NIPPY PROOFS OF P-ADIC RESULTS OF DELON AND YAO

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Abstract. Let K be an elementary extension of $\mathbb{Q}_p$, V be the set of finite $a \in K$, st be the standard part map $K^m \to \mathbb{Q}_p^m$, and $X \subseteq K^m$ be K-definable. Delon has shown that $\mathbb{Q}_p^m \cap X$ is $\mathbb{Q}_p$-definable. Yao has shown that $\dim \mathbb{Q}_p^m \cap X \leq \dim X$ and $\dim \operatorname{st}(V^n \cap X) \leq \dim X$. We give new NIP-theoretic proofs of these results and show that both inequalities hold in much more general settings. We also prove the analogous results for the expansion $\mathbb{Q}_p^{\text{an}}$ of $\mathbb{Q}_p$ by all analytic functions $\mathbb{Z}_p^m \to \mathbb{Q}_p$. As an application we show that if $(X_k)_{k\in\mathbb{N}}$ is a sequence of elements of an $\mathbb{Q}_p^{\text{an}}$-definable family of subsets of $\mathbb{Q}_p^m$ which converges in the Hausdroff topology to $X \subseteq \mathbb{Q}_p^m$ then X is $\mathbb{Q}_p^{\text{an}}$-definable and $\dim X \leq \limsup_{k\to\infty} \dim X_k$.

1. Introduction

Fix a prime p, let K be an elementary extension of $\mathbb{Q}_p$, V be the set of $a \in K$ such that $\operatorname{val}(a) \geq k$ for some $k \in \mathbb{Z}$, and st be the standard part map $V^m \to \mathbb{Q}_p^m$. Fact 1.1, a p-adic analogue of the Marker-Steinhorn theorem [18], is due to Delon [8].

Fact 1.1. *If $X \subseteq K^m$ is K-definable then $\mathbb{Q}_p^m \cap X$ is $\mathbb{Q}_p$-definable.*

Fact 1.2 follows from standard results on equicharacteristic zero Henselian valued fields as V is a Henselian valuation ring.

Fact 1.2. *If $X \subseteq K^m$ is K-definable then* $\operatorname{st}(V^m \cap X)$ *is $\mathbb{Q}_p$-definable.*

Let $\dim X$ be the dimension of a $\mathbb{Q}_p$-definable set X. Fact 1.3 is a result of Yao [36].

Fact 1.3. *Suppose that $X \subseteq K^m$ is $\mathcal{K}$-definable. Then* $\dim \mathbb{Q}_p^m \cap X \leq \dim X$ *and* $\dim \operatorname{st}(V^m \cap X) \leq \dim X$.

As $\mathbb{Q}_p^m \cap X$ is a subset of $\operatorname{st}(V^m \cap X)$ the first inequality is a corollary to the second, but we will see that they generalize in different directions. The analogue of Fact 1.3 for o-minimal expansions of $(\mathbb{R}, +, \times)$ were previously proven by van den Dries [31]. We show that Fact 1.3 and its o-minimal analogue follow easily from the theory of externally definable sets in NIP structures. The first inequality generalizes to an arbitrary elementary extension of an arbitrary NIP structure and the second inequality generalizes to any "tame extension" of dp-minimal valued fields. We also show that Fact 1.1 follows from general NIP results and Fact 1.2. In Section 8 we prove the analogues of Facts 1.1 and 1.3 for $\mathbb{Q}_p^{\text{an}}$. In Section 9 we use these results to study Hausdorff limits of $\mathbb{Q}_p^{\text{an}}$-definable sets, this is the p-adic analogue of o-minimal work of van den Dries [32].

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2. Conventions and notation

By "definable" we mean "first order definable, possibly with parameters". Throughout m, n are natural numbers, i, j, k, l are integers, and $\mathcal{M}$ is an L-structure. If X is an $\mathcal{M}$-definable set and $\mathcal{M} \prec \mathcal{N}$ then $X(\mathcal{N})$ is the $\mathcal{N}$-definable set defined by the same formula as X. Two structures on the same domain are **interdefinable** if they define the same sets. If $x = (x_1, \ldots, x_n)$ is a tuple of variables then $|x| = n$. The structure **induced** on $A \subseteq M^m$ by $\mathcal{M}$ is the structure with domain A and an n-ary relation for $A^n \cap X$ for each $\mathcal{M}$-definable $X \subseteq M^{nm}$. Our reference on NIP and dp-rank is Simon's book [27].

