathcal{H}^{\square}$ is interdefinable with the structure induced on $\mathbb{R}$ by $\mathcal{N}^{\mathrm{Sh}}$.

The completion $\mathcal{H}^{\square}$ should be "at least as tame" as $\mathcal{H}$ because $\mathcal{H}^{\square}$ is interpretable in $\mathcal{N}^{\mathrm{Sh}}$. In general $\mathcal{H}^{\mathrm{Sh}}$ is not interdefinable with the structure induced on H by $\mathcal{H}^{\square}$. Suppose $H = \mathbb{R}$ and $\mathcal{H} = (\mathbb{R}, +, <, \mathbb{Q})$, it follows from Theorem 9.2 and the quantifier elimination for $(\mathbb{R}, +, <, \mathbb{Q})$ that $(\mathbb{R}, +, <, \mathbb{Q})^{\square}$ is interdefinable with $(\mathbb{R}, +, <)$. Recall that $(\mathbb{R}, +, <, \mathbb{Q})$ has dp-rank two [12]. We expect that if $\mathcal{H}$ is dp-minimal then $\mathcal{H}^{\mathrm{Sh}}$ is interdefinable with the structure induced on H by $\mathcal{H}^{\square}$. Note that if $H = \mathbb{R}$ and $\mathcal{H}$ is dp-minimal then $\mathcal{H}$ is o-minimal by [41], so by the Marker-Steinhorn theorem $\mathcal{H}^{\square}$ is the open core of $\mathcal{H}$, so $\mathcal{H}^{\square}$ and $\mathcal{H}$ are interdefinable as any o-minimal stucture is interdefinable with its open core.

8.2. **The cyclically ordered case.** We only work over $(\mathbb{Z}, +, C_\alpha)$, but everything goes through for a cyclic order on an abelian group induced by an injective character to $\mathbb{R}/\mathbb{Z}$. Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Abusing notation we let $\psi_\alpha : \mathbb{Z}^n \to (\mathbb{R}/\mathbb{Z})^n$ be given by $\psi_\alpha(k_1, \ldots, k_n) = (\alpha k_1, \ldots, \alpha k_n)$. If $\beta \in \mathbb{R}/\mathbb{Z}$ is irrational then $C_\alpha = C_\beta$ if and only if $\alpha = \beta$, so we can recover ψ_α from $(\mathbb{Z}, +, C_\alpha)$.

Let $\mathcal{Z} \prec \mathcal{N}$ be highly saturated. We define a standard part map $\mathrm{st} : N \to \mathbb{R}/\mathbb{Z}$ by declaring $\mathrm{st}(a)$ to be the unique element of $\mathbb{R}/\mathbb{Z}$ such that for all integers k, k' we have $C(\alpha k, \mathrm{st}(a), \alpha k')$ if and only if $C_\alpha(k, a, k')$. Note that st is a homomorphism and let **Inf** be the kernal of st. We identify $N/\mathbf{Inf}$ with $\mathbb{R}/\mathbb{Z}$ and st with the quotient map. Note that **Inf** is convex, hence $\mathcal{N}^{\mathrm{Sh}}$-definable. So we consider $\mathbb{R}/\mathbb{Z}$ to be an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$.

Proposition 8.2. *Suppose $\mathcal{Z}$ is* NIP. *The following structures are interdefinable.*

1. *The structure $\mathcal{Z}^\square$ on $\mathbb{R}/\mathbb{Z}$ with an n-ary relation symbol defining the closure in $(\mathbb{R}/\mathbb{Z})^n$ of $\psi_\alpha(X)$ for every $X \subseteq \mathbb{Z}^n$ which is externally definable in $\mathcal{Z}$.*
2. *The structure on $\mathbb{R}/\mathbb{Z}$ with an n-ary relation symbol defining the image of each $\mathcal{N}$-definable $X \subseteq N^n$ under the standard part map $N^n \to (\mathbb{R}/\mathbb{Z})^n$.*
3. *The open core of the structure induced on $\mathbb{R}/\mathbb{Z}$ by $\mathcal{N}^{\mathrm{Sh}}$.*

Furthermore the structure induced on $\mathbb{Z}$ by $\mathcal{Z}^\square$ and ψ_α is a reduct of $\mathcal{Z}^{\mathrm{Sh}}$. If $\mathcal{Z}$ is strongly dependent then $\mathcal{Z}^\square$ is interdefinable with the structure induced on $\mathbb{R}/\mathbb{Z}$ by $\mathcal{N}^{\mathrm{Sh}}$ and $\mathcal{Z}^\square$ is o-minimal.

We expect that if $\mathcal{Z}$ is dp-minimal then the structure induced on $\mathbb{Z}$ by $\mathcal{Z}^\square$ and ψ_α is interdefinable with $\mathcal{Z}^{\mathrm{Sh}}$. All claims of Proposition 8.2 except o-minimality follow by slight modifications to the proof of Fact 8.1. The last claim also follows easily from the methods of [50], we provide details below. ($\mathcal{H}$ need not be o-minimal when $\mathcal{H}$ is strongly dependent, for example $(\mathbb{Q}, +, <, \mathbb{Z})$ is strongly dependent by [12, 3.1] and $(\mathbb{Q}, +, <, \mathbb{Z})^\square$ is interdefinable with $(\mathbb{R}, +, <, \mathbb{Z})$.)

We need three facts to prove the last claim. Fact 8.3 is left to the reader.

Fact 8.3. *Suppose X is a subset of $\mathbb{R}/\mathbb{Z}$. Then X is a finite union of intervals and singletons if and only if the boundary of X is finite.*

Fact 8.4 is essentially a theorem of Dolich and Goodrick [12]. They only treat linearly ordered structures, but routine alternations to their proof yield Fact 8.4.

Fact 8.4. *Suppose $(G, +, S)$ is a cyclically ordered abelian group, $\mathcal{G}$ is a strongly dependent expansion of $(G, +, S)$, and X is a $\mathcal{G}$-definable subset of G. If X is nowhere dense then X has no accumulation points.*

Fact 8.5 follows from [50, Theorem B].

Fact 8.5. *Suppose $\mathcal{Z}$ is* NIP *and X, Y are $\mathcal{Z}^\square$-definable subsets of $(\mathbb{R}/\mathbb{Z})^n$. Then X either has interior in Y or X is nowhere dense in Y.*

We now show that if $\mathcal{Z}$ is strongly dependent then $\mathcal{Z}^\square$ is o-minimal. We let $\mathrm{Bd}(X)$ be the boundary of a subset X of $\mathbb{R}/\mathbb{Z}$.

Proof. Let $\mathcal{Z}$ be strongly dependent and X be an $\mathcal{Z}^\square$-definable subset of $\mathbb{R}/\mathbb{Z}$. By Fact 8.5 X is not dense and co-dense in any interval. So $\mathrm{Bd}(X)$ is nowhere dense. By Fact 8.4 $\mathrm{Bd}(X)$ has no accumulation points, so $\mathrm{Bd}(X)$ is finite by compactness of $\mathbb{R}/\mathbb{Z}$. By Fact 8.3 X is a finite union of intervals and singletons. □

There is another way to show that $\mathcal{Z}^\square$ is o-minimal when $\mathcal{Z}$ is dp-minimal. Suppose $\mathcal{Z}$ is dp-minimal. Then $\mathcal{N}^{\mathrm{Sh}}$ is dp-minimal, so $\mathcal{Z}^\square$ is dp-minimal by Proposition 8.2. It follows from work of Simon [41] that an expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ is o-minimal if and only if it is dp-minimal.

In Section 8.3 we show $(\mathbb{Z},+,C_\alpha)^\square$ is interdefinable with $(\mathbb{R}/\mathbb{Z},+,C)$ and $(\mathbb{Z},+,C_\alpha)^{\mathrm{Sh}}$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{Z},+,C_\alpha)^\square$ and ψ_α.

8.3. **The completion of $(\mathbb{Z},+,C_\alpha)$.** Proposition 8.6 shows in particular that $(\mathbb{Q},+,<)^\square$ is the usual completion of $(\mathbb{Q},+,<)$.

Proposition 8.6. *Suppose H is a dense subgroup of $(\mathbb{R},+)$. Then $(H,+,<)^\square$ is interdefinable with $(\mathbb{R},+,<)$.*

Proposition 8.6 will require the quantifier elimination for archimedean ordered abelian groups. See Weispfennig [51] for a proof.

Fact 8.7. *Let $(H,+,<)$ be an archimedean ordered abelian group. Then $(H,+,<)$ admits quantifier elimination after adding a unary relation for every nH.*

We now prove Proposition 8.6. If $T : H^n \to H$ is a $\mathbb{Z}$-linear function given by $T(a_1,\ldots,a_n) = k_1a_1 + \ldots + k_na_n$ for integers $k_1,\ldots,k_n$ then we also let T denote the function $\mathbb{R}^n \to \mathbb{R}$ given by $(t_1,\ldots,t_n) \mapsto k_1t_1 + \ldots + k_nt_n$.

Proof. Let $(H+<) \prec (N,+,<)$ be highly saturated and let $\mathbf{Fin}, \mathrm{st} : \mathbf{Fin}^n \to \mathbb{R}^n$ be as above. As $(H,+,<)$ is NIP, it suffices by Fact 8.1 to suppose that $Y \subseteq N^n$ is $\mathcal{N}$-definable and show that $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is $(\mathbb{R},+,<)$-definable. If Z is the closure of Y in N^n then $\mathrm{st}(Z \cap \mathbf{Fin}^n) = \mathrm{st}(Y \cap \mathbf{Fin}^n)$. So we suppose that Y is closed.

As Y is closed a straightforward application of Fact 8.7 shows that Y is a finite union of sets of the form

$$\{a \in N^n : T_1(a) \le s_1, \ldots, T_k(a) \le s_k\}$$

for $\mathbb{Z}$-linear $T_1,\ldots,T_k : N^n \to N$ and $s_1,\ldots,s_k \in N$. So we may suppose that Y is of this form. If $s_i > \mathbf{Fin}$ then $\mathbf{Fin}^n$ is contained in $\{a \in N^n : T_i(a) \le s_i\}$ and if $s_i < \mathbf{Fin}$ then $\{a \in N^n : T_i(a) \le s_i\}$ is disjoint from $\mathbf{Fin}^n$. So we suppose $s_1,\ldots,s_k \in \mathbf{Fin}$. It is now easy to see that

$$\mathrm{st}(Y \cap \mathbf{Fin}^n) = \{a \in \mathbb{R}^n : T_1(a) \le \mathrm{st}(s_1), \ldots, T_k(a) \le \mathrm{st}(s_k)\}.$$

So $\mathrm{st}(Y \cap \mathbf{Fin}^n)$ is $(\mathbb{R},+,<)$-definable. □

Proposition 8.8 will be used to show that $(\mathbb{Z},+,C_\alpha)^{\mathrm{Sh}}$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z},+,C)$ and ψ_α.

