About a hundred years ago mathematics became a completely rigorous science. This required the creation of formal languages capable of expressing mathematical statements. We now only accept a proof as valid if it can be done (in theory if not in practice) in a formal language. Various theories in these languages capture different parts of mathematics. These theories are themselves mathematical objects. The basic idea of mathematical logic is that we can apply mathematics to itself by studying these theories as mathematical objects and thereby gain a deeper understanding of mathematical structures. A typical question is whether there is an algorithm that can decide if any sentence in the language is true or false, e.g. if there is one method that answers every question. Two early results are Gödel's theorem that number theory is not decidable and Tarski's theorem that Euclidean geometry is decidable.

The initial goal was to construct theories that capture most or all of mathematics. Math is hard, so these turn out to be enormously complex and plagued by incompleteness. However, many bits of mathematics can be captured by well-behaved theories. It is a remarkable empirical fact that exactly one of following holds in all theories T that we understand.

(1) T is at least as complicated as the theory of the ring $\mathbb{Z}$.
(By Gödels work all of mathematics can be encoded in the theory of $\mathbb{Z}$.)

(2) Every formula in the language of T is equivalent to a "simple" formula.

To be clear there is no general dichotomy here, one can construct all kinds of unnatural theories in between. Two typical examples. The field $\mathbb{Q}$ (more generally any number field) defines $\mathbb{Z}$ by a theorem of Julia Robinson. Tarski showed that over the ordered field $\mathbb{R}$ any formula is equivalent to a finite boolean combination of polynomial inequalities, for example

$$\exists x(ax^2 + bx + c = 0) \text{ is equivalent to } b^2 - 4ac \geq 0.$$

This implies that any subset of Euclidean space definable in the language of ordered fields is a boolean combination of polynomial inequalities, and such sets are as geometrically well-behaved as you would think. This is a general phenomenon, definable sets in logically tame structures are usually tame geometric objects in some sense. The main tasks of model theory are to determine where every theory of interest falls under the above dichotomy and to analyze what can be expressed/defined in logically tame theories. We try to think about tame theories in as uniform a way as possible, this gives a unique mathematical viewpoint.

I cannot introduce model theory without at least saying something really vague about Shelah. Shelah defined a host of combinatorial tameness properties of tame theories. These properties result in a "classification theory" of tame theories according to the combinatorics of definable objects revealing deep similarities between seemingly unrelated structures.

1. Model theory of large fields

The model theory of fields is rich subject, with a long history of applications to valuation theory, diophantine geometry, motivitic integration, Berkovich geometry, etc. This subject mainly consists of detailed studies of particular tame fields. Some examples are $\mathbb{C}$ and other algebraically closed fields, $\mathbb{R}$ and other real closed fields, $\mathbb{Q}_p$ and algebraic extensions of $\mathbb{Q}_p$, Henselian valued fields such as fields of formal power series like $L((t))$ and $L\langle\langle t\rangle\rangle$ for any characteristic zero[1] field L, infinite algebraic extensions of finite fields, pseudofinite fields[2],

[1]The status of $\mathbb{F}_p((t))$ is a huge open question.

[2]Infinite fields satisfying the theory of finite fields. They are of interest because algebraic statements about pseudofinite fields are equivalent to statements about sufficiently large finite fields.

pseudo real closed fields such as the field of totally real algebraic numbers, and (conjecturally) the maximal solvable extension of $\mathbb{Q}$.

For the past year and a half I have been working out a general theory of tame fields with (various subsets of) Minh Chieu Tran, Jinhe Ye, Will Johnson, Anand Pillay, Slyvy Anscombe, and Philip Dittman. We start with Pop's notion of largeness. All known logically tame infinite fields (such as those above) are large and the main examples of logically wild fields are the main examples of non-large fields. This has led many to feel that largeness should play a prominent role in the model theory of fields. We are finally making this a reality, the key tool is a novel topology over large fields introduced in [15].

Let K be field. Then K is **large** if whenever $f \in K[x,y]$ and $(a,a') \in K^2$ satisfies $f(a,a') = 0$, $\partial f/\partial y(a,a') \neq 0$ then f has infinitely many zeros in K^2. Equivalently: K is large if every irreducible K-curve with a smooth K-point has infinitely many K-points. The implicit function theorem implies that $\mathbb{R}$ is large and Fermat's theorem that the rational solutions of $x^4 + y^4 - 1 = 0$ are $(\pm 1, 0), (0, \pm 1)$ shows that $\mathbb{Q}$ is not large. Versions of the Mordell conjecture witness non-largeness for number fields and function fields.

