GALOIS GROUPS OF LARGE FIELDS WITH SIMPLE THEORY

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Abstract. Suppose that K is an infinite field which is large (in the sense of Pop [Pop96]) and whose first order theory is simple. We show that K is *bounded*, namely has only finitely many separable extensions of any given finite degree. We also show that any genus 0 curve over K has a K-point, if K is additionally perfect then K has trivial Brauer group, and if v is a non-trivial valuation on K then (K, v) has separably closed Henselization, so in particular the residue field of (K, v) is algebraically closed and the value group is divisible. These results give evidence towards the conjecture that large simple fields are bounded PAC. Combining our results with a theorem of Lubotzky and van den Dries we show that if K is also perfect then there is a bounded PAC field L with the same absolute Galois group as K.

1. Introduction

Throughout K is a field. Large fields were introduced by Pop [Pop96], one definition is that K is **large** if any K-curve with a smooth (nonsingular) K-point has infinitely many K-points. Finite fields, number fields, and function fields are not large. Local fields, Henselian fields, quotient fields of Henselian domains, real closed fields, separably closed fields, pseudofinite fields, infinite algebraic extensions of finite fields, p-closed fields, and fields which satisfy a local-global principle (in particular pseudo real closed and pseudo p-adically closed fields) are all large. All infinite fields whose first order theory is known to be "tame" or well-behaved in various senses, are large. Let K^{sep} be a separable closure of K. We say that K is **bounded** if for any n there are only finitely many degree n extensions of K in K^{sep}, equivalently if the absolute Galois group $\mathrm{Aut}(K^{\mathrm{sep}}/K)$ of K has only finitely many open subgroups of any given finite index. When K is also perfect, this is also called Serre's property (F). (Other authors use "bounded" to mean that K has only finitely many extensions of each degree.) Koenigsmann has conjectured that bounded fields are large, see [JK10, p. 496].

Recall that K is **pseudoalgebraically closed (**PAC**)** if any geometrically integral K-variety V has a K-point (and hence the set $V(K)$ of K-points is Zariski dense in V). Let us mention in passing that a PAC field need not be perfect, because for $a \in K$, the variety $\mathrm{Spec}\, K[x]/(x^p - a)$ need not be geometrically integral, see [Poo17, Example 2.2.9]. PAC fields are large, by definition. PAC fields were introduced by Ax [Ax68] who showed that pseudofinite fields are bounded PAC. Infinite algebraic extensions of finite fields are also bounded PAC, see [FJ05, 11.2.4]. In either case PAC follows from the Hasse-Weil estimates.

On the model theoretic side, we have various "tame" classes of first order theories T, the most "perfect" being stable theories, and some others being simple theories and NIP theories. It is a well-known theorem of Shelah that a theory is stable if and only if it is both simple and NIP. Good examples come from theories of fields. We say that a first order structure, in particular a field, is stable (simple, NIP) if its theory is stable (simple, NIP). Separably closed fields are stable and bounded PAC fields are simple. There is a considerable amount of work on NIP fields, which include real closed and p-adically closed fields, but this will not concern us in the present paper. We now recall two longstanding open conjectures.

Conjecture 1.1.

(1) Infinite stable fields are separably closed.
(2) Infinite simple fields are bounded PAC.

Our general idea is that Conjecture 1.1 is both true and tractable after making the additional assumption of largeness. It is shown in [JTWY] that a large stable field is separably closed. We describe another proof of this result in Section 4.1. Here we consider (2), and prove:

Theorem 1.2. *Suppose that K is perfect, large, and simple. Then there is a bounded* PAC *field L of the same characteristic as K such that the absolute Galois group of L is isomorphic (as a topological group) to the absolute Galois group of K.*

Theorem 1.2 follows by combining several results which we now describe.

Theorem 1.3. *If K is large and simple then K is bounded.*

The assumption that K is simple can be replaced by the more general assumption that the field K is definable in some model M of a simple theory. If we also require M to be highly saturated we can take K to be type-definable (over a small set of parameters) in M. The latter will follow from our proofs and references and we will not talk about it again. Theorem 1.3 generalizes the theorem of Chatzidakas that a simple PAC field is bounded, this is proven via quite different methods in [Cha99]. Poizat [Poi83] proved that an infinite stable bounded field is separably closed. Combining Poizat's result with Theorem 1.3 we obtain the above mentioned result of [JTWY] that large stable fields are separably closed.

Theorem 1.3 is reasonably sharp. The restriction to separable extension is necessary. If K is separably closed of infinite imperfection degree and $\mathrm{Char}(K) = p > 0$ then K is large, stable, and has infinitely many extensions of degree p. There is an emerging body of work on a generalization of simplicity known as NSOP_1. Theorem 1.3 fails over NSOP_1 fields as there are unbounded PAC NSOP_1 fields (equivalently: there are PAC fields that are NSOP_1 but not simple). For example if K is characteristic zero, PAC, and the absolute Galois group of K is a free profinite group on $\aleph_0$ generators then K is unbounded and NSOP_1 [CR16, Corollary 6.2]. We do not know if our other results on large simple fields hold for large NSOP_1 fields, see Question 5.11.

A profinite group G is **projective** if any continuous surjective homomorphism $H \to G$, H profinite, has a section. Ax showed [Ax68] that the absolute Galois group of a perfect PAC field is projective, this was proven for arbitrary PAC fields by Jarden [Jar72, Lemma 2.1].

Theorem 1.4. *If K is perfect, large, and simple then the absolute Galois group of K is projective.*

Theorem 1.2 *almost* follows from Theorem 1.3, Theorem 1.4, and the theorem of Lubotzky and van den Dries [LvdD81, p.44] that any projective profinite group is the absolute Galois group of some PAC field, we only need to ensure that the constructed field can be taken to be of the same characteristic as K, see Section 6.

We describe the proof of Theorem 1.4. It is a theorem of Gruenberg [Gru67] that a field of cohomological dimension ≤ 1 has projective absolute Galois group, and if every finite extension of K has trivial Brauer group then K has cohomological dimension ≤ 1 [Ser02, II.3.1, Proposition 5]. The class of perfect fields is closed under finite extensions. If K is simple then any finite extension L of K is simple as L is interpretable in K. Finally, a finite extension of a large field is large by Fact 2.2 below. So Theorem 1.5 implies that every finite extension of a perfect large simple field has trivial Brauer group.

Theorem 1.5. *Suppose that K is perfect, large, and simple. Then the Brauer group of K is trivial. It follows that*

(1) any finite dimensional division algebra over K is a field, and

(2) any Severi-Brauer K-variety V has a K-point.

We recall the definition of Severi-Brauer variety. Let K^{alg} be an algebraic closure of K. Given a K-variety V we let $V_{K^{\mathrm{alg}}}$ be the base change $V \times_K \operatorname{Spec} K^{\mathrm{alg}}$ of V to a K^{alg}-variety. A **Severi-Brauer variety** is a K-variety V such $V_{K^{\mathrm{alg}}}$ is isomorphic (over K^{alg}) to $\dim V$-dimensional projective space. A Severi-Brauer variety is geometrically integral, so (2) is a modest step towards the conjecture that large simple fields are PAC. Theorem 1.5 was proven for supersimple fields in [PSW98], our proof closely follows that in [PSW98], so we will not recall the definition of the Brauer group. (Supersimple fields are perfect, but large simple fields need not be perfect.) Items (1) and (2) of Theorem 1.5 are well-known consequences of triviality of the Brauer group. We refer to [Poo17, 1.5, 4.5.1] for the definition of the Brauer group and these facts.