Proposition 8.8. *Suppose H is a dense subgroup of $(\mathbb{R},+)$. Then $(H,+,<)^{\mathrm{Sh}}$ is interdefinable with the structure induced on H by $(\mathbb{R},+,<)$.*

Proof. As $(H,+,<)$ is NIP Fact 8.1 shows that the structure induced on H by $(\mathbb{R},+,<)$ is a reduct of $(H,+,<)^{\mathrm{Sh}}$. We show that $(H,+,<)^{\mathrm{Sh}}$ is a reduct of the structure induced on H by $(\mathbb{R},+,<)$. Suppose $(H,+,<) \prec (N,+,<)$ is highly saturated and $Y \subseteq N^n$ is $(N,+,<)$-definable. Applying Fact 8.7 there is a family $(X_{ij} : 1 \le i,j \le k)$ of $(N,+,<)$-definable sets such that $Y = \bigcup_{i=1}^k \bigcap_{j=1}^k X_{ij}$ and each X_{ij} is either $(N,+)$-definable or of the form $\{a \in N^n : T(a) \le s\}$ for some $\mathbb{Z}$-linear $T : N^n \to N$ and $s \in N$. As $H^n \cap Y = \bigcup_{i=1}^k \bigcap_{j=1}^k (H^n \cap X_{ij})$ it is enough to show that each $H^n \cap X_{ij}$ is definable in the structure induced on H by $(\mathbb{R},+,<)$. If X_{ij} is $(N,+)$-definable then $H^n \cap X_{ij}$ is $(H,+)$-definable by stability of abelian groups, So suppose $Y = \{a \in N^n : T(a) \le s\}$ for $\mathbb{Z}$-linear $T : N^n \to N$ and $s \in N$. Let **Fin** and st be as above. If $s > \mathbf{Fin}$ then $H^n \subseteq Y$ and if $s < \mathbf{Fin}$ then H^n is disjoint from Y. Suppose $s \in \mathbf{Fin}$. If $s \ge \mathrm{st}(s)$ then $H^n \cap Y = \{a \in H^n : T(a) \le \mathrm{st}(s)\}$ and if $s < \mathrm{st}(s)$ then $H^n \cap Y = \{a \in H^n : T(a) < \mathrm{st}(x)\}$. So in each case $H^n \cap Y$ is definable in the structure induced on H by $(\mathbb{R}, +, <)$. □

We can now compute $(\mathbb{Z}, +, C_\alpha)^\square$.

Proposition 8.9. *Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Then $(\mathbb{Z}, +, C_\alpha)^\square$ is interdefinable with $(\mathbb{R}/\mathbb{Z}, +, C)$ and $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z}, +, C)$ and ψ_α.*

Proof. Let π be the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ so $(\mathbb{R}, +, <, 1, \pi)$ is a universal cover of $(\mathbb{R}/\mathbb{Z}, +, C)$. Fix $\lambda \in \mathbb{R}$ such that $\alpha = \lambda + \mathbb{Z}$, let $H := \mathbb{Z} + \lambda\mathbb{Z}$, and let $\rho : H \to \mathbb{Z}$ be $\rho := \psi_\alpha^{-1} \circ \pi$, so that $(H, +, <, 1, \rho)$ is a universal cover of $(\mathbb{Z}, +, C_\alpha)$. Let $\rho : H^n \to \mathbb{Z}^n$ be given by $\rho(t_1, \dots, t_n) = (\rho(t_1), \dots, \rho(t_n))$. Suppose $X \subseteq \mathbb{Z}^n$ is $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$-definable. Then $Y := \rho^{-1}(X) \cap [0,1)^n$ is easily seen to be externally definable in $(H, +, <)$. Proposition 8.6 shows that $\mathrm{Cl}(Y)$ is $(\mathbb{R}, +, <)$-definable. Observe that $\pi(\mathrm{Cl}(Y))$ is the closure of $\psi_\alpha(X)$ in $(\mathbb{R}/\mathbb{Z})^n$. So the closure of $\psi_\alpha(X)$ in $(\mathbb{R}/\mathbb{Z})^n$ is definable in $(\mathbb{R}/\mathbb{Z}, +, C)$. So $(\mathbb{Z}, +, C_\alpha)^\square$ is interdefinable with $(\mathbb{R}/\mathbb{Z}, +, C)$.

We now show that $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ is interdefinable with the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z}, +, C)$ and ψ_α. By Proposition 8.2 and preceding paragraph it suffices to show that $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ is a reduct of the induced structure. Again suppose that $X \subseteq \mathbb{Z}^n$ is an $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$-definable subset of $\mathbb{Z}^n$ and $Y := \rho^{-1}(X) \cap [0,1)^n$. By Proposition 8.8 Y is definable in the structure induced on $H \cap [0,1)$ by $(\mathbb{R}, +, <)$. So $\rho(Y) = X$ is definable in the structure induced on $\psi_\alpha(\mathbb{Z})$ by $(\mathbb{R}/\mathbb{Z}, +, C)$. Hence Y is definable in the structure induced on $\mathbb{Z}$ by $(\mathbb{R}/\mathbb{Z}, +, C)$ and ψ_α. □

Corollary 8.10 now follows immediately, we leave the details to the reader.

Corollary 8.10. *Suppose $\mathbb{H}$ is a semialgebraic Mordell-Lang circle group, γ is the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{H}$, $\alpha \in \mathbb{R}/\mathbb{Z}$ is irrational, $\chi : \mathbb{Z} \to \mathbb{H}$ is the character $\chi(k) := \gamma(\alpha k)$, and $\mathcal{H}_\alpha$ is the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ. Then $\mathcal{H}_\alpha$ expands $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$. So in particular $\mathcal{G}_{\alpha,\lambda}, \mathcal{S}_\alpha$, and $\mathcal{E}_{\alpha,\eta}$ all expand $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ for any $\lambda, \eta > 1$.*

By Fact 6.6 a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ cannot add new unary sets. We suspect that any dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$ adds new binary sets.

Proposition 8.11. *Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Then $\mathcal{G}_{\alpha,\lambda}, \mathcal{S}_\alpha$, and $\mathcal{E}_{\alpha,\eta}$ all define a subset of $\mathbb{Z}^2$ which is not $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$-definable for any $\lambda, \eta > 1$.*

An open subset of a topological space is **regular** if it is the interior of its closure.

Proof. We treat $\mathcal{G}_{\alpha,\lambda}$, the other cases follow in the same way. Let S the cyclic order on $[1, \lambda)$ where $S(t, t', t'')$ if and only if either $t < t' < t''$, $t' < t'' < t$ or $t'' < t < t'$. So S is the unique (up to opposite) semialgebraic cyclic group order on $([1, \lambda), \otimes_\lambda)$. Let U be a regular open semialgebraic subset of $[1, \lambda)^2$ which is not definable in $([1, \lambda), \otimes_\lambda, S)$, e.g. an open disc contained in $[1, \lambda)^2$. Let $V := \chi_\alpha^{-1}(U)$, so V is $\mathcal{G}_{\alpha,\lambda}$-definable. Suppose that V is $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$-definable. By Proposition 8.9 the closure of $\chi_\alpha(V)$ is definable in $([1,\lambda),\otimes_\lambda,S)$. As $\chi_\alpha(\mathbb{Z}^2)$ is dense in $[1,\lambda)^2$, the closure of $\chi_\alpha(V)$ agrees with the closure of U. As U is regular U is the interior of the closure of U. So U is definable in $([1,\lambda),\otimes_\lambda,S)$, contradiction. $\square$

9. When the examples are interdefinable

In this section we describe the completions of the dp-minimal expansions of $(\mathbb{Z},+,C_\alpha)$ constructed in Section 7 and show that if two of these expansions are interdefinable then the associated semialgebraic circle groups are semialgebracially isomorphic.

Suppose $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R},+,\times)$, $\mathbb{H}$ is an $\mathcal{R}$-definable circle group. We say that a subgroup A of $\mathbb{H}$ is a **GH-subgroup** if $(\mathcal{R},A)$ is NIP, $\mathrm{Th}(\mathcal{R})$ is an open core of $\mathrm{Th}(\mathcal{R},A)$, and the structure induced on A by $\mathcal{R}$ is dp-minimal.

Proposition 9.1. *Suppose $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R},+,\times)$, $\mathbb{H}$ is an $\mathcal{R}$-definable circle group, γ is the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z}\to\mathbb{H}$, χ is an injective character $\mathbb{Z}\to\mathbb{H}$, and $\mathcal{Z}$ is the structure induced on $\mathbb{Z}$ by $\mathcal{R}$ and χ. If $\chi(\mathbb{Z})$ is a* GH*-subgroup then $\mathcal{Z}^\square$ is interdefinable with the structure induced on $\mathbb{R}/\mathbb{Z}$ by $\mathcal{R}$ and γ. So for any irrational $\alpha\in\mathbb{R}/\mathbb{Z}$ and $\lambda>1$:*

1. *A subset of $(\mathbb{R}/\mathbb{Z})^n$ is $\mathcal{G}^\square_{\alpha,\lambda}$-definable if and only if it is the image under the quotient map $\mathbb{R}^n\to(\mathbb{R}/\mathbb{Z})^n$ of a set of the form*
$$\{(t_1,\ldots,t_n):(\lambda^{t_1},\ldots,\lambda^{t_n})\in X\}$$
for a semialgebraic subset X of $[1,\lambda)^n$.
2. *A subset of $(\mathbb{R}/\mathbb{Z})^n$ is $\mathcal{S}^\square_\alpha$-definable if and only if it is the image under the quotient map $\mathbb{R}^n\to(\mathbb{R}/\mathbb{Z})^n$ of a set of the form*
$$\{(t_1,\ldots,t_n)\in[0,1)^n:(e^{2\pi i t_1},\ldots,e^{2\pi i t_n})\in X\}$$
for a semialgebraic subset X of $\mathbb{S}^n$.
3. *A subset of $(\mathbb{R}/\mathbb{Z})^n$ is $\mathcal{E}^\square_{\alpha,\lambda}$-definable if and only if it is an image under the quotient map $\mathbb{R}^n\to(\mathbb{R}/\mathbb{Z})^n$ of a set of the form*
$$\{(t_1,\ldots,t_n)\in[0,1)^n:(\mathfrak{p}_\lambda(t_1),\ldots,\mathfrak{p}_\lambda(t_n))\in X\}$$
for a semialgebraic subset X of $\mathbb{E}^0_\lambda(\mathbb{R})^n$.