Pop introduced largeness and proved the inverse Galois conjecture over $K(x)$ for K large [20]. Large fields are extensively studied in Galois theory. Essentially all fields of particular interest fall (or conjecturally fall) into one of the following mutually exclusive categories:

(1) K is large.
(2) K is finitely generated over its prime subfield (i.e. K is a number field, a function field[3] over a number field, or a function field over a finite field).
(3) K is a function field over a large field. (This is nontrivially equivalent to: K is finitely generated over a large field and not large.)

There is an old family of conjectures that combine to form a conjectural picture of the landscape of logically tame fields, these conjectures basically say that the only fields satisfying certain strong logical properties are the fields we know. The conjectures are beautiful, too hard, and quite possibly mostly false. In recent years evidence has accumulated suggesting that there are are strange logically tame fields lurking out there beyond the fields we know. I believe that this conjectural picture is not really false but rather *too general*, it should become both true and feasible when restricted to large fields. I discuss an important example.

Shelah's most important tameness notion is *stability*. A theory is stable if, very roughly, definable objects have the same combinatorial properties as definable sets in algebraically closed fields. The *stable fields conjecture* asserts that an infinite stable field is separably closed. (The converse is an old theorem of Carol Wood.) Theorem 1.1, proven in [15], is the first significant progress on this conjecture in at least 20 years, arguably 40.

Theorem 1.1. *An infinite field is large and stable if and only if it is separably closed.*

We did not set out to prove 1.1, at some point we just realized that it was basically obvious given what we had proven about the étale open topology. I now explain this topology.

The definition is in terms of étale morphisms, so recall that an étale morphism is the algebraic analogue of a local diffeomorphism. Suppose that V is a K-variety[4] and let $V(K)$ be the

[3]Finitely generated purely transendental extension.

[4]V is the coordinate-free version of a system of polynomial equations with coefficients from K and $V(K)$ is the set of solutions of the system in K.

set of K-points of V. An étale image in $V(K)$ is a set of the form $f(X(K))$ for an étale morphism $f\colon X \to V$ of K-varieties. For example if $f \in K[x]$ and $U = \{a \in K : f'(a) \neq 0\}$ then $f(U)$ is an étale image in $K = \mathbb{A}^1(K)$, e.g. $\{a^n : a \in K^\times\}$ is an étale image when n is prime to the characteristic of K. Basic closure properties of étale maps show that the collection of étale images forms a basis for a topology on $V(K)$ refining the Zariski topology. We refer to this topology as the **étale open ($\mathcal{E}_K$-) topology.**[5] The name is chosen as it is the coarsest topology where $V(K) \to W(K)$ is open when $V \to W$ is étale. My personal interest in this topology arose from the following two theorems, proven in [15].

Theorem 1.2. *Work in the étale open topology. The following are equivalent:*
(1) K is not large.
(2) K is discrete.
(3) $V(K)$ is discrete for every K-variety V.
(4) $V(K)$ is discrete for some K-variety V with $V(K)$ infinite.

The proof of Theorem 1.2 is natural and elementary.

Theorem 1.3. *(1) If K is separably closed then the $\mathcal{E}_K$-topology is the Zariski topology.*
(2) If K is real closed then the $\mathcal{E}_K$-topology is the order topology.
(3) If K is Henselian and not separably closed then the $\mathcal{E}_K$-topology is the valuation topology. (In particular the étale open topology over $\mathbb{Q}_p$ is the p-adic topology.)
Hence if K is a local field other than $\mathbb{C}$ then the $\mathcal{E}_K$-topology is the usual field topology.

Over other large fields such as psuedofinite fields and pseudo real closed fields we get an entirely new topology, in particular we don't get a field topology. However, polynomial maps are $\mathcal{E}_K$-continuous. Remarkably, definable sets in very different logically tame fields behave very similarly with respect to this topology.

Suppose that K is algebraically closed. By Tarski's quantifier elimination every definable subset of K^n is boolean combination of solution sets of polynomial equalities and inequalities. It follows that every definable subset of K^n is a finite union of Zariski open subsets of Zariski closed sets. Suppose that K is real closed (e.g. $K = \mathbb{R}$). By Tarski's other quantifier elimination every definable subset of K^n is a finite union of sets of the form

$$\{a \in K^n : f_1(a) = 0, \ldots, f_k(a) = 0, g_1(a) > 0, \ldots, g_\ell(a) > 0\} \quad \text{for} \quad f_i, g_i \in K[x_1, \ldots, x_n].$$

It follows that every definable set is a finite union of order-open subsets of Zariski closed sets. An application of Macintyre's theorem shows that if K is p-adically closed (e.g. $K = \mathbb{Q}_p$) then every definable set is a finite union of p-adically open subsets of Zariski closed sets.