Suppose $\mathrm{Char}(K) \neq 2$, then we say that a **conic** over K is a smooth irreducible projective K-curve of genus 0, one-dimensional Severi-Brauer varieties are exactly conics [Poo17, 4.5.8]. Thus Theorem 1.6 generalizes the one-dimensional case of Theorem 1.5(2) to imperfect fields.

Theorem 1.6. *Suppose that K is large and simple,* $\mathrm{Char}(K) \neq 2$*, and C is a conic over K. Then C has a K-point, hence (by largeness) $C(K)$ is infinite.*

Let us mention some other earlier work around the conjectures on stable and simple fields described above. One of the first results on deducing algebraic results from model-theoretic hypotheses was Macintyre's theorem that infinite fields with ω-stable theory are algebraically closed ([Mac71] and generalized to superstable fields in [CS80]). Macintyre's Galois-theoretic method has been used in many later works including the result on large stable fields [JTWY] mentioned above. Supersimple theories are simple theories in which there are not infinite forking chains of types, whereby any complete type has an ordinal valued dimension called the SU-rank. This gives a so-called "surgical dimension" as in [PP95] from which one deduces that an infinite field with *supersimple* theory is perfect and bounded. So in so far as Conjecture (2) is restricted to supersimple theories, it remained to prove that supersimple theories are PAC, and some partial results were obtained in [PSW98] and [MPP04] for example. A theme of the current paper is that, other than perfection of of K, any results on supersimple fields also hold over large simple fields.

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2. Large fields and definability

2.1. **Algebraic conventions.** We let K^* be the set of non-zero elements of K and $\mathrm{Char}(K)$ be the characteristic of K. A K**-variety** is a separated, reduced, K-scheme of finite type. We let $\dim V$ be the usual algebraic dimension and $V(K)$ be the set of K-points of a K-variety V. We let $\mathbb{A}^n$ be n-dimensional affine space over K, recall that $\mathbb{A}^n(K) = K^n$. We will often assume irreducibility of the relevant K-varieties. A K-curve is a one-dimensional K-variety. A **morphism** is a morphism of K-varieties.

2.2. **Largeness.** Large fields were introduced by Florian Pop. A survey appears in [Pop] which starts by saying that large fields are fields over which (or in which) one can do a lot of "interesting mathematics". So largeness looks like a field-arithmetic tameness notion. The field K is **large** if every irreducible K-curve with a smooth (also called nonsingular) K-point has infinitely many K-points. Fact 2.1 is due to Pop [Pop96].

Fact 2.1. *The following are equivalent:*

(1) K is large,
(2) K is existentially closed in $K((t))$,
(3) if an irreducible K-variety V has a smooth K-point then $V(K)$ is Zariski dense in V.

We also make use of Fact 2.2, see [Pop, Proposition 2.7].

Fact 2.2. *An algebraic extension of a large field is large.*

Fact 2.3 allows us to pass to elementary extensions, see [Pop, Proposition 2.1].

Fact 2.3. *Large fields form an elementary class.*

2.3. **Existentially étale sets.** Let W be a K-variety. The authors of [JTWY] introduced the **étale open topology** on $W(K)$. If K is not large then the étale open topology is always discrete and if K is large then the étale open topology on $W(K)$ is non-discrete whenever $W(K)$ is infinite. Our original proofs were given in terms of this topology, but at present we will mostly avoid the topology and give proofs from scratch. We will use properties of certain special existentially definable subsets of $W(K)$. As subset X of $W(K)$ is an **EE** set if there is a K-variety W and an étale morphism $f : V \to W$ such that $X = f(V(K))$. It is shown in [JTWY] that the EE subsets of $W(K)$ form a basis for the étale open topology. (In [JTWY] EE sets are referred to as "étale images".)

If W is smooth and $V \to W$ is an étale morphism then V is also smooth. At present we are mainly concerned with subsets of $K^n = \mathbb{A}^n(K)$, so we may restrict attention to smooth K-varieties. We quickly recall what we need from this setting. Let V, W be smooth irreducible K-varieties. An **étale morphism** $f : V \to W$ is a morphism such that the differential df_a is an isomorphism $TV_a \to TW_{f(a)}$ for all $a \in V$. In particular if $f : \mathbb{A}^n \to \mathbb{A}^n$ is a morphism then f is étale at $a \in K^n$ if and only if the Jacobian of f at a is invertible. The general notion of an étale morphism between not necessarily smooth varieties is more complicated but will not be needed here. Fact 2.4 is proven in [Gro67, Proposition 17.1.3].

Fact 2.4. *Suppose W_1, W_2, V are smooth K-varieties and $f_i : W_i \to V$ is an étale morphisms for $i \in \{1, 2\}$. Let W be the fibre product $W_1 \times_V W_2$ and $f : W \to V$ be the canonical map. Then W is a smooth K-variety and f is étale.*

We have $(W_1 \times_V W_2)(K) = \{(a_1, a_2) \in W_1(K) \times W_2(K) : f_1(a_1) = f_2(a_2)\}$. So the image of $(W_1 \times_V W_2)(K)$ under f agrees with $f_1(W_1(K)) \cap f_2(W_2(K))$. Corollary 2.5 follows.

Corollary 2.5. *Suppose that W is a smooth K-variety. Then the collection of EE subsets of $W(K)$ is closed under finite intersections.*

Corollary 2.5 holds for an arbitrary K-variety, but we will not need this.

Lemma 2.6. *Suppose that K is large, W is a smooth irreducible K-variety, and X is a nonempty EE subset of $W(K)$. Then X is Zariski dense in W. In particular any nonempty EE subset of K^n is Zariski dense in K^n.*

The identity morphism $W \to W$ is étale, so Lemma 2.6 generalizes the fact that if K is large and W is a smooth irreducible K-variety with $W(K) \neq \emptyset$ then $W(K)$ is Zariski dense in W.

Proof. Let V be a K-variety and $f : V \to W$ be an étale morphism such that $X = f(V(K))$. Suppose that X is not Zariski dense in V. Then X is contained in a proper closed subvariety Y of W. As W is irreducible we have $\dim Y < \dim W$. Note that $f^{-1}(Y)$ is a closed subvariety of V containing $V(K)$. As f is étale it is finite-to-one, hence $\dim V = \dim W$ and $\dim f^{-1}(Y) = \dim Y < \dim W$. So $f^{-1}(Y)$ is a proper closed subvariety of V containing $V(K)$. This contradicts Fact 2.1. $\square$

Corollary 2.7 follows from Corollary 2.5 and Lemma 2.6.