It follows from Proposition 9.1 that if $\mathcal{Z}$ is one of the expansions of $(\mathbb{Z},+,C_\alpha)$ described above then $\mathcal{Z}^\square$ defines an isomorphic copy of $(\mathbb{R},+,\times)$, so if $\mathcal{Z}\prec\mathcal{N}$ is highly saturated then $\mathcal{N}^{\mathrm{Sh}}$ interprets $(\mathbb{R},+,\times)$. So $\mathcal{Z}$ is non-modular. An adaptation of [50, Proposition 15.2] shows that $\mathcal{N}$ cannot interpret an infinite field.

We prove Theorem 9.2, a more general result on completions which covers almost all "dense pairs". It is easy to see that Proposition 9.1 follows from Theorem 9.2, we leave the details of this to the reader.

Theorem 9.2. *Let $\mathcal{S}$ be an o-minimal expansion of $(\mathbb{R},+,<)$. Suppose A is a subset of $\mathbb{R}^m$ such that $(\mathcal{S},A)$ is* NIP *and* $\mathrm{Th}(\mathcal{S})$ *is an open core of* $\mathrm{Th}(\mathcal{S},A)$. *Let $\mathcal{A}$ be the structure induced on A by $\mathcal{S}$ and X be the closure of A in $\mathbb{R}^m$. Then*

1. *the structure $\mathcal{A}^\square$ with domain X and an n-ary relation symbol defining* $\mathrm{Cl}(Y)$ *for each $\mathcal{A}^{\mathrm{Sh}}$-definable $Y\subseteq A^n$.*
2. *and the structure $\mathcal{X}$ induced on X by $\mathcal{S}$,*

are interdefinable. (Note that X is $\mathcal{S}$-definable.)

We let

$$\|a\| := \max\{|a_1|, \dots, |a_n|\} \quad \text{for all } a = (a_1, \dots, a_n) \in \mathbb{R}^n.$$

We will need a metric argument from [50] to show that $\mathcal{X}$ is a reduct of $\mathcal{A}^\square$. If $X = \mathbb{R}^m$ then one can can give a topological proof following [49, Proposition 3.4].

Proof. We first show that $\mathcal{X}$ is a reduct of $\mathcal{A}^\square$. Suppose Y is a nonempty $\mathcal{S}$-definable subset of X^n. By o-minimal cell decomposition there are definable closed subsets $E_1, F_1 \dots, E_k, F_k$ of $\mathbb{R}^{nm}$ such that $Y = \bigcup_{i=1}^k (E_i \setminus F_i)$. We have

$$Y = \bigcup_{i=1}^{k} ((X^n \cap E_i) \setminus (X^n \cap F_i))$$

so we may suppose that Y is a nonempty closed $\mathcal{S}$-definable subset of X^n. Let W be the set of $(a, a', c) \in X \times X \times X^n$ for which there is $c' \in Y$ satisfying $\|c - c'\| < \|a - a'\|$. So $W \cap (A \times A \times A^n)$ is $\mathcal{A}$-definable and $Z := \mathrm{Cl}(W \cap (A \times A \times A^n))$ is $\mathcal{A}^\square$-definable. The metric argument in the proof of [50, Lemma 13.5] shows that

$$Y = \bigcap_{a, a' \in X, a \neq a'} \{c \in X : (a, a', c) \in Z\}.$$

(This metric argument requires Y to be closed.) So Y is $\mathcal{A}^\square$-definable.

We now show that $\mathcal{X}$ is a reduct of $\mathcal{A}^\square$. Suppose Y is an $\mathcal{A}^{\mathrm{Sh}}$-definable subset of A^n. We show that $\mathrm{Cl}(Y) \subseteq X^n$ is $\mathcal{S}$-definable. As $(\mathcal{S}, A)$ is NIP, $\mathcal{A}$ is NIP, so an application of Fact 3.5 yields an $\mathcal{A}$-definable family $(Y_a : a \in A^k)$ of subsets of A^n such that for every finite $B \subseteq Y$ we have $B \subseteq Y_a \subseteq Y$ for some $a \in A^k$. As $\mathrm{Th}(\mathcal{S})$ is an open core of $\mathrm{Th}(\mathcal{S}, A)$ there is an $\mathcal{S}$-definable family $(Z_b : b \in \mathbb{R}^l)$ of subsets of $\mathbb{R}^{nm}$ such that for every $a \in A^k$ we have $\mathrm{Cl}(Y_a) = Z_b$ for some $b \in \mathbb{R}^l$. So for every finite $F \subseteq X$ there is $b \in \mathbb{R}^l$ such that $F \subseteq Z_b \subseteq \mathrm{Cl}(Y)$. A saturation argument yields an $\mathcal{R}^{\mathrm{Sh}}$-definable subset Z of X^n such that $Y \subseteq Z \subseteq \mathrm{Cl}(Y)$. An application of Fact 3.6 shows that Z is $\mathcal{S}$-definable, so $\mathrm{Cl}(Z) = \mathrm{Cl}(Y)$ is $\mathcal{S}$-definable. $\square$

The proof of Theorem 9.2 goes through for any expansion $\mathcal{S}$ of $(\mathbb{R}, +, <)$ such that $\mathcal{S}$ is NIP, every $\mathcal{S}$-definable set is a boolean combination of definable closed sets, and $\mathcal{S}^{\mathrm{Sh}}$ is interdefinable with $\mathcal{S}$. So for example Theorem 9.2 holds when $\mathcal{S} = (\mathbb{R}, +, <, \mathbb{Z})$.

Our next goal is to show that if $\mathbb{H}$ and $\mathcal{Z}$ are as in Proposition 9.1 then we can recover $\mathcal{R}$ and $\mathbb{H}$ from $\mathcal{Z}$. We show that we can recover $\mathcal{R}$ and $\mathbb{H}$ from $\mathcal{Z}^\square$. This follows from a general correspondence between

(1) non-modular o-minimal expansions $\mathcal{C}$ of $(\mathbb{R}/\mathbb{Z}, +, C)$, and
(2) pairs of the form $\langle \mathcal{R}, \mathbb{H} \rangle$, for an o-minimal expansion $\mathcal{R}$ of $(\mathbb{R}, +, \times)$ and an $\mathcal{R}$-definable circle group $\mathbb{H}$.

In this correspondence $\mathcal{C}$ is unique up to interdefinibility, $\mathcal{R}$ is unique up to interdefinibility, and $\mathbb{H}$ is unique up to $\mathcal{R}$-definable isomorphism.

Suppose that $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, \times)$ and $\mathbb{H}$ is an $\mathcal{R}$-definable circle group. We consider $\mathbb{H}$ as a topological group with $\mathcal{T}_{\mathbb{H}}$. Let γ be the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{H}$. Let $\mathcal{C}$ be the structure induced on $\mathbb{R}/\mathbb{Z}$ by $\mathcal{R}$ and γ. Note $\mathcal{C}$ is unique up to interdefinibility. It is easy to see that

$\mathcal{C}$ defines an isomorphic copy of $(\mathbb{R}, +, \times)$.

Now suppose $\mathcal{C}$ is a non-modular o-minimal expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$. Suppose I is a non-empty open interval and $\oplus, \otimes : I^2 \to I$ are $\mathcal{C}$-definable such that $(I, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, +, \times)$. Let ι be the unique isomorphism $(\mathbb{R}, +, \times) \to (I, \oplus, \otimes)$. Let $\mathcal{R}$ be the structure induced on $\mathbb{R}$ by $\mathcal{C}$ and ι. By compactness of $\mathbb{R}/\mathbb{Z}$ there is a finite $A \subseteq \mathbb{R}/\mathbb{Z}$ such that $(a + I : a \in A)$ covers $\mathbb{R}/\mathbb{Z}$. Fix a bijection $f : B \to A$ for some $B \subseteq \mathbb{R}$. Let $\tau : B \times \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the surjection given by $\tau(b, t) = f(b) + \iota(t)$. Observe that equality modulo τ is an $\mathcal{R}$-definable equivalence relation and, applying definable choice, let H be an $\mathcal{R}$-definable subset of $B \times \mathbb{R}$ which contains one element from each fiber of τ. Let $\tau' : H \to \mathbb{R}/\mathbb{Z}$ be the induced bijection and $\boxplus$ be the pullback of $+$ by τ'. Then $\mathbb{H} := (H, \boxplus)$ is an $\mathcal{R}$-definable circle group. Note that the expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ associated to $\langle \mathcal{R}, \mathbb{H} \rangle$ is interdefinable with $\mathcal{C}$.

Proposition 9.3. *For $i \in \{0, 1\}$ suppose that $\mathcal{R}_i$ is an o-minimal expansion of $(\mathbb{R}, +, \times)$, $\mathbb{H}_i$ is an $\mathcal{R}_i$-definable circle group, and $\mathcal{C}_i$ is the expansion of $(\mathbb{R}/\mathbb{Z}, +, C)$ associated to $\langle \mathcal{R}_i, \mathbb{H}_i \rangle$. If $\mathcal{C}_0$ and $\mathcal{C}_1$ are interdefinable then $\mathcal{R}_0$ and $\mathcal{R}_1$ are interdefinable and there is an $\mathcal{R}_0$-definable group isomorphism $\mathbb{H}_0 \to \mathbb{H}_1$.*

Proof. It is easy to see that $\mathcal{C}_0$ is bi-interpretable with $\mathcal{R}_0$ and $\mathcal{C}_1$ is bi-interpretable with $\mathcal{R}_1$. So if $\mathcal{C}_0$ and $\mathcal{C}_1$ are interdefinable then $\mathcal{R}_0$ and $\mathcal{R}_1$ are bi-interpretable, hence interdefinable by Fact 5.3. So we suppose $\mathcal{R}_0 = \mathcal{R}_1$ and denote $\mathcal{R}_0$ by $\mathcal{R}$.