3. Dp rank

3.1. **Definition.** To generalize Fact 1.3 we need a general notion of dimension. We use dp-rank. The dp-rank of a definable set is either a cardinal or the formal symbol ∞ which is by declared to be larger than all cardinals. Fix an $\mathcal{M}$-definable set $X \subseteq M^{|y|}$. Let λ be a cardinal. An $(\mathcal{M}, X, \lambda)$-array consists of a sequence $(\varphi_\alpha(x_\alpha; y) : \alpha < \lambda)$ of L-formulas and an array $(a_{\alpha,i} \in M^{|x_\alpha|} : \alpha < \lambda, i < \omega)$ such that for every $f : \lambda \to \omega$ there is $b \in X$ such that

$$\mathcal{M} \models \varphi_\alpha(a_{\alpha,i}; b) \quad \text{if and only if} \quad f(\alpha) = i \quad \text{for all } \alpha, i.$$

Then $\dim X \geq \lambda$ if there is $\mathcal{M} \prec \mathcal{N}$ and an $(\mathcal{N}, X(\mathcal{N}), \lambda)$-array. If $\dim X \geq \lambda$ for all cardinals λ then $\dim X := \infty$, we let $\dim X := \max\{\lambda : \dim X \geq \lambda\}$ when this maximum exists and otherwise declare

$$\dim X := \sup\{\lambda : \dim X \geq \lambda\} - 1.$$

The dp-rank of $\mathcal{M}$ is defined to be $\dim M$. Of course these definitions raise the question of what exactly $\kappa - 1$ is when κ is an infinite cardinal. There are several options, and it does not matter which we select. When the structure may not be clear from context we let $\dim_{\mathcal{M}} X$ be the dp-rank of an $\mathcal{M}$-definable set X.

We will want to avoid passing to an elementary extension, so we use finitary arrays. Let Φ be a sequence $(\varphi_\alpha(x_\alpha; y) : \alpha < \lambda)$ of L-formulas, $F \subseteq \lambda$ be finite, and $n \in \mathbb{N}$. An $(\mathcal{M}, X, \Phi, F, n)$-array is an array $(a_{\alpha,i} \in M^{|x_\alpha|} : \alpha \in F, i \leq n)$ such that for every $f : F \to n$ there is $b \in X$ such that for all $\alpha \in F, i \leq n$ we have $\mathcal{M} \models \varphi_\alpha(a_{\alpha,i}; b)$ if and only if $f(\alpha) = i$. So $\dim X \geq \lambda$ if and only if there is such a Φ so that for every finite $F \subseteq \lambda$ and n there is an $(\mathcal{M}, X, \Phi, F, n)$-array.

3.2. **Properties.** Dp-rank characterizes NIP structures, see [27].

Fact 3.1. *Let T be a complete theory and $\mathcal{M} \models T$. The following are equivalent.*

1. $\mathcal{M}$ *is* NIP,
2. $\dim \mathcal{M} < \infty$,
3. $\dim \mathcal{M} < |T|^+$,
4. $\dim X < |T|^+$ *for all $\mathcal{M}$-definable sets X.*

Fact 3.2 shows that dp-rank is a reasonable notion of dimension. The first two items follow easily from the definition and the fourth is proven in [12].

Fact 3.2. *Suppose $\mathcal{M}$ is* NIP*, X, Y are definable sets, and $f : X \to M^m$ is definable.*

1. $\dim X = 0$ *if and only if X is finite.*
2. $\dim X \cup Y = \max\{\dim X, \dim Y\}$. *(so $X \subseteq Y$ implies $\dim X \leq \dim Y$),*

(3) $\dim f(X) \leq \dim X$,
(4) *If* $\dim f^{-1}(a) \leq \lambda$ *for all* $a \in f(X)$ *then* $\dim X \leq \dim f(X) + \lambda$.

We say that $\mathcal{M}$ is **dp-minimal** when $\dim \mathcal{M} \leq 1$. O-minimal structures and $\mathbb{Q}_p$ are both dp-minimal [10]. Dp-rank is the canonical notion of dimension for definable sets in dp-minimal expansions of valued fields or divisible ordered abelian groups. Fact 3.3 is proven in [28]. Let Y be a topological space and equip Y^n with the product topology. The **naive dimension** of a nonempty $X \subseteq Y^n$ is the maximal $0 \leq k \leq n$ such that $\pi(X)$ has interior for some coordinate projection $\pi : Y^n \to Y^k$. Acl-dimension is defined in the same way as dimension is usually defined in a geometric structure (this definition makes sense in any structure).

