THE ÉTALE OPEN TOPOLOGY OVER THE FRACTION FIELD OF A HENSELIAN LOCAL DOMAIN

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Abstract. Suppose that R is a local domain with fraction field K. If R is Henselian then the R-adic topology over K refines the étale open topology. If R is regular then the étale open topology over K refines the R-adic topology. In particular the étale open topology over $L((t_1,\dots,t_n))$ agrees with the $L[[t_1,\dots,t_n]]$-adic topology for any field L and $n \geq 1$.

1. Introduction

Throughout, K and L are fields, all rings are commutative with unit, the "dimension" of a ring is the Krull dimension, and by convention a "local ring" is not a field. The étale open topology (or $\mathcal{E}_K$-topology) on the K-points $V(K)$ of a K-variety V is introduced in [JTWY]; see Section 1.1 for definitions. Fact 1.1, proven in [JTWY], describes what we have established so far concerning the relationship between the étale open and other topologies.

Fact 1.1. *Suppose that V is a K-variety and v is a non-trivial valuation on K.*

(1) The $\mathcal{E}_K$-topology on $V(K)$ refines the Zariski topology.
(2) If K is separably closed then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the Zariski topology.
(3) If $<$ is a field order on K then the $\mathcal{E}_K$-topology on $V(K)$ refines the $<$-topology.
(4) If K is real closed then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the order topology.
(5) If the Henselization of (K,v) is not separably closed then the $\mathcal{E}_K$-topology on $V(K)$ refines the v-topology. (Hence if the value group of v is not divisible or the residue field of v is not algebraically closed then the $\mathcal{E}_K$-topology on $V(K)$ refines the v-topology.)
(6) If K is not separably closed and v is Henselian then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the v-topology.

Fact 1.1.6 shows that the $\mathcal{E}_K$-topology over $L((t))$ agrees with the valuation topology. We give a generalization to the fraction field $L((t_1,\dots,t_n))$ of $L[[t_1,\dots,t_n]]$. Let R be the valuation ring $\{a \in K : v(a) \geq 0\}$ of v. Then R is a local ring and (by definition) v is Henselian if and only if R is Henselian. It is natural to ask if Fact 1.1 generalizes to fraction fields of Henselian local domains such as $L[[t_1,\dots,t_n]]$. If R is a local domain with fraction field K then $(\alpha R + \beta : \alpha \in K^\times, \beta \in K)$ is a basis for a non-discrete Hausdorff field topology [PZ78, Theorem 2.2] on K which we refer to as the **R-adic topology**. It is easy to see that the R-adic topology on R agrees with the union of the I-adic topologies, I ranging over non-zero ideals in R. When R is a valuation ring the R-adic topology is the valuation topology.

Theorem 1.2. *Suppose that R is a local domain with fraction field K and V is a K-variety.*

(1) If R is Henselian then the R-adic topology on $V(K)$ refines the $\mathcal{E}_K$-topology.
(2) If R is regular then the $\mathcal{E}_K$-topology on $V(K)$ refines the R-adic topology.

Hence the étale open topology over $L((t_1,\dots,t_n))$ agrees with the $L[[t_1,\dots,t_n]]$-adic topology.

See Section 1.1 for the definition of regularity. Examples of regular Henselian local rings are $L[[t_1, \ldots, t_n]]$ and the algebraic closure of $L[t_1, \ldots, t_n]$ in $L[[t_1, \ldots, t_n]]$ for any field L, and the ring $L\{t_1, \ldots, t_n\}$ of convergent powers series in n variables for L a local field.

A one-dimensional regular local ring is a discrete valuation ring [Eis, 11.1], so the one-dimensional case of Theorem 1.2 follows from Fact 1.1.5. If R is a Noetherian local domain of dimension at least two then the R-adic topology is not induced by a valuation (see Proposition 2.1) so Theorem 1.2 does not follow from Fact 1.1. The étale open topology on $K = \mathbb{A}^1(K)$ is non-discrete if and only if K is large [JTWY]. Thus the first claim of Theorem 1.2 can be seen as a topological refinement of the fact, proven in [Pop10], that fraction fields of Henselian local domains are large. (See [Pop, BSF14] for an account of largeness.)

Both refinements in Theorem 1.2 can be strict. For example the localization of $L[t_1, \ldots, t_n]$ at the ideal generated by $t_1, \ldots, t_n$ is a regular local ring R with non-large fraction field $L(t_1, \ldots, t_n)$, so the étale open topology over $L(t_1, \ldots, t_n)$ is discrete and hence strictly refines the R-adic topology. Theorem 1.3 shows that the other refinement may also be strict.

Theorem 1.3. *Fix a prime p. There is a subring E of $\mathbb{Z}_p$ such that:*

(1) E is a one-dimensional Noetherian Henselian local domain,
(2) $\mathbb{Q}_p$ is the fraction field of E, and
(3) the E-adic topology on $\mathbb{Q}_p$ strictly refines the p-adic topology.

By Fact 1.1.6 the étale open topology over $\mathbb{Q}_p$ agrees with the p-adic topology, hence the E-adic topology on $\mathbb{Q}_p$ strictly refines the étale open topology. Hence some assumption like regularity is needed in Theorem 1.2.2. Possibly regularity can be replaced with a different assumption on R, such as excellence or $\dim R \geq 2$. We leave this question for future work. (E has non-reduced completion and is hence not excellent; see Remark 5.10 below.)