For each $i \in \{0, 1\}$ let I_i be an interval in $\mathbb{R}/\mathbb{Z}$ and ι_i be a bijection $\mathbb{R} \to I_i$ such that $\mathcal{R}$ is interdefinable with the structure induced on $\mathbb{R}$ by $\mathcal{C}_i$ and ι_i. Let $\mathcal{F}_i$ be the pushforward of $\mathcal{R}$ by ι_i for $i \in \{0, 1\}$. So $\mathcal{F}_0$ is a $\mathcal{C}_0$-definable copy of $\mathcal{R}$ and $\mathcal{F}_1$ is a $\mathcal{C}_1$-definable copy of $\mathcal{R}$. Let $\mathbb{H}_{00}, \mathbb{H}_{01}$ be the pushforward of $\mathbb{H}_0, \mathbb{H}_1$ by ι_0, respectively. Likewise, let $\mathbb{H}_{10}, \mathbb{H}_{11}$ be the pushforward of $\mathbb{H}_0, \mathbb{H}_1$ by ι_1, respectively. So $\mathbb{H}_{00}$, $\mathbb{H}_{01}$ are $\mathcal{F}_0$-definable copies of $\mathbb{H}_0$, $\mathbb{H}_1$, respectively, and $\mathbb{H}_{10}$, $\mathbb{H}_{11}$ are $\mathcal{F}_1$-definable copies of $\mathbb{H}_0$, $\mathbb{H}_1$, respectively. Given $i \in \{0, 1\}$ let γ_i be a $\mathcal{C}_i$-definable group isomorphism $\mathbb{H}_{i0} \to \mathbb{R}/\mathbb{Z}$. (Note that $\mathcal{C}_0$ a priori does not define a group isomorphism from $\mathbb{H}_{01}$ to $\mathbb{R}/\mathbb{Z}$, likewise for $\mathcal{C}_1$ and $\mathbb{H}_{10}$.)

Now suppose that $\mathcal{C}_0$ and $\mathcal{C}_1$ are interdefinable. We show that $\mathbb{H}_0$ and $\mathbb{H}_1$ are $\mathcal{R}$-definably isomorphic. It suffices to show that $\mathbb{H}_{00}$ and $\mathbb{H}_{01}$ are $\mathcal{F}_0$-definably isomorphic. As $\mathcal{F}_0$ and $\mathcal{C}_0$ are bi-interpretable it is enough to produce a $\mathcal{C}_0$-definable group isomorphism $\mathbb{H}_{00} \to \mathbb{H}_{01}$. As $\mathcal{C}_0$ and $\mathcal{C}_1$ are interdefinable $\gamma_1^{-1} \circ \gamma_0$ is a $\mathcal{C}_0$-definable group isomorphism $\mathbb{H}_{00} \to \mathbb{H}_{11}$. By Fact 5.3 there is a $\mathcal{C}_0$-definable bijection $\xi : I_1 \to I_0$ which induces an isomorphism (up to interdefinibility) from $\mathcal{F}_1$ to $\mathcal{F}_0$. Let ζ be the $\mathcal{C}_0$-definable group isomorphism $\mathbb{H}_{11} \to \mathbb{H}_{01}$ induced by ξ. Then $\zeta \circ \gamma_1 \circ \gamma_0^{-1}$ is a $\mathcal{C}_0$-definable group isomorphism $\mathbb{H}_{00} \to \mathbb{H}_{01}$. □

Theorem 9.4 classifies our examples up to interdefinibility.

Theorem 9.4. *Let $\mathcal{R}_0, \mathcal{R}_1, \mathbb{H}_0, \mathbb{H}_1$ be as in Proposition 9.3. Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. For each $i \in \{0, 1\}$ let $\gamma_i : \mathbb{R}/\mathbb{Z} \to \mathbb{H}_i$ be the unique (up to sign) topological group isomorphism, $\chi_i : \mathbb{Z} \to \mathbb{H}_i$ be given by $\chi_i(k) := \gamma_i(\alpha k)$, and $\mathcal{Z}_i$ be the structure induced on $\mathbb{Z}$ by $\mathcal{R}_i$ and χ_i. Suppose $\chi_i(\mathbb{Z})$ is a* GH*-subgroup for $i \in \{0, 1\}$. Then $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are interdefinable if and only if $\mathcal{R}_0$ and $\mathcal{R}_1$ are interdefinable and there is an $\mathcal{R}_0$-definable group isomorphism $\mathbb{H}_0 \to \mathbb{H}_1$.*

Proof. Suppose that $\mathcal{R}_0$ and $\mathcal{R}_1$ are interdefnable and $\xi : \mathbb{H}_0 \to \mathbb{H}_1$ is an $\mathcal{R}_0$-definable group isomorphism. Then $\xi \circ \gamma_0$ is the unique (up to sign) topological group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{H}_1$. So after possibly replacing ξ with $-\xi$ we have $\gamma_1 = \xi \circ \gamma_0$, hence $\chi_1 = \xi \circ \chi_0$. It easily follows that $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are interdefinable.

Suppose $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are interdefinable. Then $\mathcal{Z}_0^\square$ and $\mathcal{Z}_1^\square$ are interdefinable. By Proposition 9.1 the expansions of $(\mathbb{R}/\mathbb{Z}, +, C)$ associated to $\langle \mathcal{R}_0, \mathbb{H}_0 \rangle$ and $\langle \mathcal{R}_1, \mathbb{H}_1 \rangle$ are interdefinable. Applying Proposition 9.3 see that $\mathcal{R}_0$ and $\mathcal{R}_1$ are interdefinable and there is an $\mathcal{R}_0$-definable group isomorphism $\mathbb{H}_0 \to \mathbb{H}_1$. □

We now see that we have constructed uncountably many dp-minimal expansions of each $(\mathbb{Z}, +, C_\alpha)^{\mathrm{Sh}}$. Corollary 9.5 follows from Theorem 9.4 and the classification of one-dimensional semialgebraic groups described above.

Corollary 9.5. *Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$ and let $\lambda, \eta > 1$. Then*

(1) no two of $\mathcal{G}_{\alpha,\lambda}, \mathcal{S}_\alpha, \mathcal{E}_{\alpha,\eta}$ are interdefinable,
(2) $\mathcal{G}_{\alpha,\lambda}$ and $\mathcal{G}_{\alpha,\eta}$ are interdefinable if and only if $\lambda/\eta \in \mathbb{Q}$,
(3) $\mathcal{E}_{\alpha,\lambda}$ and $\mathcal{E}_{\alpha,\eta}$ are interdefinable if and only if $\lambda/\eta \in \mathbb{Q}$.

Suppose for the rest of this section that Conjecture 2 holds. Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Suppose that $\mathbb{H}$ is a semialgebraic Mordell-Lang circle group, $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ is the unique (up to sign) topological group isomorphism, and $\chi : \mathbb{Z} \to \mathbb{H}$ is the character $\chi(k) := \gamma(\alpha k)$. Let $\mathcal{H}_\alpha$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times)$ and χ. For any $f \in \Lambda$ let $\mathcal{H}_{\alpha,f}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{R}, +, \times, f)$ and χ. Then $\mathcal{H}_{\alpha,f}$ is dp-minimal and $\mathcal{H}_{\alpha,f}^\square$ is interdefinable with the structure induced on $\mathbb{R}/\mathbb{Z}$ by $(\mathbb{R}, +, \times, f)$ and γ. It follows that by Proposition 9.1 that $\mathcal{H}_{\alpha,f}$ is a proper expansion of $\mathcal{H}_\alpha$ and if f, g are distinct elements of Λ then $\mathcal{H}_{\alpha,f}$ and $\mathcal{H}_{\alpha,g}$ are not interdefinable.

In this way, still assuming Conjecture 2, we can produce produce two dp-minimal expansions of $\mathcal{H}_\alpha$ which do not have a common NIP expansion. Let $h \in C^\infty(I)$ be such that $(\mathbb{R}, +, \times, h)$ is not NIP. (For example one can arrange that $I = [0, 1]$ and $\{t \in I : h(t) = 0\}$ is $0 \cup \{1/n : n \geq 1\}$.) As Λ is comeager an application of the Pettis lemma [23, Theorem 9.9] implies that there are $f, g \in \Lambda$ and $t > 0$ such that $f - g = th$. So after rescaling h we suppose $f - g = h$. Suppose that $\mathcal{Z}$ is an NIP expansion of both $\mathcal{H}_{\alpha,f}$ and $\mathcal{H}_{\alpha,g}$. Then $\mathcal{Z}^\square$ is NIP. An easy argument using the first part of the proof of Theorem 9.2 shows that $(\mathbb{R}, +, \times, f, g)$ is interpretable in $\mathcal{Z}^\square$, contradiction. (This kind of argument was previously used by Le Gal [26] to show that there are two o-minimal expansions of $(\mathbb{R}, +, \times)$ which are not reducts of a common o-minimal structure.)

10. Dp-minimal expansions of $(\mathbb{Z}, +, \mathrm{Val_p})$

Throughout p is a fixed prime. To avoid mild technical issues we assume $p \neq 2$. It is shown in [13] that $(\mathbb{Z}_p, +, \times)$ is dp-minimal.

10.1. A proper dp-minimal expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$.

We apply work of Mariaule. The first and third claims of Fact 10.1 are special cases of the results of [30]. The second claim follows from Mariaule's results and a general theorem of Boxall and Hieronymi on open cores [6]. Recall that $1 + p\mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_p^\times$.

Fact 10.1. *Suppose that A is a finitely generated dense subgroup of* $(1+p\mathbb{Z}_p,\times)$. *Then* $(\mathbb{Z}_p,+,\times,A)$ *is* NIP, $\mathrm{Th}(\mathbb{Z}_p,+,\times)$ *is an open core of* $\mathrm{Th}(\mathbb{Z}_p,+,\times,A)$*, and every* $(\mathbb{Z}_p,+,\times,A)$*-definable subset of* A^k *is of the form* $X\cap Y$ *where* X *is an* $(A,\times)$*-definable subset of* A^k *and* Y *is a semialgebraic subset of* $\mathbb{Z}_p^k$.

We let Exp be the p-adic exponential, i.e.

$$\mathrm{Exp}(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!} \quad \text{for all } a\in p\mathbb{Z}_p.$$

(The sum does not converge off $p\mathbb{Z}_p$.) Exp is a topological group isomorphism $(p\mathbb{Z}_p,+)\to(1+p\mathbb{Z}_p,\times)$. So $a\mapsto\mathrm{Exp}(pa)$ is a topological group isomorphism $(\mathbb{Z}_p,+)\to(1+p\mathbb{Z}_p,\times)$. It is easy to see that

$$\mathrm{Val_p}(\mathrm{Exp}(a)-1)=\mathrm{Val_p}(a) \quad \text{for all } a\in p\mathbb{Z}_p.$$

So for all $a\in\mathbb{Z}_p$ we have

$$\mathrm{Val_p}(\mathrm{Exp}(pa)-1)=\mathrm{Val_p}(pa)=\mathrm{Val_p}(a)+1.$$

Define $\mathbf{v}(b)=\mathrm{Val_p}(b-1)-1$ for all $b\in 1+p\mathbb{Z}_p$, so $a\mapsto\mathrm{Exp}(pa)$ is an isomorphism $(\mathbb{Z}_p,+,\mathrm{Val_p})\to(1+p\mathbb{Z}_p,\times,\mathbf{v})$.