Motivated by this we say that a subset of K^n is **éz** if it is a finite union of **é**tale open subsets of **Z**ariski closed sets, i.e. if it is a finite union of $\mathcal{E}_K$-locally Zariski closed sets. Note that an éz subset of K is a union of an $\mathcal{E}_K$-open set and a finite set. The field K is éz if K is large and every definable set is éz. By Theorem 1.2 the first condition merely ensures that the second is nontrivial. Éz fields are perfect: If K is characteristic p, large, and not perfect, then $\{a^p : a \in K\}$ has empty $\mathcal{E}_K$-interior and is hence not éz.

Conjecture 1.4. *All known logically perfect tame fields are éz.*

[5]This is not, and indeed is fundamentally different than, the classical étale topology. In particular, unlike the étale topology, it is a topology.

Conjecture 1.4 is largely proven in [23]. We showed that if K is large and perfect then every existentially definable subset[6] of K^n is éz. In most tame fields every formula is equivalent to an existential formula. Motivated by Conjecture 1.4, we are developing a uniform theory of definable sets in éz fields. In particular this gives a uniform approach to the theory of definable sets across all characteristic zero local fields. For example we give a natural dimension on definable sets and show that definable surjections cannot raise dimension [23]. I have also shown that in characteristic zero definable functions are generically smooth.

It should be noted that there are logically wild large fields. For example the fraction field $\mathbb{C}((x, y))$ of $\mathbb{C}[[x, y]]$ is large and defines $\mathbb{Z}$ [9]. However, we do see that existentially definable sets in $\mathbb{C}((x, y))$ are éz and hence geometrically tame. There should be a uniform geometric theory of low complexity definable sets in arbitrary large perfect fields, and the tame fields should be those fields in which this theory extends to all definable sets. Sylvy Anscombe and I are also working on the topology of existentially definable sets in imperfect large fields.

2. Model theory over the reals

In the previous section $\mathbb{R}$ was the real field. Here $\mathbb{R}$ is just a set as all of the structures we consider have domain $\mathbb{R}$. Here are a few of them:

(1) The ordered group $(\mathbb{R}, <, +)$.
(2) The ordered vector space $\mathbb{R}_{\mathrm{Vec}} = (\mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$.
(3) The ordered field $(\mathbb{R}, <, +, \times)$.

Given a subfield K of the real field we say that $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is **K-affine** if there is $\beta \in \mathbb{R}^n$ and K-linear $T^*\colon \mathbb{R}^m \to \mathbb{R}^n$ such that $T(x) = T^*(x) + \beta$. In each of the three structures listed above definable sets are geometrically tame objects.

(1) Definable sets are boolean combinations of solution sets of inequalities between $\mathbb{Q}$-affine functions. Such sets are said to be $\mathbb{Q}$-semilinear.
(2) Definable sets are boolean combinations of solution sets of inequalities between affine functions. Such sets are said to be semilinear.
(3) Definable sets are boolean combinations of solutions sets of polynomial inequalities. Such sets are said to be semialgebraic.

In each case the fact that definable sets are of this form has a concrete geometric meaning. Let $\mathcal{X}$ be any collection of subsets of Euclidean space. Then $(\mathbb{R}, <, +, \mathcal{X})$ is the structure constructed by adding all sets in $\mathcal{X}$ to $(\mathbb{R}, <, +)$. Note that $(\mathbb{R}, <, +, \times)$ is of this form as we can add the graphs of addition and multiplication. The collection of $(\mathbb{R}, <, +, \mathcal{X})$-definable sets is then the smallest collection of subsets of Euclidean space which contains all intervals in $\mathbb{R}$ and is closed under finite boolean combinations, finite products, and images under $\mathbb{Q}$-linear functions $\mathbb{R}^m \to \mathbb{R}^n$. It is easy to see that the collections of $\mathbb{Q}$-semilinear, semilinear, and semialgebraic sets, contain intervals and are closed under finite boolean combinations and products. Theorem 2.1 is the geometric content of the descriptions of definable sets above.