We now discuss an application to definable sets. "Definable" means "first order definable in the language of rings, possibly with parameters" and a field is *Henselian* if it admits a non-trivial Henselian valuation. It is a well-known and important fact that definable sets in characteristic zero Henselian fields are well behaved with respect to the valuation. In particular if K is a characteristic zero Henselian field then every definable subset of K^n is a finite union of valuation open subsets of Zariski closed subsets of K^n [vdD89], note that this applies to $L((t))$. Definable sets in fraction fields of characteristic zero Henselian local domains need not be well behaved. Suppose $\mathrm{Char}(L) = 0$ and $n \geq 2$. Jensen and Lenzig [JL89, Theorem 3.34] showed that $L((t_1, \ldots, t_n))$ defines $L[[t_1, \ldots, t_n]]$, Becker and Lipschitz [BL80] showed that $L[[t_1, \ldots, t_n]]$ defines $\mathbb{N}$, Delon [Del81] showed that $L[[t_1, \ldots, t_n]]$ uniformly defines all subsets of $\mathbb{N}$, hence $L((t_1, \ldots, t_n))$ defines the standard model of second order arithmetic. However we can still say something about existentially definable sets.

Corollary 1.4. *Suppose that R is a Henselian local domain with fraction field K, K is perfect, and $X \subseteq K^n$ is existentially definable. Then there are pairwise disjoint irreducible smooth subvarieties $V_1, \ldots, V_k$ of $\mathbb{A}^n$ and $O_1, \ldots, O_k$ such that each O_i is a definable R-adically open subset of $V_i(K)$ and $X = O_1 \cup \ldots \cup O_k$.*

Corollary 1.4 follows directly from Theorem 1.2.1, and the theorem, proven in [WY21], that if K is perfect then every existentially definable subset of K^n is a finite union of definable $\mathcal{E}_K$-open subsets of Zariski closed subsets of K^n.

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1.1. **Conventions and background.** A K-variety is a separated K-scheme of finite type, n is a natural number, $\mathbb{A}^n$ is n-dimensional affine space over K, and $\mathbb{G}_m$ is the scheme-theoretic multiplicative group over K, i.e. $\mathbb{A}^n = \operatorname{Spec} K[x_1, \ldots, x_n]$ and $\mathbb{G}_m = \operatorname{Spec} K[y, y^{-1}]$. We let $V(K)$ be the set of K-points of a K-variety V. Recall that $\mathbb{A}^n(K) = K^n$ and $\mathbb{G}_m(K) = K^\times$. We let $\operatorname{Frac}(R)$ be the fraction field of a domain R.

Suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Then R is **Henselian** if for any $g \in R[x]$ and $\alpha \in R$ such that $g(\alpha) \equiv 0 \pmod{\mathfrak{m}}$ and $g'(\alpha) \not\equiv 0 \pmod{\mathfrak{m}}$ there is $\alpha^* \in R$ such that $g(\alpha^*) = 0$ and $\alpha^* \equiv \alpha \pmod{\mathfrak{m}}$. Furthermore R is **regular** if R is Noetherian and $\mathfrak{m}$ admits a d-element generating set, where $d = \dim R$.

We briefly recall the étale open topology. Suppose that V is a K-variety. An **étale image** in $V(K)$ is a set of the form $f(X(K))$ for an étale morphism $f \colon X \to V$ of K-varieties. The collection of étale images in $V(K)$ forms a basis for the étale open topology on $V(K)$. Fact 1.5 below gathers some basic facts on the étale open topology from [JTWY].

Fact 1.5. *Suppose that $V \to W$ is a morphism of K-varieties. Then:*

1. *the induced map $V(K) \to W(K)$ is $\mathcal{E}_K$-continuous.*
2. *if $V \to W$ is étale then the induced map $V(K) \to W(K)$ is $\mathcal{E}_K$-open.*
3. *the map $K \to K$, $x \mapsto \alpha x + \beta$ is an $\mathcal{E}_K$-homeomorphism for any $\alpha \in K^\times, \beta \in K$.*
4. *if n is prime to* $\operatorname{Char}(K)$ *then $\{\alpha^n : \alpha \in K^\times\}$ is an étale open subset of K.*

We will need Fact 1.6 below, proven in [JTWY].

Fact 1.6. *Suppose that $\mathcal{T}$ is a Hausdorff field topology on K. If the $\mathcal{T}$-topology on each $K^n = \mathbb{A}^n(K)$ refines the $\mathcal{E}_K$-topology, then the $\mathcal{T}$-topology on $V(K)$ refines the $\mathcal{E}_K$-topology for any K-variety V. If the $\mathcal{E}_K$-topology on K refines $\mathcal{T}$, then the $\mathcal{E}_K$-topology on $V(K)$ refines the $\mathcal{T}$-topology for any K-variety V.*

We finally prove a general lemma.

Lemma 1.7. *Suppose R is a local domain and $K = \operatorname{Frac}(R)$. The following are equivalent:*

1. *The $\mathcal{E}_K$-topology on $V(K)$ refines the R-adic topology for any K-variety V.*
2. *The $\mathcal{E}_K$-topology on K refines the R-adic topology.*
3. *R is an $\mathcal{E}_K$-open subset of K.*
4. *R contains a nonempty $\mathcal{E}_K$-open subset of K.*

If K is additionally large and perfect then (1)-(4) is equivalent to the following:

5. *R contains an infinite set which is existentially definable in K.*
6. *There is a morphism $f \colon V \to \mathbb{A}^1$ of K-varieties such that $f(V(K))$ is infinite and contained in R.*

We will only use the equivalence of (1)-(4) at present. If K is not large then the étale open topology is discrete and hence trivially refines the R-adic topology.