We let $\chi:\mathbb{Z}\to\mathbb{Z}_p^\times$ be the character $\chi(k):=\mathrm{Exp}(pk)$ and let $\mathcal{P}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{Z}_p,+,\times)$ and χ. Note that $\mathcal{P}$ expands $(\mathbb{Z},+,\mathrm{Val_p})$ because χ is an isomorphism $(\mathbb{Z},+,\mathrm{Val_p})\to(\chi(\mathbb{Z}),\times,\mathbf{v})$. Let

$$\chi(k_1,\dots,k_n)=(\chi(k_1),\dots,\chi(k_n)) \quad \text{for all } (k_1,\dots,k_n)\in\mathbb{Z}^n.$$

There are $\mathcal{P}$-definable subsets of $\mathbb{Z}$ which are not $(\mathbb{Z},+,\mathrm{Val_p})$-definable. Consider $\mathbb{N}\cup\{\infty\}$ as the value set of $(\mathbb{Z},+,\mathrm{Val_p})$. It follows from the quantifier elimination for $(\mathbb{Z},+,\mathrm{Val_p})$ that the structure induced on $\mathbb{N}\cup\{\infty\}$ by $(\mathbb{Z},+,\mathrm{Val_p})$ is interdefinable with $(\mathbb{N}\cup\{\infty\},<)$. So $\mathrm{Val_p}^{-1}(2\mathbb{N})$ is $\mathcal{P}$-definable and not $(\mathbb{Z},+,\mathrm{Val_p})$-definable. Let E be the set of $a\in 1+p\mathbb{Z}_p$ such that $\mathbf{v}(a)\in 2\mathbb{N}$. Note that if $b,c\in 1+p\mathbb{Z}_p$ then $\chi^{-1}(bE)=\chi^{-1}(cE)$ if and only if $b=c$. So $\mathcal{P}$ defines uncountably many subsets of $\mathbb{Z}$, in constrast $(\mathbb{Z},+,\mathrm{Val_p})$ defines only countably many subsets of $\mathbb{Z}$.

Proposition 10.2. $\mathcal{P}$ *is dp-minimal.*

Proposition 10.2 requires some preliminaries. A formula $\vartheta(x;y)$ is **bounded** if $\mathcal{P}\models\forall y\exists^{\le n}x\vartheta(x;y)$ for some n. Let L_{Ab} be the language of abelian groups together with unary relations $(D_n)_{n\ge1}$ and L_{Val} be the expansion of L_{Ab} by a binary relation $\preccurlyeq_p$ We let D_n define $n\mathbb{Z}$ and declare $k\preccurlyeq_p k'$ if and only if $\mathrm{Val_p}(k)\le\mathrm{Val_p}(k')$. Fact 10.3 was independently proven by Alouf and d'Elbée [2], Mariaule [29], and Guignot [16].

Fact 10.3. $(\mathbb{Z},+,\mathrm{Val_p})$ *has quantifier elimination in* L_{Val}.

We let L_{In} be the language with an n-ary relation symbol defining $\chi^{-1}(X)$ for each semialgebraic $X\subseteq\mathbb{Z}_p^n$. So $Y\subseteq\mathbb{Z}^n$ is quantifier free L_{In}-definable if and only if $Y=\chi^{-1}(X)$ for semialgebraic $X\subseteq\mathbb{Z}_p^n$. Take $\mathcal{P}$ to be an $(L_{\mathrm{Val}}\cup L_{\mathrm{In}})$-structure.

We first give a description of unary $\mathcal{P}$-definable sets.

Lemma 10.4. *Suppose $\varphi(x;y), |x| = 1$ is an $(L_{\mathrm{Val}} \cup L_{\mathrm{In}})$-formula. Then $\varphi(x;y)$ is equivalent to a finite disjunction of formulas of the form $\varphi_1(x;y) \wedge \varphi_2(x;y)$ where $\varphi_2(x;y)$ is a quantifier free L_{In}-formula and $\varphi_1(x;y)$ is an L_{Ab}-formula such that either $\varphi_1(x;y)$ is bounded or there are integers k, l such that $\mathrm{Val}_{\mathrm{p}}(k) = 0$ and for every $b \in \mathbb{Z}^{|y|}$, $\varphi_1(\mathbb{Z};b) = (k\mathbb{Z} + l) \setminus A$ for finite A.*

The condition $\mathrm{Val}_{\mathrm{p}}(k) = 0$ ensures that each $\varphi_1(\mathbb{Z};b)$ is dense in the p-adic topology.

Proof. By Fact 10.1 we may suppose $\varphi(x;y) = \varphi_1(x;y) \wedge \varphi_2(x;y)$ where φ_1 is an L_{Val}-formula and φ_2 is a quantifier free L_{In}-formula. By Fact 10.3 we may suppose

$$\varphi_1(x;y) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{m} \vartheta_{ij}(x;y)$$

where each $\vartheta_{ij}(x;y)$ is an atomic L_{Val}-formula. So we may suppose

$$\varphi(x;y) = \bigvee_{i=1}^{m} \left(\varphi_2(x;y) \wedge \bigwedge_{j=1}^{m} \vartheta_{ij}(x;y) \right).$$

So we suppose $\varphi(x;y)$ is of the form $\varphi_2(x;y) \wedge \bigwedge_{j=1}^{m} \vartheta_j(x;y)$ where each $\vartheta_j(x;y)$ is an atomic L_{Val}-formula. After possibly rearranging there is $0 \le m' \le m$ such that

1. if $1 \le j \le m'$ then $\vartheta_j(x;y)$ is of the form $g \preccurlyeq_p h$ or $g \prec_p h$ where g, h are L_{Ab}-terms in the variables x, y, and
2. if $m' < j \le m$ then $\vartheta_j(x;y)$ is an atomic L_{Ab}-formula.

Note that any formula of type (1) is equivalent to a quantifier free L_{In}-formula. Now

$$\left(\varphi_2(x;y) \wedge \bigwedge_{j=1}^{m'} \vartheta_j(x;y) \right) \wedge \bigwedge_{j=m'+1}^{m} \vartheta_j(x;y).$$

The formula inside the parentheses is equivalent to a quantifier free L_{In}-formula. So we suppose that $\varphi(x;y)$ is for the form $\varphi_1(x;y) \wedge \varphi_2(x;y)$ where $\varphi_1(x;y)$ is an L_{Ab}-formula and $\varphi_2(x;y)$ is an quantifier free L_{In}-formula. An easy application of quantifier elimination shows that $\varphi_1(x;y)$ is equivalent to a formula of the form $\bigvee_{i=1}^{m} \theta_i(x;y)$ where for each i either:

1. $\theta_i(x;y)$ is bounded, or
2. there are integers $k \neq 0, l$ and an L_{Ab}-formula $\theta'_i(x;y)$ such that $\theta_i(x;y)$ is equivalent to $(x \in k'\mathbb{Z} + l) \wedge \neg\theta'_i(x;y)$ and $\theta'_i(x;y)$ is bounded.

Applying the same reasoning as above we may suppose that $\varphi_1(x;y)$ satisfies (1) or (2) above. If $\varphi_1(x;y)$ is bounded then we are done. So fix integers k', l and bounded $\varphi'_1(x;y)$ such that $\varphi_1(x;y)$ is equivalent to $(x \in k'\mathbb{Z} + l) \wedge \neg\varphi'_1(x;y)$. Let $v := \mathrm{Val}_{\mathrm{p}}(k')$ and $k := k'/p^v$. So $k'\mathbb{Z} + l = (p^v\mathbb{Z} + l) \cap (k\mathbb{Z} + l)$ and $\varphi_1(x;y)$ is logically equivalent to

$$[\mathrm{Val}_{\mathrm{p}}(x - l) \ge v] \wedge [x \in k\mathbb{Z} + l] \wedge \neg\varphi'_1(x;y).$$

After replacing $\varphi_1(x;y)$ with $[x \in k\mathbb{Z} + l] \wedge \neg\varphi'_1(x;y)$ and replacing $\varphi_2(x;y)$ with $[\mathrm{Val}_{\mathrm{p}}(x - l) \ge v] \wedge \varphi_2(x;y)$ we may suppose that for every $b \in \mathbb{Z}^{|y|}$, $\varphi_1(\mathbb{Z};b)$ agrees with $(k\mathbb{Z} + l) \setminus A$ for finite A. ∎

We also need Fact 10.5, a consequence of the quantifier elimination for $(\mathbb{Z}_p, +, \times)$.

Fact 10.5. *Suppose that $\phi(x; y), |x| = 1$ is a formula in the language of rings. Then there are formulas $\phi_1(x; y), \phi_2(x; y)$ such that*

(1) $\phi(x; y)$ and $\phi_1(x; y) \vee \phi_2(x; y)$ are equivalent in $(\mathbb{Z}_p, +, \times)$,
(2) $\phi_1(\mathbb{Z}_p; b)$ is finite and $\phi_2(\mathbb{Z}_p; b)$ is open for every $b \in \mathbb{Z}_p^{|y|}$.

Lemma 10.6 follows from inp-minimality of $(\mathbb{Z}_p, +, \times)$ and the fact that $\chi(X)$ is dense in $1 + p\mathbb{Z}_p$. We leave the verification to the reader.

Lemma 10.6. *Suppose that $\varphi(x; y), \phi(x; y), |x| = 1$ are quantifier free L_{In}-formulas such that $\varphi(\mathbb{Z}; b)$ and $\phi(\mathbb{Z}; b)$ are both open in the p-adic topology for every $b \in \mathbb{Z}^{|y|}$. Then $\varphi(x; y)$ and $\phi(x; y)$ cannot violate inp-minimality.*

We now prove Proposition 10.2.