Fact 3.3. *Let $\mathcal{M}$ be a dp-minimal expansion of a valued field or a divisible ordered abelian group and $X \subseteq M^n$ be definable and nonempty. The following are equal:*
(1) The dp-rank of X,
(2) The acl*-dimension of X, and*
(3) The naive dimension of X.

So in particular dp-rank agrees with the canonical dimension for definable sets in a p-adically closed field or an o-minimal expansion of an ordered abelian group. It follows from Fact 3.3 that in this setting the dp-rank of X depends only on X and the topology, not on $\mathcal{M}$. We will also apply Fact 3.4, proven in [28].

Fact 3.4. *Suppose that $\mathcal{K}$ is a dp-minimal expansion of a valued field. Then every $\mathcal{K}$-definable set is a boolean combination of closed $\mathcal{K}$-definable sets.*

4. Externally definable sets

Throughout this section $\mathcal{M} \prec \mathcal{N}$, $\mathcal{N}$ is highly saturated, and $X \subseteq M^n$. We say that X is **externally definable** if $X = M^n \cap Y$ for some $\mathcal{N}$-definable Y. By saturation the collection of externally definable sets does not depend on choice of $\mathcal{N}$. We say that $\mathcal{M}$ is **Shelah complete** if every externally definable set is definable. The **Shelah completion** $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ is the structure induced on M by $\mathcal{N}$. We say that $Y \subseteq N^n$ is an **honest definition** of X if Y is $\mathcal{N}$-definable, $M^n \cap Y = X$, and whenever $Z \subseteq M^n$ is $\mathcal{M}$-definable such that $Z \cap X = \emptyset$ then $Z(\mathcal{N}) \cap Y = \emptyset$. The second claim of Fact 4.1 is a theorem of Shelah [26]. The first is due to Chernikov and Simon [3]. The second claim is a corollary to the first.

Fact 4.1. *Suppose $\mathcal{M}$ is* NIP*. Every externally definable subset has an honest definition. Every $\mathcal{M}^{\mathrm{Sh}}$-definable set is externally definable.*

It follows easily from Fact 4.1 that the Shelah completion of an NIP structure is Shelah complete, this justifies our terminology. Shelah observed that Fact 4.1 implies the first claim of Fact 4.2. The second claim is due to Onshuus and Usvyatsov[22].

Fact 4.2. *If $\mathcal{M}$ is* NIP *then $\mathcal{M}^{\mathrm{Sh}}$ is* NIP*. If $\mathcal{M}$ is dp-minimal then $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal.*

The first claim of Fact 4.3 is elementary. The second claim follows from the first, Fact 4.1, and saturation.

Fact 4.3. *Suppose $\mathcal{M} \prec \mathcal{O}$. If $X \subseteq O^n$ is externally definable in $\mathcal{O}$ then $M^n \cap X$ is externally definable in $\mathcal{M}$. If $\mathcal{M}$ is* NIP *then the structure induced on M by $\mathcal{O}^{\mathrm{Sh}}$ is interdefinable with $\mathcal{M}^{\mathrm{Sh}}$.*

Lemma 4.4 is easy and left to the reader.

Lemma 4.4. *Suppose that* $\mathcal{M}$ *is* NIP*,* $\mathcal{M}$ *is Shelah complete, and* $\mathcal{O} \prec \mathcal{M}$*. Then every* $\mathcal{O}^{\mathrm{Sh}}$*-definable set is of the form* $O^n \cap X$ *for* $\mathcal{M}$*-definable* $X \subseteq M^n$*.*

5. The first inequality

We generalize the first inequality to arbitrary NIP structures.

Proposition 5.1. *Suppose that* $\mathcal{M}$ *is* NIP*,* $\mathcal{M} \prec \mathcal{N}$*, and* $X \subseteq N^m$ *is* $\mathcal{N}$*-definable. Then* $\dim_{\mathcal{M}^{\mathrm{Sh}}} M^n \cap X \leq \dim_{\mathcal{N}} X$

Taking $X = M$ we get $\dim \mathcal{M} = \dim \mathcal{M}^{\mathrm{Sh}}$. So Proposition 5.1 generalizes Fact 4.2. The proof below is essentially the same as Onshuus and Usvyatsov's proof that $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal when $\mathcal{M}$ is dp-minimal [22].