Proof. Fact 1.6 shows that (1)⇔(2). It is clear that (2)⇒(3). The definition of the R-adic topology and Fact 1.5.3 together show that (3)⇒(2). The equivalence of (3) and (4) follows by Fact 1.5.3 as R is an additive subgroup of K. Suppose that K is large and perfect. By [JTWY] the $\mathcal{E}_K$-topology on K is not discrete, so any nonempty étale image in

K is infinite. If (4) holds then R contains a nonempty étale image, (6) follows immediately and (5) follows as étale images are existentially definable. (6) implies (5) as $f(V(K))$ is existentially definable. Finally, an infinite existentially definable subset of K has nonempty $\mathcal{E}_K$-interior [WY21], hence (5) implies (4). ☐

2. R-adic topologies and V-topologies

Let $\mathfrak{T}$ be a Hausdorff field topology on K. Then $\mathfrak{T}$ is a **V-topology** if for every neighbourhood O of 0 there is $a \in K$ such that for every $b \in K$ either $b \in O$ or $a/b \in O$, and $\mathfrak{T}$ is a V-topology if and only if $\mathfrak{T}$ is induced by a valuation or absolute value on K [EP05, Theorem B.1].

Proposition 2.1. *Suppose that R is an n-dimensional Noetherian local domain with fraction field K. If $n \geq 2$ then the R-adic topology on K is not a V-topology. Hence if R is in addition regular then the R-adic topology is a V-topology if and only if $n = 1$. In particular, the $L[[t_1, \ldots, t_n]]$-adic topology on $L((t_1, \ldots, t_n))$ is a V-topology if and only if $n = 1$.*

A Noetherian local ring has finite Krull dimension [Eis, 8.2.2], so our use of "n" is justified.

Lemma 2.2. *Suppose that R is a local domain, K is the fraction field of R, $v, v^* \colon K^\times \to \mathbb{Z}$ are homomorphisms such that v, v^* are both non-negative on $R \setminus \{0\}$, and some $\beta \in K^\times$ satisfies $v(\beta) < 0 < v^*(\beta)$. Then the R-adic topology on K is not a V-topology.*

Before proving Lemma 2.2, we use it to show that the $L[[t, t^*]]$-adic topology on $L((t, t^*))$ is not a V-topology. Let v be the t-adic valuation and v^* be the t^*-adic valuation on $L((t, t^*))$. Set $\beta = t^*/t$. Then v, v^* are both non-negative on $L[[t, t^*]]$ and $v(\beta) < 0 < v^*(\beta)$.

Proof (of Lemma 2.2). Suppose otherwise. Fix $c \in K$ such that for any $a \in K$, either a or c/a is in R. For every $n \geq 1$ we have $v(\beta^n) < 0$, and if n is sufficiently large then $v^*(c/\beta^n) = v^*(c) - nv^*(\beta) < 0$. Thus there is n such that $\beta^n, c/\beta^n \notin R$, a contradiction. ☐

Lemma 2.3. *Suppose R is a Noetherian domain with dimension ≥ 2. Then R contains infinitely many distinct prime ideals of height one.*

Proof. Assume otherwise. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be all the primes of height 1 and let $\mathfrak{m}$ be a maximal ideal of R of height ≥ 2. By prime avoidance, $\mathfrak{m} \not\subseteq \bigcup_i \mathfrak{p}_i$. Take $a \in \mathfrak{m} \setminus \bigcup_i \mathfrak{p}_i$. Let $\mathfrak{p}^*$ be a minimal prime ideal containing a. By Krull's principal ideal theorem [Eis, Theorem 10.1], $\mathfrak{p}^*$ has height one. But $\mathfrak{p}^* \neq \mathfrak{p}_i$ for any $i \in \{1, \ldots, n\}$, a contradiction. ☐

We now prove Proposition 2.1. We let $R_\mathfrak{p}$ be the localization of a ring R at a prime ideal $\mathfrak{p}$.

Proof. The second claim follows from the first as a one-dimensional regular local ring is a DVR. We prove the first claim. Applying Lemma 2.3 fix height one prime ideals $\mathfrak{p} \neq \mathfrak{p}^*$ in R. Then $R_\mathfrak{p}, R_{\mathfrak{p}^*}$ are one-dimensional Noetherian domains. Let $v \colon R \setminus \{0\} \to \mathbb{Z}$ be given by declaring $v(a)$ to be the length of $R_\mathfrak{p}/aR_\mathfrak{p}$. Then v extends to a map $K^\times \to \mathbb{Z}$ by declaring $v(a/b) = v(a) - v(b)$. We likewise define $v^* \colon K^\times \to \mathbb{Z}$ using $\mathfrak{p}^*$. By [Ful98, Definition A.3] v, v^* are well-defined homomorphisms. Note that v, v^* are non-negative on R as the length is non-negative. As $\mathfrak{p}, \mathfrak{p}^*$ are distinct height one ideals neither is contained in the other. Fix $\alpha \in \mathfrak{p} \setminus \mathfrak{p}^*$, $\alpha^* \in \mathfrak{p}^* \setminus \mathfrak{p}$, and set $\beta = \alpha^*/\alpha$. Note that $v(\beta) < 0 < v^*(\beta)$. Apply Lemma 2.2. ☐

The valuational approach to $L((t, t^*))$ generalizes to the regular case. Suppose that R is a regular local ring of dimension ≥ 2. Let $\mathfrak{p}, \mathfrak{p}^*$ be distinct prime ideals of R. A localization of a regular ring is regular [Eis, Corollary 19.14] so $R_\mathfrak{p}, R_{\mathfrak{p}^*}$ are one-dimensional regular local rings, hence DVR's. Note that the induced discrete valuations on K satisfy Lemma 2.2.