Proof. We equip $\mathbb{Z}$ with the p-adic topology. Fact 10.1 shows that $\mathcal{P}$ is NIP so it is enough to show that $\mathcal{P}$ is inp-minimal. Suppose towards a contradiction that $\varphi(x; y)$, $\phi(x; z)$, and n violate inp-minimality. Applying Lemma 10.4, Fact 3.1, and Fact 3.2 we may suppose there are $\varphi_1(x; y), \varphi_2(x; y)$ and k, l such that

(1) $\varphi(x; y) = \varphi_1(x; y) \wedge \varphi_2(x; y)$,
(2) $\varphi_2(x; y)$ is a quantifier free L_{In}-formula, and
(3) $\mathrm{Val}_{\mathrm{p}}(k) = 0$ and for all $b \in \mathbb{Z}^{|y|}$, $\varphi_1(\mathbb{Z}; b) = (k\mathbb{Z} + l) \setminus A$ for finite A.

Applying Fact 10.5 we get L_{In}-formulas $\varphi_2'(x; y)$ and $\varphi_2''(x; y)$ such that $\varphi_2'(x; y)$ is bounded, $\varphi_2''(\mathbb{Z}; b)$ is open for all $b \in \mathbb{Z}^{|y|}$, and $\varphi_2(x; y) = \varphi_2'(x; y) \vee \varphi_2''(x; y)$. Applying Facts 3.1 and 3.2 we may suppose that $\varphi_2(\mathbb{Z}; b)$ is open for all $b \in \mathbb{Z}^{|y|}$.

We have reduced to the case when $\varphi(x; y) = \varphi_1(x; y) \wedge \varphi_2(x; y)$ where $\varphi_2(x; y)$ is a quantifier free L_{In}-formula such that each $\varphi_2(\mathbb{Z}; b)$ is open and there are k, l such that $\mathrm{Val}_{\mathrm{p}}(k) = 0$ and for all $b \in \mathbb{Z}^{|y|}$ we have $\varphi_1(\mathbb{Z}; b) = (k\mathbb{Z} + l) \setminus A$ for finite A. By the same reasoning we may suppose that there are $\phi_1(x; z), \phi_2(x; z)$ and k', l' which satisfy the same conditions with respect to $\phi(x; z)$.

We show that $\varphi_2(x; y), \phi_2(x; z)$ and n violate inp-minimality and thereby obtain a contradiction with Lemma 10.6. Fix $a_1, \ldots, a_m \in \mathbb{Z}^{|y|}$ and $b_1, \ldots, b_m \in \mathbb{Z}^{|z|}$ such that $\varphi(x; a_1), \ldots, \varphi(x; a_m)$ and $\phi(x; b_1), \ldots, \phi(x; b_m)$ are both n-inconsistent and $\mathcal{P} \models \exists x \varphi(x; a_i) \wedge \phi(x; a_j)$ for all i, j. So $\mathcal{P} \models \exists x \varphi_2(x; a_i) \wedge \phi_2(x; b_j)$ for all i, j. It suffices to show that $\varphi_2(x; a_1), \ldots, \varphi_2(x; a_m)$ and $\phi_2(x; b_1), \ldots, \phi_2(x; b_m)$ are both n-inconsistent. We prove this for φ_2, the same argument works for ϕ_2. Fix a subset I of $\{1, \ldots, m\}$ such that $|I| = n$. Let $U := \bigcap_{i \in I} \varphi_2(\mathbb{Z}; a_i)$ and $F := \bigcap_{i \in I} \varphi_1(\mathbb{Z}; a_i)$. So $F \cap U$ is empty as $\varphi(x; a_1), \ldots, \varphi(x; a_m)$ is n-inconsistent. Observe that U is open and $F = (k\mathbb{Z} + l) \setminus A$ for finite A. So F is dense in $\mathbb{Z}$ as $\mathrm{Val}_{\mathrm{p}}(k) = 0$. So $F \cap U$ is the intersection of a dense set and an open set, so U is empty. Thus $\varphi_2(x; a_1), \ldots \varphi_2(x; a_m)$ is n-inconsistent. □

11. The p-adic completion

Among other things we show that $(\mathbb{Q}_p, +, \times)$ is interpretable in the Shelah completion of a highly saturated elementary extension of $\mathcal{P}$, so $\mathcal{P}$ is non-modular.

We construct a p-adic completion $\mathcal{Z}^\square$ of a dp-minimal expansion $\mathcal{Z}$ of $(\mathbb{Z}, +, \mathrm{Val}_{\mathrm{p}})$. We show that $\mathcal{Z}^\square$ is dp-minimal, but in contrast with the situation over $(\mathbb{Z}, +, C_\alpha)$

we do not obtain an explicit description of unary definable sets. So we first show that definable sets and functions in dp-minimal expansions of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ behave similarly to definable sets and functions in o-minimal structures.

11.1. **Dp-minimal expansions of** $(\mathbb{Z}_p, +, \mathrm{Val_p})$**.** Let $\mathcal{Y}$ expand $(\mathbb{Z}_p, +, \mathrm{Val_p})$.

Fact 11.1. *Suppose $\mathcal{Y}$ is dp-minimal. Then the following are satisfied for any $\mathcal{Y}$-definable subset X of $\mathbb{Z}_p^n$ and $\mathcal{Y}$-definable function $f : X \to \mathbb{Z}_p^m$.*

1. *X is a boolean combination of $\mathcal{Y}$-definable closed subsets of $\mathbb{Z}_p^n$.*
2. *If $n = 1$ then X is the union of a definable open set and a finite set.*
3. *The dp-rank of X, the* acl*-dimension of X, and the maximal $0 \le d \le n$ for which there is a coordinate projection $\pi : \mathbb{Z}_p^m \to \mathbb{Z}_p^d$ such that $\pi(X)$ has interior are all equal. (We denote the resulting dimension by $\dim X$.)*
4. *There is a $\mathcal{Y}$-definable $Y \subseteq X$ such that $\dim X \setminus Y < \dim X$ and f is continuous on Y.*
5. *The frontier inequality holds, i.e. $\dim \mathrm{Cl}(X) \setminus X < \dim X$.*

Furthermore the same properties hold in any elementary extension of $\mathcal{Y}$.

Fact 11.1 is a special case of the results of [44]. Every single item of Fact 11.1 fails in $(\mathbb{Z}, +, \mathrm{Val_p})$ because of the presence of dense and co-dense definable sets.

There are dp-minimal expansions of valued groups in which algebraic closure does not satisfy the exchange property [5, 24], but this cannot happen over $(\mathbb{Z}_p, +, \mathrm{Val_p})$

Proposition 11.2. *Suppose $\mathcal{Y}$ is dp-minimal. Then $\mathcal{Y}$ is a geometric structure, i.e. $\mathcal{Y}$ eliminates $\exists^\infty$ and algebraic closure satisfies the exchange property.*

Proof. Elimination of $\exists^\infty$ follows from Fact 6.8. We show that algebraic closure satisfies exchange. By [44, Proposition 5.2] exactly one of the following is satisfied,

1. algebraic closure satisfies exchange, or
2. there is definable open $U \subseteq \mathbb{Z}_p$, definable $F \subseteq U \times \mathbb{Z}_p$ such that each F_a is finite, for every $a \in U$ there is an open $a \in V \subseteq U$ such that $F_b = F_a$ for all $b \in V$, and the family $(F_a : a \in U)$ contains infinitely many distinct sets.

Suppose (2) holds. Let E be the set of $(a, b) \in U^2$ such that $F_a = F_b$. Then E is a definable equivalence relation, every E-class is open, and there are infinitely many E-classes. Suppose $A \subseteq U$ contains exactly one element from each E-class. As $\mathbb{Z}_p$ is separable $|A| = \aleph_0$. Let $D := \bigcup_{a \in U} F_a = \bigcup_{a \in A} F_a$. So $D \subseteq \mathbb{Z}_p$ is definable and $|D| = \aleph_0$. This contradicts Fact 11.1(2). □

Finally, Fact 11.3 is proven in [43].

Fact 11.3. *A dp-minimal expansion of $(\mathbb{Z}_p, +, \times)$ is $(\mathbb{Z}_p, +, \times)$-minimal.*

It is an open question whether the theory of a dp-minimal expansion of $(\mathbb{Z}_p, +, \times)$ is $\mathrm{Th}(\mathbb{Z}_p, +, \times)$-minimal (equivalently: P-minimal).

11.2. **The p-adic completion.** Suppose $\mathcal{Z}$ is an expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$. Let $\mathcal{S} \prec \mathcal{N}$ be highly saturated. We define a standard part map $\mathrm{st} : N \to \mathbb{Z}_p$ by declaring $\mathrm{st}(a)$ to be the unique element of $\mathbb{Z}_p$ such that for all non-zero integers k, k' we have $\mathrm{Val_p}(a - k) \ge k'$ if and only if $\mathrm{Val_p}(\mathrm{st}(a) - k) \ge k'$. Note that st is a homomorphism and let **Inf** be the kernal of st. We identify $N/\mathbf{Inf}$ with $\mathbb{Z}_p$ and identify st with the quotient map. Note that **Inf** is the set of $a \in N$ such that

$\mathrm{st}(a) \geq k$ for all integers k, so **Inf** is externally definable and we consider $\mathbb{Z}_p$ as an imaginary sort of $\mathcal{N}^{\mathrm{Sh}}$.

Proposition 11.4. *Suppose $\mathcal{Z}$ is* NIP*. Then the following are interdefinable.*

1. *The structure $\mathcal{Z}^{\square}$ on $\mathbb{Z}_p$ with an n-ary relation symbol defining the closure in $\mathbb{Z}_p^n$ of every $\mathcal{Z}^{\mathrm{Sh}}$-definable subset of $\mathbb{Z}^n$.*
2. *The structure on $\mathbb{Z}_p$ with an n-ary relation symbol defining the image of each $\mathcal{N}$-definable subset of N^n under the standard part map $N^n \to \mathbb{Z}_p^n$.*
3. *The open core of the structure induced on $\mathbb{Z}_p$ by $\mathcal{N}^{\mathrm{Sh}}$.*

The structure induced on $\mathbb{Z}$ by $\mathcal{Z}^{\square}$ is a reduct of $\mathcal{Z}^{\mathrm{Sh}}$. If $\mathcal{Z}$ is dp-minimal then $\mathcal{Z}^{\square}$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $\mathcal{N}^{\mathrm{Sh}}$.

So in particular $\mathcal{Z}^{\square}$ is dp-minimal when $\mathcal{Z}$ is dp-minimal. All claims of Proposition 11.4 except the last follow by easy alternations to the proof of Fact 8.1.

We prove the last claim of Proposition 11.4.