Corollary 2.7. *Suppose that K is large. Let W be a smooth irreducible K-variety and $X_1, \ldots, X_n$ be EE subsets of $W(K)$ with $\bigcap_{i=1}^{k} X_i \neq \emptyset$. Then $\bigcap_{i=1}^{k} X_i$ is Zariski dense in W. In particular if $X_1, \ldots, X_k$ are EE subsets of K^n with $\bigcap_{i=1}^{k} X_i \neq \emptyset$ then $\bigcap_{i=1}^{k} X_i$ is Zariski dense in K^n.*

Fact 2.8 is proven in [JTWY] for arbitrary K-varieties.

Fact 2.8. *Let W be a smooth K-variety, $g : W \to W$ be a K-variety isomorphism, and X be an EE subset of $W(K)$. Then $g(X)$ is also an EE subset of $W(K)$.*

Proof. Let V be a smooth K-variety and $f : V \to W$ be an étale morphism such that $X = f(V(K))$. Note that g is étale as any K-variety isomorphism is étale. So $g \circ f : V \to W$ is étale as a composition of étale morphisms is étale. $\square$

We will apply Corollary 2.9 below.

Corollary 2.9. *Suppose that X is an EE subset of K^n, $a = (a_1, \ldots, a_n) \in K^n$, and $b = (b_1, \ldots, b_n) \in (K^*)^n$. Then*

$$X + a = \{(c_1 + a_1, \ldots, c_n + a_n) : (c_1, \ldots, c_n) \in X\}$$

and

$$bX = \{(b_1 c_1, \ldots, b_n c_n) : (c_1, \ldots, c_n) \in X\}$$

are EE subsets of K^n.

Proof. The morphisms $\mathbb{A}^n \to \mathbb{A}^n$ given by $(x_1, \ldots, x_n) \mapsto (x_1 + a_1, \ldots, x_n + a_n)$ and $(x_1, \ldots, x_n) \mapsto (b_1 x_1, \ldots, b_n x_n)$ are K-variety isomorphisms. Apply Fact 2.8. $\square$

3. Fields with simple theory

We recall some basic results about fields K whose first order theory is simple, and then make an additional observation under the assumption of largeness. For simple theories see [KP97] and [Cas11], and for groups definable in (models of) simple theories, see in addition [Pil98] and [PSW98]. We recall the relevant portions of this theory.

3.1. **Conventions and basic definitions.** Our model theoretic notation is standard. We let L be a first order language, T be a complete consistent L-theory, and $\overline{M}$ be a highly saturated model of T. For now, $x, y, z, \ldots$ range over finite tuples of variables, $a, b, c, \ldots$ range over finite tuples of parameters from $\overline{M}$, and $A, B, C, \ldots$ range over small subsets of $\overline{M}$. We will sometimes identify definable sets with the formulas defining them.

Given an L-formula $\phi(x, y)$ and a suitable tuple b we say that $\phi(x, b)$ **divides over** a set A of parameters if $\{\phi(x, b_i) : i < \omega\}$ is inconsistent for some infinite A-indiscernible sequence $(b_i : i < \omega)$ with $b_0 = b$. A partial type $\Sigma(x)$ divides over A if some formula in Σ divides over A. The theory T is **simple** if for any small set A of parameters and complete type $\Sigma(x)$ there is $A_0 \subseteq A$ such that $|A_0| \leq |T|$ and $\Sigma(x)$ **does not divide** over A_0. Simplicity may also be defined in terms of the combinatorial **tree property**, but we will not need this. It is worth mentioning that simplicity is incompatible with the existence of a definable partial ordering which contains an infinite chain. It follows that real closed fields and non-separably closed Henselian fields are not simple. Non-dividing yields a good notion of independence in simple theories: a is **independent** from B over A if $\mathrm{tp}(a/B, A)$ does not divide over A.

3.2. **Generics in definable groups.** In this section we summarize [Pil98, Section 3], although we introduce things in a different order and use somewhat different terminology. Suppose that T is simple and G is an infinite group definable over $\emptyset$ in $\overline{M}$. A definable subset X of G is **(left)** f**-generic** if every left translate gX of X does not divide over $\emptyset$ and a complete type $\Sigma(x)$ concentrated on G is (left) f-generic if every formula in $\Sigma(x)$ is left f-generic. Note that if a definable $X \subseteq G$ is f-generic then aX is f-generic for any $a \in G$. Note that in [Pil98] "generic" is used for "f-generic". (The language was changed after some more recent work on groups definable in NIP theories.)

Fact 3.1. *Suppose that T is simple, G is an $\emptyset$-definable group in $\overline{M}$, A, B are small sets of parameters, and $a \in G$. Then*

1. *Left f-genericity is equivalent to right f-genericity, so we just say f-generic.*
2. $\mathrm{tp}(a/A)$ *is left f-generic if whenever $b \in G$ is independent from a over A then the product ba is independent of $A \cup \{b\}$ over $\emptyset$.*
3. *if $A \subseteq B$ and a is independent from B over A, then* $\mathrm{tp}(a/B)$ *is f-generic if and only if* $\mathrm{tp}(a/A)$ *is f-generic.*
4. *if $b \in B$ then* $\mathrm{tp}(a/A, b)$ *is f-generic if and only if* $\mathrm{tp}(ba/A, b)$ *is f-generic.*
5. *an A-definable subset X of G is f-generic if and only if it is contained in an f-generic complete type over A.*

Fact 3.2 is immediate from the definitions.

Fact 3.2. *If $X \subset G$ is not f-generic then there are $g_1, \ldots, g_k \in G$ such that $\bigcap_{i=1}^{k} g_i X = \emptyset$.*

Lemma 3.3. *Suppose that T is simple, M is a model of T, G is an $\emptyset$-definable group in M, H is a subgroup of G with $|G/H| \geq \aleph_0$, and X is a definable subset of G such that $X \subseteq aH$ for some $a \in G$. Then X is not f-generic. In particular an infinite index definable subgroup of G is not f-generic.*

Proof. Let $(g_i : i < \omega)$ be a sequence of elements of G which lie in distinct cosets of H. So $g_i X \cap g_j X = \emptyset$ when $i \neq j$. After passing to a highly saturated elementary extension and applying Ramsey and saturation we obtain an indiscernible sequence $(h_i : i < \omega)$ of elements of G such that $h_i X \cap h_j X = \emptyset$ when $i \neq j$. So X is not f-generic. □

Lemma 3.4. *Suppose that T is simple, X is a definable subset of G, $\approx$ is a definable equivalence relation on X, and each $\approx$-class is f-generic. Then there are only finitely many $\approx$-classes.*

Proof. Suppose towards a contradiction that there are infinitely many $\approx$-classes. Let c be a finite tuple of parameters over which X and $\approx$ are definable. Then there is an $\approx$-class D with canonical parameter d such that $d \notin \mathrm{acl}(c)$. Let $\phi(x, d, c)$ be a formula defining D and $(d_i : i < \omega)$ be an infinite sequence of realizations of $\mathrm{tp}(d/c)$ which is indiscernible over c and satisfies $d_0 = d$. Then $((c, d_i) : i < \omega)$ is indiscernible, and the formulas $\phi(x, d_i, c)$ are pairwise inconsistent, so $\phi(x, d, c)$ divides over $\emptyset$. This contradicts that $\phi(x, d, c)$ defines the set D which is an f-generic subset of K^n. □

We prove the easy Lemma 3.5, which we could not find in the literature.