Proof. Let L^{Sh} be the language of $\mathcal{M}^{\mathrm{Sh}}$. If $\mathcal{N} \prec \mathcal{O}$ is highly saturated then we have $M^m \cap X = M^m \cap X(\mathcal{O})$, so after possibly replacing $\mathcal{N}$ with $\mathcal{O}$ we suppose that $\mathcal{N}$ is highly saturated. Let $Y := M^m \cap X$ and λ be a cardinal. Suppose that $\dim_{\mathcal{M}^{\mathrm{Sh}}} Y \geq \lambda$. Let $|y| = m$ and fix a sequence $\Phi := (\varphi_\alpha(x_\alpha; y) : \alpha < \lambda)$ of L^{Sh}-formulas such that for every finite $F \subseteq \lambda$ and n there is a $(\mathcal{M}^{\mathrm{Sh}}, Y, \Phi, F, n)$-array. By Fact 4.1 we have for each $\alpha \leq \lambda$ an L-formula $\theta_\alpha(x_\alpha; y)$ such that

$$\mathcal{M} \models \varphi_\alpha(a; b) \quad \text{if and only if} \quad \mathcal{N} \models \theta_\alpha(a; b) \quad \text{for all} \quad a \in M^{|x_\alpha|}, b \in M^m.$$

Fix finite $F \subseteq \lambda$ and n. Let Θ be the sequence $(\theta_\alpha(x_\alpha; y) : \alpha < \lambda)$. Observe that if $\mathcal{A} := (a_{\alpha,i} \in M^{|x_\alpha|} : \alpha \in F, i \leq n)$ is an $(\mathcal{M}^{\mathrm{Sh}}, Y, \Phi, F, n)$-array then $\mathcal{A}$ is also an $(\mathcal{N}, X, \Theta, F, n)$-array. So $\dim_{\mathcal{N}} X \geq \lambda$. $\square$

6. The second inequality

We generalize the second inequality. The results of this section are easily adapted to expansions of divisible ordered abelian groups, we leave that to the reader.

Let (K, val) be a valued field, $\mathcal{K}$ be an expansion of (K, val), and $\mathcal{K} \prec \mathcal{L}$. Let V be the set of $a \in L$ such that $\mathrm{val}(a) \geq \mathrm{val}(b)$ for some $b \in K$. Then $\mathcal{L}$ is a **tame extension** of $\mathcal{K}$ if for every $a \in V$ there is $b \in K$ such that $\mathrm{val}(a - b) \geq \mathrm{val}(a - b')$ for all $b' \in K$. It is easy to see that b must be unique, so if $\mathcal{K} \prec \mathcal{L}$ is tame then we let $\mathrm{st} : L \to K$ be the map taking each a to the unique $b = \mathrm{st}(a)$ with this property. If (K, val) is locally compact then any elementary extension is tame.

In this section $\mathcal{K}$ is NIP **and $\mathcal{K} \prec \mathcal{L}$ is tame**. Let $\mathrm{st}(\mathcal{L})$ be the structure on K with an m-ary relation defining $\mathrm{st}(V^m \cap X)$ for each $\mathcal{L}$-definable $X \subseteq L^m$.

Proposition 6.1. *If* $X \subseteq L^m$ *is* $\mathcal{L}$*-definable then* $\dim_{\mathrm{st}(\mathcal{L})} \mathrm{st}(V^m \cap X) \leq \dim_{\mathcal{L}} X$.

So in particular $\mathrm{st}(\mathcal{L})$ is NIP and $\mathrm{st}(\mathcal{L})$ is dp-minimal when $\mathcal{K}$ is dp-minimal.

It is easy to see that V is a subring of L and if $a \in L \setminus V$ then $1/a \in V$, so V is a valuation subring of L. The maximal ideal $\mathfrak{m}$ of V is the set of $a \in L$ such that $\mathrm{val}(a) \geq \mathrm{val}(b)$ for all $b \in K^\times$. Observe that $\{\mathrm{st}(a)\} = (a + \mathfrak{m}) \cap K$ for all $a \in V$, so we may identify K with $V/\mathfrak{m}$. It is easy to see that $\mathrm{st} : V \to K$ is the residue map. We describe the associated valuation. Let Γ_K, Γ_L be the value group of $(K, \mathrm{val}), (L, \mathrm{val})$, respectively. Let O be the convex hull of Γ_K in Γ_L and w be the valuation on L given by composing val with the quotient $\Gamma_L \to \Gamma_K/O$. Then V is the valuation ring of w. We now prove Proposition 6.1.