3. Proof of Theorem 1.2.1

In this section we suppose that R is a Henselian local domain with maximal ideal $\mathfrak{m}$ and fraction field K. We will need the following variant of Hensel's lemma.

Lemma 3.1. *Suppose that $g \in R[x]$ and $\alpha \in R$ satisfy $g'(\alpha) \neq 0$ and $g(\alpha) \equiv 0 \pmod{g'(\alpha)^2\mathfrak{m}}$. Then there is $\alpha^* \in R$ such that $g(\alpha^*) = 0$ and $\alpha^* - \alpha \in (g(\alpha)/g'(\alpha))R$.*

We let $\overline{a} \in R/\mathfrak{m}$, $\overline{p}(y) \in (R/\mathfrak{m})[y]$ be the reduction mod $\mathfrak{m}$ of an $a \in R$, $p \in R[y]$, respectively.

Proof. After possibly replacing $g(x)$ with $g(x+\alpha)$ we suppose $\alpha = 0$. Fix $c_0, \ldots, c_m \in R$ with $g(x) = c_0 + c_1 x + \ldots + c_n x^n$. Then $c_1 = g'(0) \neq 0$ and $c_0/c_1^2 = g(\alpha)/g'(\alpha)^2 \in \mathfrak{m}$. Let $p \in R[x]$ be given by

$$p(x) = x + \sum_{i=2}^{n} c_i c_1^{i-2} \left(\frac{c_0}{c_1^2}\right)^{i-1} (x-1)^i.$$

As $c_0/c_1^2 \in \mathfrak{m}$ we have $\overline{p}(x) = x$. Hence $\overline{p}(0) = 0$ and $\overline{p'}(0) = 1$. As R is Henselian there is $\beta \in \mathfrak{m}$ such that $p(\beta) = 0$. Then

$$0 = p(\beta) = \beta + \sum_{i=2}^{n} c_i c_1^{i-2} \left(\frac{c_0}{c_1^2}\right)^{i-1} (\beta-1)^i = 1 + (\beta - 1) + \sum_{i=2}^{n} c_i \left(\frac{c_0^{i-1}}{c_1^i}\right) (\beta-1)^i.$$

Hence

$$\begin{aligned} 0 = c_0 p(\beta) &= c_0 + c_0(\beta-1) + \sum_{i=2}^{n} c_i \left(\frac{c_0}{c_1}\right)^i (\beta-1)^i \\ &= c_0 + c_1 \left(\frac{c_0}{c_1}\right)(\beta-1) + \sum_{i=2}^{n} c_i \left(\frac{c_0}{c_1}\right)^i (\beta-1)^i \\ &= \sum_{i=0}^{n} c_i \left(\frac{c_0}{c_1}\right)^i (\beta-1)^i \end{aligned}$$

Set $\alpha^* = (c_0/c_1)(\beta - 1)$. Then $\alpha^* \in (c_0/c_1)R$ as $\beta \in R$. Note that $g(\alpha^*) = 0$. □

We proceed with the proof of Theorem 1.2.1. We assume some familiarity with the notion of a standard étale morphism; see for example [Poo17, Definition 3.5.38]. Given an affine K-variety V, a **standard étale image** in $V(K)$ is a set of the form $f(X(K))$ for a standard étale morphism $f\colon X \to V$ of K-varieties. If V is affine, then every étale image is a union of standard étale images. This holds because for any étale morphism $f\colon X \to V$, we can cover X by affine opens $U \subseteq X$ such that $U \to V$ is standard étale [Sta20, Lemma 02GT].

We show that the R-adic topology on $V(K)$ refines the $\mathcal{E}_K$-topology for any K-variety V. By Fact 1.6 it suffices to fix n and show that the R-adic topology on K^n refines the $\mathcal{E}_K$-topology. It suffices to fix an $\mathcal{E}_K$-open neighbourhood $O \subseteq K^n$ of the origin and show that 0 lies in the R-adic interior of O. We may suppose that O is a standard étale image. By the definition of a standard étale morphism there are $f, g \in K[x_1, \ldots, x_n, y]$ such that:

(1) $O = \{\alpha \in K^n : (\exists \beta \in K) f(\alpha, \beta) = 0 \neq g(\alpha, \beta)\}$, and
(2) $\frac{\partial f}{\partial y}(\alpha, \beta) \neq 0$ for all $(\alpha, \beta) \in K^n \times K$ such that $f(\alpha, \beta) = 0 \neq g(\alpha, \beta)$.

Fix $b \in K$ such that $f(0,b)=0$. Replacing y with $y+b$, we may suppose $b=0$, so that $f(0,0)=0 \neq g(0,0)$. After clearing denominators we may suppose $f \in R[x_1,\ldots,x_n,y]$. This implies that $\frac{\partial f}{\partial y} \in R[x_1,\ldots,x_n]$, hence $\frac{\partial f}{\partial y}(0,0) \in R$. Because $g(a,b)$ is R-adically continuous and $g(0,0) \neq 0$, there is some R-adic neighborhood U of 0 in $K^n \times K$ such that $g(a,b) \neq 0$ whenever $(a,b) \in U^n \times U$. Because R is R-adically bounded, there is an R-adic neighborhood U^* such that $U^* \cdot R \subseteq U$. For each $a \in K^n$ we let $f_a \in K[y]$ be given by $f_a(y) = f(a,y)$. Note that $f'_a(y) = \frac{\partial f}{\partial y}(a,y)$. Thus $f_0 \in R[y]$, $f_0(0)=0$, and $f'_0(0) \neq 0$. By continuity there is an R-adic neighbourhood U' of 0 in K^n such that $U' \subseteq U^n$ and if $a \in U'$ then:

(1) $f_a \in R[y]$,
(2) $f'_a(0) \neq 0$,
(3) $f_a(0)/f'_a(0)^2 \in \mathfrak{m}$,
(4) $f_a(0)/f'_a(0) \in U^*$.