Theorem 2.1 (Tarski)**.** *If $X \subseteq \mathbb{R}^m$ is $\mathbb{Q}$-semilinear, semilinear, semialgebraic and $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is $\mathbb{Q}$-linear then $T(X)$ is $\mathbb{Q}$-semilinear, semilinear, semialgebraic, respectively. (In the latter two cases we may suppose T is linear.)*

[6] Any existentially definable set is of the form $\{\alpha \in K^n : \exists \beta \in K^m[f(\alpha, \beta) = 0]\}$ for a polynomial $f \in K[x_1, \dots, x_n, y_1, \dots, y_m]$ when K is not algebraically closed. Such sets are often called "diophantine".

Theorem 2.1 is foundational for the geometric theories of semilinear and semialgebraic sets. This is best explained by describing (one version of) the main proof technique in the subject. We want to prove something about semialgebraic subsets of $\mathbb{R}^n$. We apply induction on n. Then case $n = 1$ is trivial because a semialgebraic subset of $\mathbb{R}$ is a finite union of intervals. Supposing $X \subseteq \mathbb{R}^n$ semialgebraic we let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a coordinate projection. By Theorem 2.1 $\pi(X)$ is semialgebraic so we apply induction to $\pi(X)$. Sets of the form $X \cap \pi^{-1}(p)$ for $p \in \mathbb{R}^{n-1}$ are again finite unions of intervals, so we combine what we know about $\pi(X)$ and $X \cap \pi^{-1}(p)$ to handle X. For example a (much more complicated) argument of this form is used to show that semialgebraic sets admit semialgebraic triangulations [21, Chapter 8].

The structures listed above where all understood to be tame a hundred years ago. More recently Gabrielov, Wilkie, van den Dries, and others around them, showed that a large array of structures on $\mathbb{R}$ are tame. For example the expansion $\mathbb{R}_{\mathrm{an}}$ of the real field by all functions $[0,1]^m \to \mathbb{R}$ that admit analytic continuations on some open neighbourhood of $[0,1]^m$ is tame, and Wilkie famously showed that $(\mathbb{R}, <, +, \times, \exp)$ is tame.

Question. *Is any structure on $\mathbb{R}$ that defines geometrically pathological sets logically wild?*

I have done a lot of work on this with Hieronymi. A set $X \subseteq \mathbb{R}^m$ is a **fractal** if it is closed and the Hausdorff dimension of X is strictly greater than the topological dimension of X.

Theorem 2.2. *If $X \subseteq \mathbb{R}^m$ is a fractal then $(\mathbb{R}_{\mathrm{Vec}}, X)$ defines all bounded Borel subsets of all $\mathbb{R}^n$, hence $(\mathbb{R}_{\mathrm{Vec}}, X)$ is maximally logically wild. Equivalently: every bounded Borel subset of $\mathbb{R}^n$ may be built up from intervals and X by finitely many boolean combinations, products, and linear images.*

The proof of Theorem 2.2 makes crucial use of tools from geometric measure theory. One many wonder why Theorem 2.2 is not stated over $(\mathbb{R}, <, +)$. This is because it fails. If C is the classical middle thirds Cantor set then the subsets of Euclidean space definable in $(\mathbb{R}, <, +, C)$ are exactly those sets X such that the set of ternary expansions of elements of X is recognizable by an automaton. This remains true when X is replaced by the Menger sponge or the Sierpiński triangle [3, 6, 1]. This surprising fact opens a connection between model theory over $\mathbb{R}$ and automata theory. Hieronymi and I have shown that a structure over $\mathbb{R}$ which defines a closed $X \subseteq \mathbb{R}^m$ such that the C^k-smooth points of X are not dense in X for some k interprets $(\mathbb{R}, <, +, C)$, so any such structure is at least as complex as $(\mathbb{R}, <, +, C)$ in a precise sense [13, 14]. This shows that if X is basically any fractal of interest then $(\mathbb{R}, <, +, X)$ is at least as complicated as $(\mathbb{R}, <, +, C)$.

Conjecture 2.3. *Suppose that $\mathcal{X}$ is a collection of subsets of Euclidean space. Then exactly one of the following holds in $(\mathbb{R}, <, +, \mathcal{X})$:*

1. *There is an interval I and definable $\oplus, \otimes\colon I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(\mathbb{R}, <, +, \times)$.*
2. *If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is continuous and definable then there is dense open subset U of $\mathbb{R}^m$ such that f is affine on each connected component of U.*

It (very nontrivially) follows that if $(\mathbb{R}, <, +, \mathcal{X})$ defines a fractal and does not define every bounded Borel set then (2) above holds, in fact there is a countable subfield K of the real field such that every continuous definable function $\mathbb{R}^m \to \mathbb{R}^n$ is generically locally K-affine.