Lemma 3.5. *Suppose T is simple and G, H are $\emptyset$-definable groups in $\overline{M}$. Fix a small set A of parameters and $(a, b) \in G \times H$. Then $\mathrm{tp}((a, b)/A)$ is f-generic in $G \times H$ if and only if the following conditions hold:*

(1) $\mathrm{tp}(a/A)$ *is an f-generic type of G,*
(2) $\mathrm{tp}(b/A)$ *is an f-generic type of H,*
(3) and a is independent from b over A.

Proof. The definitions and "forking calculus" easily show that (1), (2), and (3) together imply the $\mathrm{tp}((a, b)/A)$ is f-generic in $G \times H$. The difficulty lies in showing that all f-generic types of $G \times H$ are of this form. We suppose that $\mathrm{tp}((a, b)/A)$ is f-generic in $G \times H$. It follows directly that $\mathrm{tp}(a/A)$, and $\mathrm{tp}(b/A)$ are f-generic types of G, H respectively. It remains to prove that a is independent from b over A. Suppose that $(c, d) \in G \times H$, $\mathrm{tp}(c/A), \mathrm{tp}(d/A)$ is f-generic in G, H, respectively, and (c, d) is independent from (a, b) over A. By Fact 3.1 ca is independent from db over $\emptyset$. As $\mathrm{tp}((a, b)/A)$ is f-generic in $G \times H$, and (a, b) is independent from (c, d) over A, we see that that (ca, db) is independent from A, c, d over $\emptyset$. It follows that a is independent from b over A, c, d, and then that a is independent from b over A. □

3.3. **Generics in definable fields.** Now suppose K is an infinite field definable (say over $\emptyset$) in $\overline{M} \models T$. Everything we say remains true for K a type-definable field in $\overline{M}$. We have two attached groups, the additive group $(K, +)$ and the multiplicative group $(K^*, \times)$, recall that $K^* = K \setminus \{0\}$. A definable $X \subseteq K$ is **additively f-generic** if it is f-generic in $(K, +)$ and is **multiplicatively f-generic** if $X \cap K^*$ it is an f-generic in $(K^*, \times)$, and we make the analogous definitions for a type concentrated on K. The first claim of Fact 3.6 is [PSW98, Proposition 3.1]. Uniqueness of f-generic types in stable fields is [Poi01, Theorem 5.10].

Fact 3.6. *Suppose that T is simple. Let X be a definable subset of K, A be a small set of parameters, and $p = \mathrm{tp}(a/A)$ for some $a \in K$. Then X is an additive f-generic if and only if it is a multiplicative f-generic and p is an additive f-generic if and only if it is multiplicative f-generic. Furthermore if T is stable then there is a unique additive f-generic type over K.*

We let D_n be the group $((K^*)^n, \times)$. Corollary 3.7 is a higher dimensional version of Fact 3.6. The first claim of Corollary 3.7 follows from Fact 3.6, Lemma 3.5, and induction on n. The second claim follows from the first claim and Fact 3.1(5).

Corollary 3.7. *Suppose that T is simple, A is a small set of parameters, $a = (a_1, .., a_n) \in K^n$, and $p(x) = \mathrm{tp}(a/A)$. Then p is an f-generic type of $(K^n, +)$ if and only if p is an f-generic type of D_n. So if $X \subseteq K^n$ is definable, then X is f-generic in $(K^n, +)$ if and only if $X \cap D_n$ is f-generic in D_n.*

Proposition 3.8 is our main tool when dealing with large simple fields.

Proposition 3.8. *Suppose that T is simple and K is large. Let X be a nonempty EE subset of K^n. Then X is f-generic for $(K^n, +)$, and is hence f-generic for D_n.*

Proof. Suppose towards a contradiction that X is not f-generic for $(K^n, +)$. By Corollary 3.7, $X \cap D_n$ is not f-generic for D_n. We may suppose that X contains $\overline{0} = (0, \dots, 0)$ as both EE subsets and f-generic subsets of K^n are closed under additive translation (by Corollary 2.9 and definitions). Let $X' = X \cap D_n$. By Corollary 3.7, X' is not f-generic in D_n. By Fact 3.2 there are $g_1, .., g_k \in D_n$ such that $\bigcap_{i=1}^k g_i X' = \emptyset$. Then $\bigcap_{i=1}^k g_i X$ is nonempty, as it contains $\overline{0}$, but is contained in $K^n \setminus D^n$ and is hence not Zariski dense in K^n. This contradicts Corollary 2.7. $\square$

Fact 3.9 will be crucial for Theorem 1.5. It is proven in [PSW98].

Fact 3.9. *Suppose that T is simple. Let H be a finite index definable subgroup of $(K^*, \times)$ and H_1, H_2 be cosets of H. Then $H_1 + H_2$ contains K^*, namely every nonzero element of K is of the form $a + b$ where $a \in H_1$ and $b \in H_2$.*

4. Proof of Theorem 1.3

This section is the proof of Theorem 1.3. Our proof follows the "Remarque" at the end of [PP95] which outlines another proof of the main result of that paper (that fields equipped with a certain "surgical dimension" are bounded) and which was suggested by Zoe Chatzidakis. Remember that when we say that K is bounded we mean that for every n, K has only finitely many extensions of any given degree, inside K^{sep}. We first make a few reductions. Fact 4.1 is well-known, we include a proof for the sake of completenes.

Fact 4.1. *The following are equivalent:*

(1) *K is bounded,*
(2) *for any n there are only finitely many degree n separable extensions of K up to K-algebra isomorphism.*

Proof. By the primitive element theorem a degree n separable extension L of K is of the form $L = K(\alpha)$ where α is a root of a separable irreducible monic degree n polynomial $p(x) \in K[x]$. So L has at most n distinct conjugates over K in K^{sep}, the fact easily follows. $\square$

We set some notation. Given $a = (a_0, ..., a_{n-1}) \in K^n$ we let $p_a(x)$ denote the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. We let U be the set of $a \in K^n$ such that p_a is separable and irreducible in $K[x]$. Note that U is definable. Given $p \in K[x]$ we let (p) be the ideal in $K[x]$ generated by p. For each $a \in U$ the field extension $K(\alpha)$ generated over K by a root α of p_a is isomorphic to $K[x]/(p_a)$. For $a, b \in U$, we write $a \approx b$ if $K[x]/(p_a)$ is isomorphic over K to $K[x]/(p_b)$. So K has finitely many separable extensions of degree n if and only if there are only finitely many $\approx$-classes.

Remark 4.2. The equivalence relation $\approx$ on U is definable in K.