We show that $U' \subseteq O$. Fix $\alpha \in U'$. Then we have $f'_\alpha(0) \neq 0$ and $f'_\alpha(0) \equiv 0 \pmod{f'_\alpha(0)^2\mathfrak{m}}$. By Lemma 3.1 there is $\beta \in (f_\alpha(0)/f'_\alpha(0))R$ such that $f_\alpha(\beta)=0$. By (4) we have $\beta \in U^* \cdot R$. Hence $\beta \in U$, so $(\alpha,\beta) \in U^n \times U$, and $g(\alpha,\beta) \neq 0$. Note $f(\alpha,\beta)=0$, so β witnesses $\alpha \in O$.

4. Proof of Theorem 1.2.2

In this section we suppose that R is a regular local ring with fraction field K. We first prove several algebraic lemmas. Lemma 4.1 is presumably unoriginal.

Lemma 4.1. *Suppose that S is the Henselization of R. Then $S \cap K = R$.*

Proof. It suffices to show $S \cap K \subseteq R$. Suppose $\beta \in S \cap K$; we may suppose $\beta \neq 0$. Fix $\alpha, \alpha^* \in R$ such that $\beta = \alpha/\alpha^*$. We may assume that α, α^* have no common prime factors in R as a regular local ring is a UFD [Eis, Theorem 19.19]. Let $I = \{r \in R : \beta r \in R\}$ and $I' = \beta I$. Both I and I' are ideals of R, with $\alpha^* \in I$ and $\alpha \in I'$. Note that $I'S \subseteq IS$ as $\beta \in S$. By faithful flatness of S over R [Sta20, Lemma 07QM], we have $I' = I'S \cap R \subseteq I = IS \cap R$. Then $\alpha \in I'$ implies $\alpha \in I$, so $\beta\alpha \in R$. Note that $\alpha^*(\beta\alpha) = \alpha^2$, so α^* divides α^2 in R. As α, α^* have no common prime factors, α^* must be invertible. Hence $\beta \in R$. ☐

Fact 4.2 is from [JL89, pg 52, 55].

Fact 4.2. *Suppose that R is in addition Henselian and* $\dim R \geq 2$*.*

(1) If $\mathrm{Char}(R/\mathfrak{m}) \neq 2$ *and $\alpha, \beta \in K$ satisfy $1+\alpha^4 = \beta^2$ then $\alpha \in R$ or $1/\alpha \in R$.*
(2) If $\mathrm{Char}(R/\mathfrak{m}) = 2$ *and $\alpha, \beta \in K$ satisfy $1+\alpha^3 = \beta^3$ then $\alpha \in R$ or $1/\alpha \in R$.*

We now remove the assumption of Henselianity from Fact 4.2.

Lemma 4.3. *Suppose that* $\dim R \geq 2$*.*

(1) If $\mathrm{Char}(R/\mathfrak{m}) \neq 2$ *and $\alpha, \beta \in K$ satisfy $1+\alpha^4 = \beta^2$ then $\alpha \in R$ or $1/\alpha \in R$.*
(2) If $\mathrm{Char}(R/\mathfrak{m}) = 2$ *and $\alpha, \beta \in K$ satisfy $1+\alpha^3 = \beta^3$ then $\alpha \in R$ or $1/\alpha \in R$.*

Proof. We treat (1); (2) follows in the same manner. Suppose that $\mathrm{Char}(R/\mathfrak{m}) \neq 2$ and $\alpha, \beta \in K$ satisfy $1+\alpha^4 = \beta^2$. Let S be the Henselization of R. Then S is a regular local ring of the same dimension as R by [Sta20, Lemmas 06LJ, 06LK, 06LN]. The maximal ideal of S is $\mathfrak{m}S$ and $S/\mathfrak{m}S = R/\mathfrak{m}$ [Sta20, Lemma 07QM]. Hence $\mathrm{Char}(S/\mathfrak{m}S) \neq 2$. Thus by Fact 4.2 $\alpha \in S$ or $1/\alpha \in S$. An application of Lemma 4.1 shows that $\alpha \in R$ or $1/\alpha \in R$. ☐