Hieronymi and I showed that Conjecture 2.3 holds whenever $(\mathbb{R}, <, +, \mathcal{X})$ satisfies any conceivable Shelah-style combinatorial tameness notion [12]. However, this does not cover

$(\mathbb{R}, <, +, C)$. We used automata theory to show that every continuous definable function in $(\mathbb{R}, <, +, C)$ is generically locally $\mathbb{Q}$-affine. This was a consequence of the following.

Theorem 2.4. . *Fix $r \in \mathbb{N}, r \geq 1$. Suppose that $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is continuous and suppose the set of r-ary expansions of elements of the graph of f is recognizable by a Büchi automaton. Then f is generically locally $\mathbb{Q}$-affine. If f is differentiable then f is $\mathbb{Q}$-affine.*

Theorem 2.4 was proven via a combination of automaton theory and fractal geometry in [10]. It strengthened several results from computer science [16, 19, 2] and allowed us to answer a question of Chaudhuri, Sankaranarayanan, and Vardi on recognizable functions [7].

3. Weak interpretations

I mentioned interpretations above at a few points. A structure $\mathcal{M}$ interprets a structure $\mathcal{N}$ if, roughly speaking, there is a definable copy of $\mathcal{N}$ in $\mathcal{M}$. The usual construction of the integers from the naturals shows that $(\mathbb{Z}, +)$ is interpretable in $(\mathbb{N}, +)$, the usual construction of complex numbers shows that $(\mathbb{C}, +, \times)$ is interpretable in $(\mathbb{R}, +, \times)$, and Descartes essentially showed that $(\mathbb{R}, +, \times)$ and Euclidean plane geometry are biinterpretable. We expect that any structure interpretable in a logically tame structure $\mathcal{M}$ is "similar to $\mathcal{M}$", for example every group interpretable $(\mathbb{R}, +, \times)$, $(\mathbb{Q}_p, +, \times)$ is a Lie group, p-adic analytic group, respectively.

It is much harder to show that $(\mathbb{Z}, +)$ does not interpret $(\mathbb{N}, +)$ and $(\mathbb{C}, +, \times)$ does not interpret $(\mathbb{R}, +, \times)$. This follows as $(\mathbb{C}, +, \times)$ and $(\mathbb{Z}, +)$ are stable, $(\mathbb{N}, +)$ and $(\mathbb{R}, +, \times)$ are not, and stability is preserved under interpretations. All of Shelah's combinatorial notions are preserved under interpretations, indeed non-interpretability results are an often-stated motivation of classification theory. Interpretability is a very rigid notion, Shelah's tameness notions are more pliable. The notion of a first order structure is extremely broad, Shelah's work is very much about finding structure underneath the noise produced by this broadness.

I have been working to develop a weak notion of interpretability called *trace definibility* and an associated weak notion of equivalence for first order structures and theories called *trace equivalence* [22]. These notions ignore much of the "noise". The structure $\mathcal{M}$ trace defines $\mathcal{N}$ if, up to isomorphism, the domain N of $\mathcal{N}$ is a subset of M^m and for every $\mathcal{N}$-definable $X \subseteq N^n$ there is $\mathcal{M}$-definable $Y \subseteq M^{mn}$ such that $X = Y \cap M^n$. As is often the case, when we work up to a weaker notion of equivalence, more structure becomes visible. For example, many of Shelah's combinatorial tameness notions are equivalent to non-trace-definibility of a specific theory. For example, stability is equivalent to non-trace-definibility of the theory of dense linear orders. (Pseudofinite fields are unstable and do not interpret linear orders.) There are surprising trace equivalences, e.g. the theory of real closed fields is trace equivalent to the theory of real closed fields equipped with order-compatible valuations.

A structure $\mathcal{M}$ is homogeneous if any partial isomorphism between finite subsets of M extends to an automorphism of $\mathcal{M}$. We assume all homogeneous structures are countable in a finite relational language. There is a long line of work on classificaton of particular kinds of homogeneous structures: graphs [18], tournaments [17], directed graphs [8], permutation structures [5, 4], etc. These classifications become complicated to the point where it is clear that they will not extend to a general classification. When we move to trace equivalence the classifications become simple to the point where it seems likely that they will extend to a general classification. For example, all homogeneous permutation structures are trace equivalent to $(\mathbb{Q}, <)$. The trace equivalence classification of homogeneous structures has consequences in abstract model theory, see [11, 22].

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