Proof. By definition O is a convex subset of Γ_L so O is definable in $\mathcal{L}^{\mathrm{Sh}}$. So w is an $\mathcal{L}^{\mathrm{Sh}}$-definable valuation and we can regard K as an imaginary sort of $\mathcal{L}^{\mathrm{Sh}}$, thus $\mathrm{st} : V^m \to K^m$ is $\mathcal{L}^{\mathrm{Sh}}$-definable. The proposition now follows from Fact 3.2(3). $\square$

What is not clear at the moment is how $\mathrm{st}(\mathcal{L})$ relates to $\mathcal{K}$.

Proposition 6.2. $\mathrm{st}(\mathcal{L})$ *is a reduct of* $\mathcal{K}^{\mathrm{Sh}}$.

Proof. Suppose $Y \subseteq L^m$ is $\mathcal{L}$-definable. Let $Z := Y + \mathfrak{m}^m$, so Z is $\mathcal{L}^{\mathrm{Sh}}$-definable. Note that $\mathrm{st}(V^m \cap Y) = Z \cap K^m$. So $\mathrm{st}(V^m \cap Y)$ is $\mathcal{K}^{\mathrm{Sh}}$-definable by Fact 4.3. $\square$

In general $\mathcal{K}$ is not a reduct of $\mathrm{st}(\mathcal{L})$. By [35] $\mathrm{st}(\mathcal{L})$ cannot define a subset of $\mathbb{Q}_p^m$ which is dense and co-dense in a nonempty open set, but there are NIP expansions of $\mathbb{Q}_p$ which define such sets. For example Mariaule [17] shows that if H is a dense finitely generated subgroup of $(1 + p\mathbb{Z}_p, \times)$ then $(\mathbb{Q}_p, H)$ is NIP. We expect that in this case $\mathrm{st}(\mathcal{L})$ is interdefinable with $\mathbb{Q}_p$ but we have not carefully checked this.

Proposition 6.3. *Suppose that* $\mathcal{K}$ *is dp-minimal. Then* $\mathcal{K}$ *is a reduct of* $\mathrm{st}(\mathcal{L})$ *and* $\mathcal{K}^{\mathrm{Sh}}$ *is interdefinable with* $\mathrm{st}(\mathcal{L}^{\mathrm{Sh}})$.

Proof. The proof of Proposition 6.2 shows that $\mathrm{st}(\mathcal{L}^{\mathrm{Sh}})$ is a reduct of $\mathcal{K}^{\mathrm{Sh}}$. We first show that $\mathcal{K}^{\mathrm{Sh}}$ is a reduct of $\mathrm{st}(\mathcal{L}^{\mathrm{Sh}})$. Suppose that $X \subseteq K^n$ is $\mathcal{K}^{\mathrm{Sh}}$-definable. We show that X is $\mathrm{st}(\mathcal{L}^{\mathrm{Sh}})$-definable. By Facts 4.2 and 3.4 we may suppose that X is closed. Let $\mathcal{L} \prec \mathcal{N}$ be highly saturated, $Z \subseteq N^n$ be an honest definition of X, and $Y := L^n \cap Z$. So Y is $\mathcal{L}^{\mathrm{Sh}}$-definable, we show that $\mathrm{st}(V^n \cap Y) = X$. As $X \subseteq Y$ and st is the identity of K^n we have $X \subseteq \mathrm{st}(V^n \cap Y)$. Fix $p \in \mathrm{st}(V^n \cap Y)$. We show that $p \in X$. As X is closed it suffices to fix a val-ball $B \subseteq K^n$ containing p and show that $B \cap X \neq \emptyset$. Fix $q \in V^n \cap Y$ such that $\mathrm{st}(q) = p$. Then $q \in B(\mathcal{N})$ so $B(\mathcal{N}) \cap Z \neq \emptyset$. As Z is honest $B \cap X$ is nonempty.