We proceed with the proof of Theorem 1.2.2. If R is one dimensional then K is a discretely valued field; this case follows by Fact 1.1.6. Hence we suppose that $\dim R \geq 2$. By Lemma 1.7 it is enough to produce an $\mathcal{E}_K$-neighbourhood of 0 contained in R. First suppose $\mathrm{Char}(R/\mathfrak{m}) \neq 2$. This implies that $\mathrm{Char}(K) \neq 2$. Let S be $\{b^2 : b \in K^\times\}$, $f\colon K \to K$ be given by $f(a) = 1 + a^4$, and $\Omega = f^{-1}(S)$. Note $0 \in \Omega$. By Fact 1.5, S and Ω are $\mathcal{E}_K$-open sets. By Lemma 4.3, $\Omega \subseteq R \cup R^{-1}$. Fix a height one prime ideal $\mathfrak{p}$ of R. The localization $R_\mathfrak{p}$ is a regular local ring of dimension 1, hence a discrete valuation ring. Let $v\colon K^\times \to \mathbb{Z}$ be the associated valuation. By Fact 1.1.5, the $\mathcal{E}_K$-topology on K refines the v-topology. Therefore the valuation ideal $\mathfrak{p}R_\mathfrak{p} = \{a \in K : v(a) > 0\}$ is $\mathcal{E}_K$-open. The intersection $\Omega \cap \mathfrak{p}R_\mathfrak{p}$ is an $\mathcal{E}_K$-open neighborhood of 0. We show that $\Omega \cap \mathfrak{p}R_\mathfrak{p} \subseteq R$. Suppose $a \in \Omega \cap \mathfrak{p}R_\mathfrak{p}$. As $a \in \mathfrak{p}R_\mathfrak{p}$ we have $v(a) > 0$, hence $v(a^{-1}) < 0$, hence $a^{-1} \notin R_\mathfrak{p}$, so $a^{-1} \notin R$. As $a \in \Omega$, we have $a \in R$.

The case when $\mathrm{Char}(R/\mathfrak{m}) = 2$ is similar, using $S = \{b^3 : b \in K^\times\}$ and $f(a) = 1 + a^3$.

5. Proof of Theorem 1.3

Fix a prime p, let $\mathrm{res}\colon \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$ be the residue map, and $v\colon \mathbb{Q}_p^\times \to \mathbb{Z}$ be the p-adic valuation. Fix a $\mathbb{Q}$-linear derivation $\partial\colon \mathbb{Q}_p \to \mathbb{Q}_p$ that is not constant zero. We can construct ∂ in the following fashion: Fix $t \in \mathbb{Q}_p$ transcendental over $\mathbb{Q}$, let ∂^* be the unique $\mathbb{Q}$-linear derivation $\mathbb{Q}(t) \to \mathbb{Q}(t)$ such that $\partial^* t = 1$, and ∂ be a $\mathbb{Q}$-linear derivation $\mathbb{Q}_p \to \mathbb{Q}_p$ extending ∂^*. Such extensions exist by [Wei46, Chapter 1 §5]. By $\mathbb{Q}$-linearity $\partial r = 0$ for all $r \in \mathbb{Q}$.

We declare E to be $\{a \in \mathbb{Z}_p : \partial a \in \mathbb{Z}_p\}$. It is easy to see that E is a subring of $\mathbb{Z}_p$. In this section we show that E is a one-dimensional Noetherian Henselian local ring, $\mathbb{Q}_p$ is the fraction field of E, and the E-adic topology on $\mathbb{Q}_p$ strictly refines the p-adic topology. We also show that E has non-reduced completion, so E is neither regular nor excellent.

Lemma 5.1. *The graph $\{(a, \partial a) : a \in \mathbb{Q}_p\}$ of ∂ is p-adically dense in $\mathbb{Q}_p^2$.*

Proof. Fix $t \in \mathbb{Q}_p$ with $\partial t \neq 0$. Recall that $\mathbb{Q}^2$ is dense in $\mathbb{Q}_p^2$. Let $f\colon \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ be the affine transformation $f(x, y) = (x + yt, (\partial t)y)$. Then f is invertible as $\partial t \neq 0$, hence $f(\mathbb{Q}^2)$ is p-adically dense in $\mathbb{Q}_p^2$. We have $\partial(a + bt) = \partial(a) + b\partial(t) = b\partial(t)$ for all $a, b \in \mathbb{Q}$, hence

$$f(\mathbb{Q}^2) = \{(a + bt, \partial(a + bt)) : a, b \in \mathbb{Q}\} \subseteq \{(c, \partial c) : c \in \mathbb{Q}_p\}. \qquad \square$$

Lemma 5.2. *The fraction field of E is $\mathbb{Q}_p$.*

Proof. It is enough to show that $\mathbb{Z}_p$ is contained in the fraction field of E. Fix $a \in \mathbb{Z}_p$. We produce $b \in \mathbb{Z}_p$ such that $b, ab \in E$. Take $\gamma \in \mathbb{N}$ such that $\gamma + v(\partial a) \geq 0$. By Lemma 5.1 there is $b \in \mathbb{Q}_p$ satisfying $v(b) = v(\partial b) = \gamma$; in particular $b \neq 0$. Then $b, \partial b \in \mathbb{Z}_p$ as $\gamma \geq 0$, hence $b \in E$. As $a, b \in \mathbb{Z}_p$ we have $ab \in \mathbb{Z}_p$, so it remains to show that $\partial(ab) \in \mathbb{Z}_p$. We have

$$v(\partial(b)a) = v(\partial b) + v(a) = \gamma + v(a) \geq 0$$
$$v(\partial(a)b) = v(\partial a) + v(b) = v(\partial a) + \gamma \geq 0.$$

Hence $a\partial(b), b\partial(a) \in \mathbb{Z}_p$, so $\partial(ab) = a\partial(b) + b\partial(a)$ is in $\mathbb{Z}_p$. $\square$

Lemma 5.3. *E is a local ring with maximal ideal $E \cap p\mathbb{Z}_p = \{a \in E : \mathrm{res}(a) = 0\}$.*