Proof. The field $K[x]/(p_a)$ is uniformly interpretable in K (as a varies), as an n-dimensional vector space over K (with basis $1, \alpha, \ldots, \alpha^{n-1}$ for α a root of $p_a(x)$ and the appropriate multiplication). Now note that if $a, b \in U$ then $a \approx b$ if and only if p_b has a root in $K[x]/(p_a)$. $\square$

The main result we have to prove to obtain Theorem 1.3 is:

Theorem 4.3. *Suppose that $a \in U$ and let D be the $\approx$-class of a. Then there is an EE subset X of K^n such that $a \in X \subseteq D$.*

We now work towards the proof of Theorem 4.3.

Fix $a \in U$, and let $\alpha \in K^{\text{sep}}$ be a root of $p_a(x)$. Let $\overline{x} = (x_0, \ldots, x_{n-1})$ be a tuple of variables and let $\beta(\overline{x}) = x_0 + \alpha x_1 + \ldots + x_{n-1}\alpha^{n-1}$. Let $\alpha = \alpha_1, \ldots, \alpha_n$, be the K-conjugates of α, namely the roots of $p_a(x)$ (which are distinct). And let us write $\beta_i(\overline{x})$ for $x_0 + x_1\alpha_i + \ldots + x_{n-1}\alpha_i^{n-1}$. So, for $b \in K^n$, $\beta_1(b), \ldots, \beta_n(b)$ are the K-conjugates of $\beta(b)$.

Let V be the set of $b = (b_0, b_1, \ldots, b_{n-1}) \in K^n$ such $K(\beta(b)) = K(\alpha)$. Note that $b \in V$ if and only if $\beta(b)$ is a root of $p_c(x)$ for some (in fact unique) $c \in U$ such that $c \approx a$. Note further that $b \in V$ if and only if $\beta(b), \ldots, \beta(b)^{n-1}$ are linearly independent over K, hence V is a Zariski open subset of K^n.

Let $e_1, \ldots, e_n \in \mathbb{Z}[\overline{x}]$ be the elementary symmetric polynomials in n variables, i.e.

$$e_k(\overline{x}) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} \ldots x_{i_k}.$$

Given $b = (b_0, \ldots, b_{n-1}) \in K^n$ we let

$$G(b) = (-e_1(\beta_1(b), \ldots, \beta_n(b)), e_2(\beta_1(b), \ldots, \beta_n(b)), \ldots, (-1)^n e_n(\beta_1(b), \ldots, \beta_n(b))).$$

Claim 4.4. There are $G_1, \ldots, G_n \in K[\overline{x}]$ such that $G(b) = (G_1(b), \ldots, G_n(b))$ for all $b \in K^n$, and if $b \in V$ then $G(b) \approx a$.

The first claim of Claim 4.4 follows as G is symmetric in $\alpha_1, \ldots, \alpha_n$. The second claim follows as $p_{G(b)}$ is the monic polynomial with roots $\beta_1(b), \ldots, \beta_n(b)$. Claim 4.5 below is crucial.

Claim 4.5. $G(0, 1, 0, ..., 0) = a$ and the Jacobian of G at $(0, 1, 0, .., 0)$ is invertible.

Given a polynomial function $f : K^n \to K^n$ we let $\text{Jac}_f(a)$ be the Jacobian of f and $|\text{Jac}_f(a)|$ be the jacobian determinant of f at $a \in K^n$.

Proof. It is clear that $G(0,1,0,..,0) = a$ and $(0,1,0,..,0) \in V$. Let $L = K(\alpha)$. To show that the Jacobian of G at $(0,1,0,..,0)$ is invertible we first produce maps $D, E, F : L^n \to L^n$ such that G agrees with the restriction of $D \circ E \circ F$ to V. We define $F : L^n \to L^n$ by

$$F(b_0, \dots, b_{n-1}) = (b_0 + b_1\alpha_1 + \dots + b_{n-1}\alpha_1^{n-1}, \dots, b_0 + b_1\alpha_n + \dots + b_{n-1}\alpha_n^{n-1}),$$

$E : L^n \to L^n$ is given by

$$E(b) = (e_1(b), \dots, e_n(b)),$$

and $D : L^n \to L^n$ is given by

$$D(b_0, \dots, b_{n-1}) = (-b_0, b_1, -b_2, \dots, (-1)^n b_{n-1}).$$

So if $b \in V$ then $G(b) = (D \circ E \circ F)(b)$. Note that F and D are linear, so Jac_F and Jac_D are constant. Applying the chain rule we have

$$\begin{aligned} \mathrm{Jac}_G(0,1,0,\dots,0) &= \mathrm{Jac}_D \, \mathrm{Jac}_E(F(0,1,0,\dots,0)) \, \mathrm{Jac}_F \\ &= \mathrm{Jac}_D \, \mathrm{Jac}_E(\alpha_1, \dots, \alpha_n) \, \mathrm{Jac}_F . \end{aligned}$$

It is clear that $|\,\mathrm{Jac}_D\,| \in \{-1, 1\}$. Furthermore Jac_F is a Vandermonde matrix

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{pmatrix}.$$

So Jac_F is invertible as $\alpha_1, \dots, \alpha_n$ are distinct. Finally by [LP02]

$$|\,\mathrm{Jac}_E(\alpha_1, \dots, \alpha_n)| = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j).$$

This is non-zero as the α_i are distinct, so $\mathrm{Jac}_E(\alpha_1, \dots, \alpha_n)$ is invertible. ☐

We now deduce Theorem 4.3. Let O be the open subvariety of $\mathbb{A}^n$ given by $|\,\mathrm{Jac}_G(\overline{x})| \neq 0$. So G gives an étale morphism $O \to \mathbb{A}^n$. Then $O(K) \cap V$ is a Zariski open subset of K^n, which is nonempty by Claim 4.4. Let W be an open subvariety of $\mathbb{A}^n$ such that $W(K) = O(K) \cap V$. The restriction of G to W is an étale morphism $W \to \mathbb{A}^n$. Let $X = G(W(K))$. So X is a nonempty EE subset of K^n contained in the $\approx$-class of a. As a was an arbitrary member of U, this concludes the proof of Theorem 4.3.

Finally we can complete the proof of Theorem 1.3.

Proof. Let T be a simple theory, M be a model of T, and K be a large field definable in M. As remarked at the beginning of this section, it suffices to fix n and show that K has only finitely many separable extensions of degree n up to K-algebra isomorphism, and thus that the definable equivalence relation $\approx$ on the definable set $U \subset K^n$ has only finitely many classes. After possibly passing to an elementary extension we may suppose that M is highly saturated. By Theorem 4.3 and Proposition 3.4, every $\approx$-class is f-generic for $(K^n, +)$. By Lemma 3.4 there are only finitely many $\approx$-classes. ☐

4.1. **Another proof that large stable fields are separably closed.** We give a proof that large stable fields are separably closed that avoids Macintyre's Galois theoretic argument. We first prove Lemma 4.6. We continue to use the notation of the previous section.