It remains to show that $\mathcal{K}$ is a reduct of $\mathrm{st}(\mathcal{K})$. Suppose $X \subseteq K^n$ is $\mathcal{K}$-definable. By Fact 3.4 we may suppose X is closed. Let Y be the subset of L^n defined by the same formula as X. The proceeding paragraph shows that $\mathrm{st}(V^n \cap Y) = X$. $\square$

7. Delon's Theorem

Fact 7.1 is a well-known consequence of Pas's quantifier elimination [23].

Fact 7.1. *Let* (M, v) *be a Henselian valued field of equicharacteristic zero with residue field* R. *Every* (M, v)*-definable subset every of* R^m *is* R*-definable.*

We now give a proof of Delon's theorem that $\mathbb{Q}_p$ is Shelah complete.

Proof. Let $\mathbb{Q}_p \prec \mathcal{L}$ be highly saturated. So $\mathcal{L}$ is a tame extension as $\mathbb{Q}_p$ is locally compact. Let w be the valuation on $\mathcal{N}$ with residue map st. By the observations above w is a coarsening of the p-adic valuation on L, so w is Henselian as a coarsening of a Henselian valuation is always Henselian. An application of Fact 7.1 shows that $\mathrm{st}(\mathcal{L})$ is interdefinable with $\mathbb{Q}_p$. Slight modifications to the proof of Proposition 6.3 show that $\mathbb{Q}_p^{\mathrm{Sh}}$ and $\mathrm{st}(\mathcal{L})$ are interdefinable. $\square$

A subfield of $\mathbb{Q}_p$ is an elementary substructure if and only if it is algebraically closed in $\mathbb{Q}_p$ [11, Lemma 6.2.1]. So Corollary 7.2 follows from Fact 1.1 and Lemma 4.4.

Corollary 7.2. *Suppose that K is a subfield of $\mathbb{Q}_p$ which is algebraically closed in $\mathbb{Q}_p$ (e.g. the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$). Then $X \subseteq K^m$ is K^{Sh}-definable if and only if $X = K^m \cap Y$ for some $\mathbb{Q}_p$-definable $Y \subseteq \mathbb{Q}_p^m$.*

So the Shelah completion of K is the structure induced on K by its valuation-theoretic completion. There are several similar results. If R is a real closed subfield of $(\mathbb{R}, +, \times)$ then every R^{Sh}-definable set is of the form $R^m \cap X$ for $(\mathbb{R}, +, \times)$-definable $X \subseteq \mathbb{R}^m$. More generally, suppose that R is a divisible subgroup of $(\mathbb{R}, +)$ and $\mathcal{R}$ is an o-minimal expansion of $(R, <, +)$. Laskowski and Steinhorn [14] show that there is a unique o-minimal expansion $\mathcal{R}^{\square}$ of $(\mathbb{R}, <, +)$ such that $\mathcal{R} \prec \mathcal{R}^{\square}$. By Marker-Steinhorn [18] $\mathcal{R}^{\square}$ is Shelah complete, so every $\mathcal{R}^{\mathrm{Sh}}$-definable set is of the form $R^m \cap X$ for $\mathcal{R}^{\square}$-definable $X \subseteq \mathbb{R}^m$. Finally, if H is a dense subgroup of $(\mathbb{R}, +)$ then every $(H, +, <)^{\mathrm{Sh}}$-definable set is a boolean combination of $(H, +)$-definable sets and sets of the form $H^n \cap X$ for $(\mathbb{R}, <, +)$-definable $X \subseteq \mathbb{R}^m$, see [34]. (If H is not n-divisible then $nH \neq X \cap \mathbb{R}$ for any $(\mathbb{R}, <, +)$-definable $X \subseteq \mathbb{R}$.)

8. $\mathbb{Q}_p^{\mathrm{an}}$ is Shelah complete

Let $\mathbb{Q}_p^{\mathrm{an}}$ be the expansion of $\mathbb{Q}_p$ by all analytic functions $\mathbb{Z}_p^m \to \mathbb{Q}_p$ for all m. There is a well-developed theory of $\mathbb{Q}_p^{\mathrm{an}}$-definable sets beginning with Denef and van den Dries [9]. It is shown in [33] that every definable unary set in every elementary extension of $\mathbb{Q}_p^{\mathrm{an}}$ is definable in the underlying field. Fact 8.1 easily follows.