Proof. Suppose $\mathcal{Z}$ is dp-minimal. We show that $\mathcal{Z}^{\square}$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $\mathcal{N}^{\mathrm{Sh}}$. It suffices to show that the induced structure on $\mathbb{Z}_p$ is interdefinable with its open core. The structure induced on $\mathbb{Z}_p$ by $\mathcal{N}^{\mathrm{Sh}}$ is dp-minimal as $\mathcal{N}^{\mathrm{Sh}}$ is dp-minimal. So by Fact 11.1 any $\mathcal{N}^{\mathrm{Sh}}$-definable set is a boolean combination of closed $\mathcal{N}^{\mathrm{Sh}}$-definable sets. □

One can show that $(\mathbb{Z}, +, \mathrm{Val_p})^{\square}$ is interdefinable with $(\mathbb{Z}_p, +, \mathrm{Val_p})$. We omit this for the sake of brevity.

We now give the p-adic analogue of Theorem 9.2. The proof is essentially the same as that of Theorem 9.2 so we leave the details to the reader. (One applies Fact 3.7 at the same point that Fact 3.6 is applied in the proof of Theorem 9.2.)

Proposition 11.5. *Suppose that A is a subset of $\mathbb{Z}_p^n$, $(\mathbb{Z}_p, +, \times, A)$ is* NIP*, and* $\mathrm{Th}(\mathbb{Z}_p, +, \times)$ *is an open core of* $\mathrm{Th}(\mathbb{Z}_p, +, \times, A)$*. Let $\mathcal{A}$ be the structure induced on A by $(\mathbb{Z}_p, +, \times)$ and X be the closure of A in $\mathbb{Z}_p^n$. Then*

1. *The structure $\mathcal{A}^{\square}$ with domain X and an n-ary relation for the closure in X^n of each $\mathcal{A}^{\mathrm{Sh}}$-definable subset of A^n,*
2. *and the structure $\mathcal{X}$ induced on X by $(\mathbb{Z}_p, +, \times)$,*

are interdefinable. (Note that X is semialgebraic.)

Fact 10.1 and Proposition 11.5 together easily yield Proposition 11.6.

Proposition 11.6. *The completion $\mathcal{P}^{\square}$ of $\mathcal{P}$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Z}_p, +, \times)$ and $a \mapsto \mathrm{Exp}(pa)$. So a subset of $\mathbb{Z}_p^n$ is $\mathcal{P}^{\square}$-definable if and only if it is of the form $\{(a_1, \ldots, a_n) \in \mathbb{Z}_p^n : (\mathrm{Exp}(pa_1), \ldots, \mathrm{Exp}(pa_n)) \in X\}$ for a semialgebraic subset X of $(\mathbb{Z}_p^{\times})^n$.*

Proposition 11.6 shows that $\mathcal{P}^{\square}$ defines an isomorphic copy of $(\mathbb{Q}_p, +, \times)$. So if $\mathcal{P} \prec \mathcal{N}$ is highly saturated then $\mathcal{N}^{\mathrm{Sh}}$ interprets $(\mathbb{Q}_p, +, \times)$, hence $\mathcal{P}$ is non-modular. We expect that $\mathcal{N}$ does not interpret an infinite field, but we do not have a proof.

11.3. A p-adic completion conjecture.

Conjecture 3. *Suppose $\mathcal{Z}$ is a dp-minimal expansion of $(\mathbb{Z}, +, \mathrm{Val_p})$. Then the structure induced on $\mathbb{Z}$ by $\mathcal{Z}^\square$ is interdefinable with $\mathcal{Z}^{\mathrm{Sh}}$ and every $\mathcal{Z}^{\mathrm{Sh}}$-definable subset of $\mathbb{Z}^n$ is of the form $X \cap Y$ where X is a $\mathcal{Z}^\square$-definable subset of $\mathbb{Z}_p^n$ and Y is a $(\mathbb{Z}, +)$-definable subset of $\mathbb{Z}^n$.*

The analogue of Conjecture 3 for dp-minimal expansions of divisible archimedean ordered groups is proven in [43]. We can prove a converse to Conjecture 3.

Proposition 11.7. *Let $\mathcal{Y}$ be an expansion of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ and $\mathcal{Z}$ be the structure induced on $\mathbb{Z}$ by $\mathcal{Y}$. Suppose $\mathcal{Y}$ is dp-minimal and every $\mathcal{Z}$-definable subset of $\mathbb{Z}^n$ is of the form $X \cap Y$ where X is a $\mathcal{Y}$-definable subset of $\mathbb{Z}_p^n$ and Y is a $(\mathbb{Z}, +)$-definable subset of $\mathbb{Z}^n$. Then $\mathcal{Z}$ is dp-minimal.*

Proof. NIP formulas are closed under conjunctions so $\mathcal{Z}$ is NIP. So it suffices to show that $\mathcal{Z}$ is inp-minimal. Inspection of the proof of Proposition 10.2 reveals that our proof on inp-minimality for $\mathcal{P}$ only uses the following facts about $(\mathbb{Z}_p, +, \times)$:

(1) $(\mathbb{Z}_p, +, \times)$ is inp-minimal, and
(2) every definable unary set in every elementary extension of $\mathbb{Z}_p$ is the union of a finite set and a definable open set.

It follows from Fact 11.1 that any dp-minimal expansion of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ satisfies (2). So the proof of Proposition 10.2 shows that $\mathcal{Z}$ is inp-minimal. ☐

12. p-adic elliptic curves?

We give a conjectural construction of uncountably many dp-minimal expansions of $(\mathbb{Z}, +, \mathrm{Val_p})$. Fix $\beta \in p\mathbb{Z}_p$. Then $\beta^{\mathbb{Z}}$ is a closed subgroup of $\mathbb{Q}_p^\times$. It is a well-known theorem of Tate [45] that there is an elliptic curve $\mathbb{E}_\beta$ defined over $\mathbb{Q}_p$ and a surjective p-adic analytic group homomorphism $\xi_\beta : \mathbb{Q}_p^\times \to \mathbb{E}_\beta(\mathbb{Q}_p)$ with kernel $\beta^{\mathbb{Z}}$. Note that ξ_β is injective on $1+p\mathbb{Z}_p$ as $(1+p\mathbb{Z}_p) \cap \beta^{\mathbb{Z}} = \{1\}$. We let χ_β be the injective p-adic analytic homomorphism $(\mathbb{Z}_p, +) \to \mathbb{E}_\beta(\mathbb{Q}_p)$ given by $\chi_\beta(a) := \xi_\beta(\mathrm{Exp}(pa))$, $\mathcal{Y}_\beta$ be the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Q}_p, +, \times)$ and χ_β, and $\mathcal{E}_\beta$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{Q}_p, +, \times)$ and χ_β. So $\mathcal{E}_\beta$ is the structure induced on $\mathbb{Z}$ by $\mathcal{Y}_\beta$.

Proposition 12.1. *$\mathcal{Y}_\beta$ expands $(\mathbb{Z}_p, +, \mathrm{Val_p})$ and $\mathcal{E}_\beta$ expands $(\mathbb{Z}, +, \mathrm{Val_p})$.*

Proposition 12.1 requires some p-adic metric geometry. We let

$$\mathrm{Val_p}(a) = \min\{\mathrm{Val_p}(a_1), \dots, \mathrm{Val_p}(a_m)\} \quad \text{for all } a = (a_1, \dots, a_m) \in \mathbb{Q}_p^m.$$

If X, Y are subsets of $\mathbb{Q}_p^m$ then $f : X \to Y$ is an isometry if f is a bijection and

$$\mathrm{Val_p}(f(a) - f(a')) = \mathrm{Val_p}(a - a') \quad \text{for all } a, a' \in X.$$

Suppose X, Y are p-adic analytic submanifolds of $\mathbb{Q}_p^m$. We let $T_a X$ be the tangent space of X at $a \in X$. Given a p-adic analytic map $f : X \to Y$ we let $\mathbf{d}f(a) : T_a X \to T_{f(a)} Y$ be the differential of f at $a \in X$.

Fact 12.2. *Suppose $f : X \to Y$ is a p-adic analytic map between p-adic analytic submanifolds X, Y of $\mathbb{Q}_p^m$. Fix $a \in X$ and set $b := f(a)$. Suppose that $\mathbf{d}f(a)$ is an isometry $T_a X \to T_b Y$. Then there is an open neighbourhood U of p such that $f(U)$ is open and f gives an isometry $U \to f(U)$.*

See [14, Proposition 7.1] for a proof of Fact 12.2 when X, Y are open subsets of $\mathbb{Q}_p^m$. This generalizes to p-adic analytic submanifolds as any d-dimensional p-adic analytic submanifold of $\mathbb{Q}_p^m$ is locally isometric to $\mathbb{Q}_p^d$, see for example [18, 5.2] (Halupczok only discusses smooth p-adic algebraic sets but everything goes through for p-adic analytic submanifolds).

We now prove Proposition 12.1.

Proof. To simplify notion we drop the subscript "β". It is enough to prove the first claim. It is easy to see that $\mathcal{Y}$ defines $+$. We need to show that the set of $(a, a') \in \mathbb{Z}_p^2$ such that $\mathrm{Val_p}(a) \le \mathrm{Val_p}(a')$ is definable in $\mathcal{E}$. Note that if A is a finite subset of $\mathbb{E}(\mathbb{Q}_p)$ and $f : \mathbb{E}(\mathbb{Q}_p) \setminus A \to \mathbb{Q}_p^m$ is a semialgebraic injection then $\mathcal{E}$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Q}_p, +, \times)$ and $f \circ \chi$. So we can replace $\mathbb{E}(\mathbb{Q}_p)$ and χ with $f(\mathbb{E}(\mathbb{Q}_p) \setminus A)$ and $f \circ \chi$.

We consider $\mathbb{E}(\mathbb{Q}_p)$ as a subset of $\mathbb{P}^2(\mathbb{Q}_p)$ via the Weierstrass embedding. Let $\iota : \mathbb{Q}_p^2 \to \mathbb{P}^2(\mathbb{Q}_p)$ be the inclusion $\iota(a, a') = [a : a' : 1]$, U be the image of ι, and $E := \iota^{-1}(\mathbb{E}(\mathbb{Q}_p))$. Recall that $\mathbb{E}(\mathbb{Q}_p) \setminus U$ is a singleton and E is a p-adic analytic submanifold of $\mathbb{Q}_p^2$. Let $\zeta : \mathbb{Z}_p \to E$ be $\zeta := \iota^{-1} \circ \chi$. So $\mathcal{E}$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Q}_p, +, \times)$ and ζ.