Lemma 4.6. *Let Y be the set of $a \in K^n$ such that p_a has n distinct roots in K. Then Y is an EE subset of K^n.*

Proof. Let V be the open subvariety of $\mathbb{A}^n$ given by $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Let $H : K^n \to K^n$ be given by $H(b) = (-e_1(b), e_2(b), \ldots, (-1)^n e_n(b))$. So $p_{H(a)}$ is the polynomial with roots $a_1, \ldots, a_n$ for any $a = (a_1, \ldots, a_n) \in V(K)$. It follows from [LP02] that $|\operatorname{Jac}_H(a)|$ agrees up to sign with $\prod_{1 \leq i < j \leq n}(a_i - a_j)$ for any $a = (a_1, \ldots, a_n) \in K^n$. So $\operatorname{Jac}_H(a)$ is invertible for all $a \in V(K)$. Thus $H(V(K))$ is an EE subset of K^n. $\square$

We now show that a large stable field is separably closed.

Proof. Suppose that K is large and not separably closed. Fact 3.6 and Lemma 3.5 together show that if K is stable then for each $n \geq 1$ there is a unique n-ary type over K which is generic for $(K^n, +)$. It follows by Proposition 3.8 and Fact 3.1(5) that if K is stable then any two nonempty EE subsets of K^n intersect. As K is not separably closed there is a separable, irreducible, and non-constant $p \in K[x]$. Suppose that p is monic and fix $a \in K^n$ such that $p = p_a$. By Theorem 4.3 there is an EE subset X of K^n such that $a \in X$ and p_b is separable and irreducible for any $b \in X$. Let Y be the set of $b \in K^n$ such that p_b has n distinct roots in K, by Lemma 4.6 Y is an EE subset of K^n. So X, Y are disjoint nonempty EE subsets of K^n, hence K is unstable. $\square$

The proof above easily adapts to show that an infinite superstable field is algebraically closed. We describe this proof, assuming some familiarity with superstability. We let $\dim_U Z$ be the U-rank of a definable set Z. Suppose that K is infinite and superstable. A superstable field is perfect, so it suffices to show that K is separably closed. Suppose otherwise and fix n such that there is a nonconstant separable irreducible $p \in K[x]$. Let X, Y be as in the proof above. Note that both X and Y contain a set of the form $f(W(K))$ where W is a dense open subvariety of $\mathbb{A}^n$ and $f : W \to \mathbb{A}^n$ is étale. So $\dim_U W(K) = \dim_U K^n$ and the induced map $W(K) \to K^n$ has finite fibers as f is étale. Hence $\dim_U X = \dim_U K^n = \dim_U Y$. So X, Y are both f-generic in $(K^n, +)$, which contradicts uniqueness of generic types.

4.2. **Topological corollaries.** Suppose that v is a non-trivial Henselian valuation on K. It follows from the classical Krasner's lemma that each $\approx$-classs is open in the v-adic topology on K^n. See for example [Poo17, 3.5.13.2] for a treatment of the case when K is a local field which easily generalizes to the Henselian case. It is shown in [JTWY] that if K is not separably closed then the v-adic topology on each K^n agrees with the étale open topology. So Corollary 4.7 generalizes this consequence of Krasner's lemma.

Corollary 4.7. *Fix $a \in K^n$ such that p_a is separable and irreducible. Then the set of $b \in K^n$ such that $K[x]/(p_b)$ is K-algebra isomorphic to $K[x]/(p_a)$ is an étale open neighbourhood of a. So the set of $a \in K^n$ such that p_a is separable and irreducible is étale open.*

Fact 4.8 is proven in [JTWY] by an application of Macintyre's Galois-theoretical argument.

Fact 4.8. *If K is not separably closed then the étale open topology on K is Hausdorff.*

If K is separably closed then the étale open topology agrees with the Zariski topology on $V(K)$ for any K-variety V, equivalently every EE subset of $V(K)$ is Zariski open. We give a proof of Fact 4.8 with avoids Galois theory. We apply the fact that if $V \to W$ is a morphism between K-varieties then the induced map $V(K) \to W(K)$ is étale open continuous.

Proof. Equip K with the étale open topology. Any affine transformation $x \mapsto ax + b$, $a \in K^*, b \in K$ gives a homeomorphism $K \to K$. So it is enough to produce two disjoint nonempty étale open subsets of K. The argument of Section 4.1 yields two disjoint nonempty étale open subsets X, Y of K^n. Fix $p \in X$ and $q \in Y$ and let $f : K \to K^n$ be given by $f(t) = (1-t)p + tq$. Then f is a continuous map between étale open topologies so $f^{-1}(X), f^{-1}(Y)$ are disjoint nonempty étale open subsets of K. □

Finally, we characterize bounded PAC fields amongst PAC fields.

Corollary 4.9. *Suppose that K is* PAC*, equip each K^n with the étale open topology. Then K is bounded if and only if any definable equivalence relation on K^n has only finitely many classes with interior.*

Note that Corollary 4.9 fails when "PAC" is replaced by "large". For example $\mathbb{Q}_p$ is bounded, the étale open topology on $\mathbb{Q}_p$ agrees with the p-adic topology, and the equivalence relation E where $E(a,b)$ if and only if $a, b \in \mathbb{Q}_p$ have the same p-adic valuation is definable and has infinitely many open classes.

Proof. Suppose that K is not bounded. Fix n such that K has infinitely many separable extensions of degree n. Let U and $\approx$ be as in the proof of Theorem 1.3. Then each $\approx$-class is open and there are infinitely many $\approx$-classes. Now suppose that K is bounded and E is a definable equivalence relation on K^n. Note that K is simple. By Proposition 3.8 any E-class with interior is f-generic. The proof of Lemma 3.4 shows that there are only finitely many f-generic E-classes. □

5. Additional remarks and results

We will discuss a few related topics and results, and prove Theorem 1.4. If $\mathrm{Char}(K) = p > 0$ then we let $\wp : K \to K$ be the **Artin-Schreier map** $\wp(x) = x^p - x$. This map is an additive homomorphism, so $\wp(K)$ is a subgroup of $(K, +)$. In this section we let $P_n = \{a^n : a \in K^*\}$ for each n. Some of our proofs below could be simplified by apply Scanlon's theorem [KSW11] that an infinite stable field is Artin-Schreier closed, but we will avoid this.

5.1. **Boundedness and large stable fields.** It is a theorem of Poizat that an infinite bounded stable field is separably closed. Poizat's result and Theorem 1.3 together show that large stable fields are separably closed. Poizat's result is mentioned somewhat informally at the bottom of p. 347 in [Poi83] and does not appear to be well-known, so we will take the opportunity to clarify the matter. Fact 5.1 is [Poi83, Lemma 4].

Fact 5.1. *Suppose that L is a finite Galois extension of K. Then the following holds.*

1. *If $q \neq \mathrm{Char}(K)$ is a prime then there are only finitely many cosets H of P_q in $(K^*, \times)$ such that some (equivalently: any) $a \in H$ is of the form b^q for some $b \in L$.*
2. *Suppose that $\mathrm{Char}(K) > 0$. Then there are only finitely many cosets H of $\wp(K)$ in $(K, +)$ such that some (equivalently: any) $a \in H$ is of the form $b^p - b$ for some $b \in L$.*

Fact 5.2 follows from Fact 5.1.