Proof. Let $\mathfrak{m} = E \cap p\mathbb{Z}_p$. Then $\mathfrak{m}$ is a proper ideal of E. It suffices to fix $a \in E \setminus \mathfrak{m}$ and show that a is invertible in E. We have $a \in \mathbb{Z}_p$ and $\mathrm{res}(a) \neq 0$, hence $v(a) = v(a^{-1})$. Then

$$v(\partial(a^{-1})) = v(-\partial(a)/a^2) = v(\partial a) - 2v(a) = v(\partial a) \geq 0.$$

Therefore $a^{-1}, \partial(a^{-1}) \in \mathbb{Z}_p$, hence $a^{-1} \in E$. $\square$

Lemma 5.4. *Suppose that R, R^* are local domains with the same fraction field K. Then the R-adic topology on K refines the R^*-adic topology if and only if R^* is R-adically open. If $R \subseteq R^*$ then the R-adic topology refines the R^*-adic topology.*

Proof. The first claim is easy and left to the reader. Suppose $R \subseteq R^*$. Then R is an additive subgroup of R^*, hence $R^* = \bigcup_{a \in R^*}(a + R)$, hence R^* is R-adically open. $\square$

Proposition 5.5. *The E-adic topology on $\mathbb{Q}_p$ strictly refines the p-adic topology.*

Proof. We have $E \subseteq \mathbb{Z}_p$, so the E-adic topology refines the p-adic topology by Lemma 5.4. We show that E is not open in the p-adic topology. Suppose O is a p-adically open subset of $\mathbb{Q}_p$. By Lemma 5.1 there is $a \in O$ such that $\partial a \in \mathbb{Q}_p \setminus \mathbb{Z}_p$. Then $a \notin E$, hence $O \not\subseteq E$. $\square$

Proposition 5.6. *E is Henselian.*

Given $g \in \mathbb{Q}_p[x]$, $g(x) = a_0 + a_1 x + \ldots + a_d x^d$ we let $\partial g \in \mathbb{Q}_p[x]$ be $\partial(a_0) + \partial(a_1)x + \ldots + \partial(a_d)x^d$. Note that if $g \in E[x]$ then $\partial g \in \mathbb{Z}_p[x]$. As above we let $\mathfrak{m}$ be the maximal ideal of E.

Proof. Fix $g \in E[x]$ and $a \in E$ such that $g(a) \equiv 0 \pmod{\mathfrak{m}}$ and $g'(a) \not\equiv 0 \pmod{\mathfrak{m}}$. We will produce $a^* \in E$ such that $g(a^*) = 0$ and $a^* \equiv a \pmod{\mathfrak{m}}$. As $\mathfrak{m} = E \cap p\mathbb{Z}_p$ we have $g(a) \equiv 0 \pmod{p}$ and $g'(a) \not\equiv 0 \pmod{p}$. As $\mathbb{Z}_p$ is Henselian there is $a^* \in \mathbb{Z}_p$ such that $g(a^*) = 0$ and $a^* \equiv a \pmod{p}$. We claim that $a^* \in E$; as $\mathfrak{m} = E \cap p\mathbb{Z}_p$ this yields $a^* \equiv a \pmod{\mathfrak{m}}$, so E is Henselian. It is enough to show that $\partial a^* \in \mathbb{Z}_p$. We have

$$0 = \partial(0) = \partial(g(a^*)) = (\partial g)(a^*) + g'(a^*)\partial(a^*).$$

hence

$$\partial(a^*) = \frac{(\partial g)(a^*)}{-g'(a^*)}.$$

As $g \in E[x]$, we have $\partial g \in \mathbb{Z}_p[x]$, hence $(\partial g)(a^*) \in \mathbb{Z}_p$. As $a^* \equiv a \pmod{p}$ we have $g'(a^*) \equiv g'(a) \not\equiv 0 \pmod{p}$, and so $g'(a^*) \in \mathbb{Z}_p^\times$. Therefore $\partial a^* \in \mathbb{Z}_p$. $\square$

It remains to show that E is one-dimensional Noetherian. We first prove a technical lemma.

Lemma 5.7. *Suppose $a, a', a'' \in \mathbb{Q}_p$, $v(a) \le \min\{v(a'), v(a'')\}$ and $v(\partial(a'/a)) \le v(\partial(a''/a))$. Then $a'' \in aE + a'E$.*

Proof. We may assume $a'' \neq 0$. Then $v(a) \le v(a'') < \infty$, so $a \neq 0$. After replacing a, a', a'' by $a/a, a'/a, a''/a$, we may suppose $a = 1$. Then $0 = v(a) \le \min\{v(a'), v(a'')\}$, so $a', a'' \in \mathbb{Z}_p$. Additionally, $v(\partial a') \le v(\partial a'')$. This yields $b \in \mathbb{Z}_p$ such that $\partial(a'') = b\partial(a')$. By continuity of multiplication there is a p-adically open neighbourhood $U \subseteq \mathbb{Z}_p$ of b such that if $b^* \in U$ then $\partial(a'') - b^*\partial(a') \in \mathbb{Z}_p$. By Lemma 5.1 there is $b^* \in U$ such that $\partial(b^*) \in \mathbb{Z}_p$. Then $b^* \in E$. Declare $c = a'' - b^*a'$. Then $a'' = c + b^*a'$. We claim that $c \in E$, so that $a'' \in E + a'E$. We have $c \in \mathbb{Z}_p$, as $a', a'', b^* \in \mathbb{Z}_p$. We have

$$\partial(c) = \partial(a'') - \partial(b^*a') = \partial(a'') - b^*\partial(a') - a'\partial(b^*).$$

Note that $\partial(a'') - b^*\partial(a')$ and $a'\partial(b^*)$ are both in $\mathbb{Z}_p$. Hence $\partial c \in \mathbb{Z}_p$, so $c \in E$. $\square$

The fact that E is one-dimensional Noetherian follows from Fact 5.8 and Lemma 5.9 below. Fact 5.8 is a theorem of Cohen [Coh50].