Let $e := \zeta(0)$ and identify $T_0\mathbb{Z}_p$ with $\mathbb{Q}_p$. Note that $\mathbf{d}\zeta(0)$ is a bijection $\mathbb{Q}_p \to T_eE$. After making an affine change of coordinates if necessary we suppose $\mathbf{d}\zeta(0)$ is an isometry $\mathbb{Q}_p \to T_eE$. Applying Fact 12.2 we obtain n such that the restriction of ζ to $p^n\mathbb{Z}_p$ is an isometry onto its image. So for all $a \in \mathbb{Z}_p$ we have

$$\mathrm{Val_p}(\zeta(p^n a) - e) = \mathrm{Val_p}(p^n a - 0) = \mathrm{Val_p}(a) + n.$$

So for all $a, a' \in \mathbb{Z}_p$ we have

$$\mathrm{Val_p}(a) \le \mathrm{Val_p}(a') \quad \text{if and only if} \quad \mathrm{Val_p}(\zeta(p^n a) - e) \le \mathrm{Val_p}(\zeta(p^n a') - e).$$

Let X be the set of $(a, a') \in \mathbb{Z}^2$ such that $\mathrm{Val_p}(\zeta(a) - e) \le \mathrm{Val_p}(\zeta(a') - e)$, so X is definable in $\mathcal{E}$. So for all $(a, a') \in \mathbb{Z}_p^2$ we have $\mathrm{Val_p}(a) \le \mathrm{Val_p}(a')$ if and only if $(p^n a, p^n a') \in X$. So $\{(a, a') \in \mathbb{Z}_p^2 : \mathrm{Val_p}(a) \le \mathrm{Val_p}(a')\}$ is definable in $\mathcal{Y}$. $\square$

We denote the group operation on $\mathbb{E}_\beta(\mathbb{Q}_p)$ by $\oplus$.

Conjecture 4. *Suppose A is a finite rank subgroup of $\mathbb{E}_\beta(\mathbb{Q}_p)$. Then $(\mathbb{Q}_p, +, \times, A)$ is* NIP*, $\mathrm{Th}(\mathbb{Q}_p, +, \times)$ is an open core of $\mathrm{Th}(\mathbb{Q}_p, +, \times, A)$, and every $(\mathbb{Q}_p, +, \times, A)$-definable subset of A^k is of the form $X \cap Y$ where X is an $(A, \oplus)$-definable subset of A^k and Y is a semialgebraic subset of $\mathbb{E}_\beta(\mathbb{Q}_p)^k$.*

Suppose Conjecture 4 holds. Under this assumption, Proposition 11.7 shows that $\mathcal{E}_\beta$ is dp-minimal, an application of Proposition 11.5 shows that $\mathcal{E}_\beta^\square$ is interdefinable with $\mathcal{Y}_\beta$, and an adaptation the proof of Theorem 9.4 shows that if $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are interdefinable then there is a semialgebraic group isomorphism $\mathbb{E}_\alpha(\mathbb{Q}_p) \to \mathbb{E}_\beta(\mathbb{Q}_p)$. So we obtain an uncountable collection of dp-minimal expansions of $(\mathbb{Z}, +, \mathrm{Val_p})$ no two of which are interdefinable.

Conjecture 4 should hold for any one-dimensional p-adic semialgebraic group satisfying a Mordell-Lang condition. One dimensional p-adic semialgebraic groups are classified in [27].

Suppose that $\mathbb{H}$ is a one-dimensional $(\mathbb{Q}_p, +, \times)$-definable group. By [19] there is an open subgroup V of $\mathbb{H}$, a one-dimensional abelian algebraic group W defined over $\mathbb{Q}_p$, an open subgroup U of $W(\mathbb{Q}_p)$, and a $(\mathbb{Q}_p, +, \times)$-definable group isomorphism $V \to U$. So we suppose that $\mathbb{H}$ is $W(\mathbb{Q}_p)$, so in particular $\mathbb{H}$ is a p-adic analytic group. Let e be the identity of $\mathbb{H}$ and identify $T_e\mathbb{H}$ with $\mathbb{Q}_p$. For sufficiently large n there is an open subgroup U of $\mathbb{H}$ and a p-adic analytic group isomorphism $\Xi : (p^n\mathbb{Z}_p, +) \to U$, this Ξ is the Lie-theoretic exponential, see [38, Corollary 19.9]. Let $\mathcal{H}$ be the structure induced on $\mathbb{Z}$ by $(\mathbb{Q}_p, +, \times)$ and $k \mapsto \Xi(p^n k)$. It follows in the same way as above that $\mathcal{H}$ expands $(\mathbb{Z}, +, \mathrm{Val_p})$. We expect that if $\mathbb{H}$ is semiabelian then $\mathcal{H}$ is dp-minimal and $\mathcal{H}^\square$ is interdefinable with the structure induced on $\mathbb{Z}_p$ by $(\mathbb{Q}_p, +, \times)$ and $a \mapsto \Xi(p^n a)$.

13. A general question

We briefly discuss the following question raised to us by Simon: Is there an abstract approach to $\mathcal{Z}^\square$? There are many ways in which one might try to make this more precise. For example: Given a sufficiently well behaved NIP structure $\mathcal{M}$ (perhaps dp-minimal, perhaps distal, perhaps expanding a group) can one construct a canonical structure $\mathcal{M}^\square$ containing $\mathcal{M}$ such that $\mathcal{M}^{\mathrm{Sh}}$ is the structure induced on M by $\mathcal{M}^\square$, $\mathcal{M}^\square$ is somehow "close to o-minimal", and $\mathcal{M}^\square$ is not too "big" relative to $\mathcal{M}$? In the completion of an NIP expansion $\mathcal{H}$ of an archimedean ordered abelian group, **Fin** is $\bigvee$-definable, **Inf** is $\bigwedge$-definable, and the resulting logic topology on $\mathbb{R}$ agrees with the usual topology. The same thing happens for the other completions discussed above. So perhaps there is a highly saturated $\mathcal{M} \prec \mathcal{N}$, a set X which both externally definable and $\bigvee$-definable in $\mathcal{N}$, an equivalence relation E on X which is both externally definable and $\bigwedge$-definable in $\mathcal{N}$, such that $\mathcal{M}^\square$ is the structure induced on X/E by $\mathcal{N}^{\mathrm{Sh}}$.

The completions defined above are not always the "right" notion. Let P be the set of primes and fix $q \in P$. Consider $(\mathbb{Z}, +, (\mathrm{Val_p})_{p\in P})$ as an expansion of $(\mathbb{Z}, +, \mathrm{Val_q})$, one can show that $(\mathbb{Z}, +, (\mathrm{Val_p})_{p\in P})^\square$ is interdefinable with $(\mathbb{Z}_q, +, \mathrm{Val_q})$. However the "right" completion of $(\mathbb{Z}, +, (\mathrm{Val_p})_{p\in P})$ is $(\widehat{\mathbb{Z}}, +, (\lhd_p)_{p\in P})$ where $(\widehat{\mathbb{Z}}, +)$ is the profinite completion $\prod_{p\in P}(\mathbb{Z}_p, +)$ of $(\mathbb{Z}, +)$ and we have $a \lhd_p b$ if and only if $\mathrm{Val_p}(\pi_p(a)) < \mathrm{Val_p}(\pi_p(b))$, where π_p is the projection $\widehat{\mathbb{Z}} \to \mathbb{Z}_p$. Likewise, if I is a $\mathbb{Z}$-linearly independent subset of $\mathbb{R} \setminus \mathbb{Q}$ then the completion of $(\mathbb{Z}, +, (C_\alpha)_{\alpha\in I})$ should be the torus $((\mathbb{R}/\mathbb{Z})^I, +, S_\alpha)$ where we have $S_\alpha(a, b, c)$ if and only if $C(\pi_\alpha(a), \pi_\alpha(b), \pi_\alpha(c))$ where π_α is the projection $(\mathbb{R}/\mathbb{Z})^I \to \mathbb{R}/\mathbb{Z}$ onto the αth coordinate.

When $\mathcal{M}$ expands a group it is tempting to try to define $\mathcal{M}^\square$ via general ideas from NIP group theory. Fix irrational $\alpha \in \mathbb{R}/\mathbb{Z}$, let $\mathcal{Z}$ be a dp-minimal expansion of $(\mathbb{Z}, +, C_\alpha)$, and $\mathcal{Z} \prec \mathcal{N}$ be highly saturated. One can show that $\mathcal{N}^0/\mathcal{N}^{00}$ is isomorphic as a topological group to $\mathbb{R}/\mathbb{Z}$ and it seems likely that $\mathcal{Z}^\square$ is interdefinable with the structure on $\mathbb{R}/\mathbb{Z}$ with an n-ary relation defining the image of $X \cap (\mathcal{N}^0)^n$

under the quotient map $(\mathcal{N}^0)^n \to (\mathbb{R}/\mathbb{Z})^n$, for each $\mathcal{N}$-definable $X \subseteq N^n$. But this breaks in the p-adic setting, if $(\mathbb{Z}, +, \mathrm{Val_p}) \prec \mathcal{N}$ is highly saturated then $\mathcal{N}^0 = \mathcal{N}^{00}$.

Is there some general class of "complete" structures for which $\mathcal{M}$ and $\mathcal{M}^\square$ are interdefinable? Suppose H is a dense subgroup of $(\mathbb{R}, +)$ and $\mathcal{H}$ is an NIP expansion of $(H, +, <)$. Then $\mathcal{H}$ and $\mathcal{H}^\square$ are interdefinable if and only if $H = \mathbb{R}$ and $\mathcal{H}$ is interdefinable with the open core of $\mathcal{H}^{\mathrm{Sh}}$. The Marker-Steinhorn theorem shows that these conditions are satisfied when $\mathcal{H}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$. If $\mathcal{H}$ is o-minimal then $\mathcal{H}^\square$ is (up to interdefinibility) the unique elementary extension of $\mathcal{H}$ which expands $(\mathbb{R}, +, <)$ [50, 13.2.1]. (Laskowski and Steinhorn [25] showed that there is such an extension.) If $\mathcal{H}$ is weakly o-minimal then $\mathcal{H}^\square$ is an elementary expansion of $\mathcal{H}$ if and only if $\mathcal{H}$ is o-minimal. Should $\mathcal{M}$ be "complete" if $\mathcal{M}^\square$ is (up to interdefinability) an elementary extension of $\mathcal{M}$? If $\mathcal{Z}$ is a dp-minimal expansion of $(\mathbb{Z}_p, +, \mathrm{Val_p})$ then must $\mathcal{Z}^\square$ and $\mathcal{Z}$ be interdefinable, i.e. is there a p-adic Marker-Steinhorn generalizing Fact 3.7?

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