Fact 5.2. *Suppose that K is bounded. Then*

(1) if $q \neq \mathrm{Char}(K)$ is prime then P_q has finite index in $(K^*, \times)$, and
(2) if $\mathrm{Char}(K) > 0$ then $\wp(K)$ has finite index in $(K, +)$.

We sketch a proof. See [FJ16, Lemma 2.2] for a proof of Fact 5.2(1) via Galois cohomology.

Proof. We only prove (1) as the proof of (2) is similar. Suppose $a \in K^*$ and $\alpha \in K^{\mathrm{sep}}$ satisfies $\alpha^q = a$. Then α and its conjugates generate a degree $\leq q$ Galois extension of K. As K is bounded there are only finitely many such extensions. So by Fact 5.2 P_q has finite index in $(K^*, \times)$. □

Finally, Fact 5.3 is essentially proven in [Mac71] via a Galois-theoretic argument.

Fact 5.3. *Suppose that the following holds for any finite Galois extension L of K:*

(1) the qth power map $L^* \to L^*$ is surjective for any prime $q \neq \mathrm{Char}(K)$, and
(2) if $\mathrm{Char}(K) > 0$ then the Artin-Schreier map $L \to L$ is surjective.

Then K is separably closed.

We now give a proof of Poizat's theorem.

Corollary 5.4. *Suppose that K is infinite, bounded, and stable. Then K is separably closed.*

Proof. We verify the conditions of Fact 5.3. Suppose that L is a finite Galois extension of K. Then L is bounded and stable (the latter holds as L is interpretable in K). As L is stable there is a unique additive (multiplicative) generic type over K, see Fact 3.6. It follows that there are no proper finite index definable subgroups of $(L^*, \times)$ or $(L, +)$. So by Fact 5.2 the qth power map $L^* \to L^*$ is surjective for any prime $q \neq \mathrm{Char}(K)$ and if $\mathrm{Char}(K) > 0$ then the Artin-Schreier map $L \to L$ is surjective. □

We repeat that it follows from Fact 5.1 and Theorem 1.3 that:

Corollary 5.5. *Suppose that K is large and simple. Then*

(1) if $q \neq \mathrm{Char}(K)$ is prime then P_q has finite index in $(K^*, \times)$, and
(2) if $\mathrm{Char}(K) > 0$ then $\wp(K)$ has finite index in $(K, +)$.

Corollary 5.5(2) is proven more generally for infinite simple fields in [KSW11]. We take the opportunity to sketch a direct proof of Corollary 5.5. We let $\mathbb{G}_m$ be the scheme-theoretic multiplicative group $\mathrm{Spec}\, K[x, x^{-1}]$, so $\mathbb{G}_m(K) = K^*$.

Proof. We first fix a prime $q \neq \mathrm{Char}(K)$. The morphism $\mathbb{G}_m \to \mathbb{A}^1$ given by $x \mapsto ax^q$ is étale for any $a \in K^*$. So any coset of P_q is an EE subset of K. By the special (and easier) case of Proposition 3.8 when $n = 1$, any coset of P_q is f-generic in $(K^*, \times)$. By Lemma 3.3 P_q has finite index in $(K^*, \times)$. Item (2) follows by a similar argument and the fact that the Artin-Schreier morphism $\mathbb{A}^1 \to \mathbb{A}^1$ is étale. □

Fehm and Jahnke construct an unbounded PAC field K such that the group of nth powers has finite index in each finite extension of K [FJ16, Proposition 4.4], so Theorem 1.3 does not follow from Corollary 5.5.

5.2. **Conics and the Brauer group.** Corollary 5.6 follows from Corollary 5.5 and Fact 3.9.

Corollary 5.6. *Suppose that K is large and simple, $a, b \in K^*$, and $p \neq \mathrm{Char}(K)$ is a prime. Then there are $c, d \in K$ such that $c^p + ad^p = b$.*

The proof in [PSW98] that conics over (infinite) supersimple fields have points now extends to proving Theorem 1.6.

Proof of Theorem 1.6. Let C be a conic, i.e. a smooth projective irreducible K-curve of genus 0. As $\mathrm{Char}(K) \neq 2$ we may assume that C is a closed subvariety of $\mathbb{P}^2$ given by the homogenous equation $ax^2 + by^2 = z$ for some $a, b \in K^*$. By Corollary 5.6 there are $c, d \in K$ such that $ac^2 + bd^2 = 1$. So $C(K)$ is nonempty.
Fact 5.7 follows by the proof of [PSW98, Theorem 4.6].

Fact 5.7. *Suppose that:*

1. *(1) K is perfect, and*
2. *(2) if L is a finite extension of K, p is a prime, and $a \in L^*$, then $\{b^p + ac^p : b, c \in L^*\}$ contains L^*.*

Then the Brauer group of K is trivial.

We now prove Theorem 1.5.

Proof. It suffices to show that the second condition of Fact 5.7 is satisfied. Let L be a finite extension of K and p be a prime. Note that L is perfect as a finite extension of a perfect field is perfect, the case when $p = \mathrm{Char}(K)$ follows. Suppose that $p \neq \mathrm{Char}(K)$. Note that L is simple as L is interpretable in K and L is large by Fact 2.2. Apply Corollary 5.6. ☐

Finally we record a consequence of Corollary 5.6 and the fact from [PSW98] that Corollary 5.6 holds over a supersimple field.

Remark 5.8. Suppose that $\mathrm{Char}(K) \neq 2$ and K is either infinite and supersimple or large and simple. Then there are $a, b \in K$ such that $a^2 + b^2 = -1$. So if K is either supersimple or large and simple then K is not formally real.

We do not know if there is a formally real simple field.

5.3. **Henselizations.** The conclusions of Theorems 1.6 and 1.5 are properties of PAC fields. Another well-known consequence of a field K being PAC is that the Henselization of any non-trivial valuation on K is separably closed, see [FJ05, Corollary 11.5.9]. We refer to [EP05, Chapter 5] for an account of Henselizations. Prestel [Pre91] proved that if K is Henselian and not separably closed then a basis for the valuation topology on K is (uniformly) definable in the field language. So a non-separably closed Henselian field is not simple. Proposition 5.9 extends this fact and supports the conjecture that large, simple, fields are PAC. If v is a valuation on K then we let k_v and Γ_v be the residue field and value group of v, respectively.

Proposition 5.9. *Suppose that K is large and simple and let v be a non-trivial valuation on K. Then the Henselization of (K, v) is separably closed. In particular k_v is algebraically closed and Γ_v is divisible.*

The second claim follows from the first as the Henselization of (K, v) has residue field k_v and value group Γ_v, and a non-trivially valued separably closed field has algebraically closed residue field and divisible value group [EP05, 3.2.11]. We apply Fact 5.10, proven in [JTWY].