Fact 5.8. *Suppose that R is a domain, R is not a field, and there is n such that every ideal in R admits an n element generating set. Then R is one-dimensional Noetherian.*

Lemma 5.9. *Every ideal in E has a two element generating set.*

Proof. Let I be an ideal in E. We may suppose $I \neq \{0\}$. As $E \subseteq \mathbb{Z}_p$ we have $v(a) \geq 0$ for all $a \in I$. Fix $a \in I$ minimizing $v(a)$; note that $a \neq 0$. For any $a^* \in I$ we have

$$v(\partial(a^*/a)) = v\left(\frac{a\partial(a^*) - a^*\partial(a)}{a^2}\right) = v(a\partial(a^*) - a^*\partial(a)) - 2v(a).$$

As $a, a^*, \partial(a), \partial(a^*) \in \mathbb{Z}_p$ we have $v(a\partial(a^*) - a^*\partial(a)) \geq 0$, hence $v(\partial(a^*/a)) \geq -2v(a)$. Therefore we may select $a' \in I$ minimizing $v(\partial(a'/a))$. We show that $I = aE + a'E$. Fix $a'' \in I$. Then $v(a) \leq \min\{v(a'), v(a'')\}$ and $v(\partial(a'/a)) \leq v(\partial(a''/a))$. Apply Lemma 5.7. ☐

Remark 5.10. Our ring E is similar to Ferrand and Raynaud's example of a Noetherian 1-dimensional local domain with non-reduced completion [FR70, Proposition 3.1]. Their example probably satisfies an analogue of Theorem 1.3, with $\mathbb{C}\{t\}$ replacing $\mathbb{Q}_p$. Conversely, E has non-reduced completion. This implies that E is not excellent [Sta20, 07QT, 07GH, 07QK] and not regular [Sta20, 07NY, 00NP]. In fact the completion of E is $\mathbb{Z}_p[x]/(x^2)$. It is enough to produce a ring embedding $\tau\colon E \to \mathbb{Z}_p[x]/(x^2)$ such that τ gives a dense topological embedding from the $\mathfrak{m}$-adic topology on E to the p-adic topology on $\mathbb{Z}_p[x]/(x^2)$. Let $\tau\colon E \to \mathbb{Z}_p[x]/(x^2)$ be $\tau(a) = a + \partial(a)x$. Note that τ is an injective ring homomorphism. The p-adic topology on $\mathbb{Z}_p[x]/(x^2)$ agrees with the product topology given by the natural bijection $\mathbb{Z}_p[x]/(x^2) \to \mathbb{Z}_p^2$. By Lemma 5.1 the image of τ is dense. The $\mathfrak{m}$-adic topology on E agrees with the restriction of the E-adic topology on $\mathbb{Q}_p$ to E. (It suffices to show that for any non-zero ideal I in E we have $\mathfrak{m}^n \subseteq I$ for some n. This holds because E/I is a local Artinian ring, as $\dim E = 1$.) By the proof of [Joh, Proposition 8.21] the collection of sets of the form $\{\alpha \in E : v(\beta - \alpha) > \gamma \text{ and } v(\beta^* - \partial\alpha) > \gamma^*\}$ for $\beta, \beta^* \in E$ and $\gamma, \gamma^* \in \mathbb{Z}$ is a basis for the E-adic topology on E, hence τ is a topological embedding.

Remark 5.11. We discuss our usage of the axiom of choice. Note that ∂ is a discontinuous additive homomorphism $\mathbb{Q}_p \to \mathbb{Q}_p$. This implies that ∂ is not measurable, hence existence of ∂ requires a strong application of the axiom of choice; see for example [Ros09, Section 2]. One can avoid this. In fact, our argument goes through for R a characteristic zero Henselian DVR, $K = \mathrm{Frac}(R)$, and $\partial\colon K \to K$ a derivation with dense graph. For example fix $t \in \mathbb{Q}_p$ transendental over $\mathbb{Q}$ and let F be the algebraic closure of $\mathbb{Q}(t)$ in $\mathbb{Q}_p$. Then F is a dense subfield of $\mathbb{Q}_p$ and $F \cap \mathbb{Z}_p$ is a Henselian DVR with fraction field F. Let ∂^* be the unique $\mathbb{Q}$-linear derivation $\mathbb{Q}(t) \to \mathbb{Q}(t)$ with $\partial^* t = 1$. Then ∂^* uniquely extends to a $\mathbb{Q}$-linear derivation $\partial\colon F \to F$ as $F/\mathbb{Q}(t)$ is algebraic [Wei46, Proposition 1.15]. This extension does not require choice. Our example goes through with $\mathbb{Z}_p, \mathbb{Q}_p$ replaced by $F \cap \mathbb{Z}_p, F$, respectively.

6. Final remark

We showed in [JTWY] that the étale open topology over K is induced by a V-topology if and only if K is infinite, t-Henselian, and not separably closed (t-Henselianity is a topological generalization of Henselianity, see [PZ78]). We have produced examples such as $L((t_1, \ldots, t_n))$ where the étale open topology is induced by a field topology that is not a V-topology. We do not know a general criterion for when the étale open topology is induced by a field topology.

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