Fact 5.10. *Let v be a non-trivial valuation on K. If the Henselization of (K, v) is not separably closed then étale open topology refines the v-adic topology on K.*

Proof of Proposition 5.9

Proof. Suppose that the Henselization of (K, v) is not separably closed. Let O be the valuation ring of v and $O^* = \{a \in O : a^{-1} \in O\}$. By Fact 5.10 O^* is an étale open neighbourhood of 1, so there is an EE subset X of K satisfying $1 \in X \subset O^*$. By Proposition 3.8 X is f-generic for $(K^*, \times)$. Finally $(K^*, \times)/O^*$ is isomorphic to Γ_v so in particular O^* is an infinite index subgroup of $(K^*, \times)$. This contradicts Lemma 3.3. $\square$

It is also shown in [JTWY] that if $<$ is a field order on K then the étale open topology refines the $<$-topology. A similar argument may be used to give another proof that large simple fields are not formally real, see Remark 5.8. Finally, we pose a question.

Question 5.11. *Suppose that K is* NSOP$_1$ *and large and let v be a non-trivial valuation on K. Must the Henselization of (K, v) be separably closed?*

Question 5.11 has a positive answer if every large NSOP$_1$ field is PAC, and all known examples of infinite NSOP$_1$ fields are PAC. Question 5.11 seems much more accessible then the question of whether a large NSOP$_1$ field is PAC. At present we do not have a theory of generics in NSOP$_1$ groups.

6. Proof of Theorem 1.2

Finally, we describe the proof of Theorem 1.2. By Theorems 1.3 and 1.4 it is enough to suppose that G is a projective profinite group with only finitely many open subgroups of any given index, p is 0 or a prime, and construct a PAC field L of characteristic p with absolute Galois group G (note that L is necessarily bounded). Lubotzky and van den Dries [LvdD81] do this in the case when $p = 0$. We take a slightly different approach below.

Let $\mathbb{F}_\omega$ be the free profinite group on $\aleph_0$-generators. Note that G has only countably many open subgroups and is hence separable. By a result of Douady [Dou95, Proposition 17.1.1] there is a continuous surjective homomorphism $\pi : \mathbb{F}_\omega \to G$. As G is projective π has a section $\tau : G \to \mathbb{F}_\omega$, so τ is a continuous injective homomorphism, and hence has closed image. So we may suppose that G is a closed subgroup of $\mathbb{F}_\omega$. By [FJ05, 27.4] there is a characteristic p PAC field F with absolute Galois group $\mathrm{Aut}(F^{\mathrm{sep}}/F) = \mathbb{F}_\omega$. Let L be the fixed field of G. Note that L is PAC as the class of PAC fields is closed under algebraic extensions [FJ05, Corollary 11.2.5].

References

[Ax68] James Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271. MR 229613

[Cas11] Enrique Casanovas, *Simple theories and hyperimaginaries*, Lecture Notes in Logic, vol. 39, Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2011. MR 2814891

[Cha99] Zoé Chatzidakis, *Simplicity and independence for pseudo-algebraically closed fields*, Models and computability (Leeds, 1997), London Math. Soc. Lecture Note Ser., vol. 259, Cambridge Univ. Press, Cambridge, 1999, pp. 41–61. MR 1721163

[CR16] Artem Chernikov and Nicholas Ramsey, *On model-theoretic tree properties*, J. Math. Log. **16** (2016), no. 2, 1650009, 41. MR 3580894

[CS80] G. Cherlin and S. Shelah, *Superstable fields and groups*, Ann. Math. Logic **18** (1980), no. 3, 227–270. MR 585519

[Dou95] Adrien Douady, *Cohomologie des groupes compacts totalement discontinus (d'après Tate)*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, pp. Exp. No. 189, 287–298. MR 1603474

[EP05] Antonio J. Engler and Alexander Prestel, *Valued fields*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. MR 2183496

[FJ05] Michael D. Fried and Moshe Jarden, *Field arithmetic*, Springer Berlin Heidelberg, 2005.

[FJ16] Arno Fehm and Franziska Jahnke, *Fields with almost small absolute galois group*, Israel Journal of Mathematics **214** (2016), no. 1, 193–207.

[Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 238860

[Gru67] K. W. Gruenberg, *Projective profinite groups*, J. London Math. Soc. **42** (1967), 155–165. MR 209362

[Jar72] Moshe Jarden, *Elementary statements over large algebraic fields*, Trans. Amer. Math. Soc. **164** (1972), 67–91. MR 302651

[JK10] Markus Junker and Jochen Koenigsmann, *Schlanke Körper (slim fields)*, J. Symbolic Logic **75** (2010), no. 2, 481–500. MR 2648152

[JTWY] Will Johnson, Minh Chieu Tran, Erik Walsberg, and Vinncent Ye, *Étale open topology and the stable field conjecture*, arXiv:2009.02319.

[KP97] Byunghan Kim and Anand Pillay, *Simple theories*, vol. 88, 1997, Joint AILA-KGS Model Theory Meeting (Florence, 1995), pp. 149–164. MR 1600895

[KSW11] Itay Kaplan, Thomas Scanlon, and Frank O. Wagner, *Artin-schreier extensions in NIP and simple fields*, Israel Journal of Mathematics **185** (2011), no. 1, 141–153.

[LP02] Alain Lascoux and Piotr Pragacz, *Jacobians of symmetric polynomials*, Ann. Comb. **6** (2002), no. 2, 169–172. MR 1955517

[LvdD81] A. Lubotzky and L. van den Dries, *Subgroups of free profinite groups and large subfields of* $\mathbf{Q}$, Israel J. Math. **39** (1981), no. 1-2, 25–45. MR 617288

[Mac71] Angus Macintyre, *On* ω_1*-categorical theories of fields*, Fund. Math. **71** (1971), no. 1, 1–25. (errata insert). MR 290954

[MPP04] Amador Martin-Pizarro and Anand Pillay, *Elliptic and hyperelliptic curves over supersimple fields*, J. London Math. Soc. (2) **69** (2004), no. 1, 1–13. MR 2025323

[Pil98] Anand Pillay, *Definability and definable groups in simple theories*, J. Symbolic Logic **63** (1998), no. 3, 788–796. MR 1649061

[Poi83] Bruno Poizat, *Groupes stables, avec types génériques réguliers*, J. Symbolic Logic **48** (1983), no. 2, 339–355. MR 704088

[Poi01] ———, *Stable groups*, Mathematical Surveys and Monographs, vol. 87, American Mathematical Society, Providence, RI, 2001, Translated from the 1987 French original by Moses Gabriel Klein. MR 1827833

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR 3729254

[Pop] Florian Pop, *Little survey on large fields - old & new*, Valuation Theory in Interaction, European Mathematical Society Publishing House, pp. 432–463.

[Pop96] Florian Pop, *Embedding problems over large fields*, Ann. of Math. (2) **144** (1996), no. 1, 1–34. MR 1405941

[PP95] Anand Pillay and Bruno Poizat, *Corps et chirurgie*, J. Symbolic Logic **60** (1995), no. 2, 528–533. MR 1335134

[Pre91] Alexander Prestel, *Algebraic number fields elementarily determined by their absolute galois group*, Israel Journal of Mathematics **73** (1991), no. 2, 199–205.

[PSW98] A. Pillay, T. Scanlon, and F. O. Wagner, *Supersimple fields and division rings*, Math. Res. Lett. **5** (1998), no. 4, 473–483. MR 1653312

[Ser02] Jean-Pierre Serre, *Galois cohomology*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. MR 1867431

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