Fact 8.1. *$\mathbb{Q}_p^{\mathrm{an}}$ is dp-minimal.*

So in particular dp-rank agrees with the canonical dimension on $\mathbb{Q}_p^{\mathrm{an}}$-definable sets.

Suppose that $\mathbb{Q}_p^{\mathrm{an}} \prec \mathcal{L}$ is highly saturated and let L be the underlying field of $\mathcal{L}$. Note that $\mathbb{Q}_p^{\mathrm{an}} \prec \mathcal{L}$ is tame. Let $\mathrm{val_p}$ be the p-adic valuation, V be the set of $a \in L$ such that $\mathrm{val_p}(a) \geq k$ for some k, and $\mathrm{st} : V^m \to \mathbb{Q}_p^m$ be the standard part map. As above V is a valuation subring of L, the associated valuation is a coarsening of $\mathrm{val_p}$, and $\mathrm{st} : V \to \mathbb{Q}_p$ is the residue map. Fact 8.2 is the analytic analogue of Fact 7.1. Fact 8.2 follows easily from a theorem of Rideau [25, Theorem 3.10]. (This is closely related to the work of Cluckers, Lipshitz, and Robinson on the model theory of valued fields with analytic structure [5, 6, 7].)

Fact 8.2. *A subset X of $\mathbb{Q}_p^m$ is $(\mathcal{L}, V)$-definable if and only if it is $\mathbb{Q}_p^{\mathrm{an}}$-definable.*

Following the argument of Section 7, applying Fact 8.1 when necessary, and applying Fact 8.2 in place of Fact 7.1 we obtain Theorem 8.3.

Theorem 8.3. *If $X \subseteq L^m$ is $\mathcal{L}$-definable then $\mathbb{Q}_p^m \cap X$ is $\mathbb{Q}_p^{\mathrm{an}}$-definable.*

So $\mathbb{Q}_p^{\mathrm{an}}$ is Shelah complete. (This was proven in unpublished work of Hrushovski, see [21, Fact 2.6].) Proposition 8.4 follows by Proposition 6.1 and Fact 8.2.

Proposition 8.4. *If X is an $\mathcal{L}$-definable subset of L^m then $\dim_{\mathbb{Q}_p^{\mathrm{an}}} \mathrm{st}(X) \leq \dim_{\mathcal{L}} X$.*

Proposition 6.3 and Theorem 8.3 together yield a strengthening of Fact 8.2.

Corollary 8.5. *The structure induced on $\mathbb{Q}_p$ by $\mathcal{L}^{\mathrm{Sh}}$ is interdefinable with $\mathbb{Q}_p^{\mathrm{an}}$.*

9. A geometric application

Following work of Bröcker [1, 2] and van den Dries [32] in the semialgebraic and o-minimal settings, respectively, we give a geometric application of the results above. We let $|,|$ be the usual absolute value on $\mathbb{Q}_p$ and declare

$$\|a\| := \max\{|a_1|, \ldots, |a_m|\} \quad \text{for all} \quad a = (a_1, \ldots, a_m) \in \mathbb{Q}_p^m.$$

The Hausdorff distance $d_H(X, X')$ between bounded subsets X, X' of $\mathbb{Q}_p^m$ is the infimum of $t \in \mathbb{R}_{>0}$ such that for every $a \in X$ there is $a' \in X'$ such that $\|a - a'\| < t$ and for every $a' \in X'$ there is $a \in X$ such that $\|a - a'\| < t$. The Hausdorff distance between a bounded set and its closure is always zero. If $\mathcal{X}$ is a family of bounded subsets of $\mathbb{Q}_p^m$ then $X \subseteq \mathbb{Q}_p^m$ is a **Hausdorff limit** of $\mathcal{X}$ if X is compact and there is a sequence $(X_k)_{k \in \mathbb{N}}$ of elements of $\mathcal{X}$ such that $d_H(X_k, X) \to 0$ as $k \to \infty$.

Theorem 9.1. *Suppose that $\mathcal{X}$ is an $\mathbb{Q}_p^{\mathrm{an}}$-definable family of bounded subsets of $\mathbb{Q}_p^m$. Any Hausdorff limit of $\mathcal{X}$ is $\mathbb{Q}_p^{\mathrm{an}}$-definable. If $(X_k)_{k \in \mathbb{N}}$ is a sequence of elements of $\mathcal{X}$ which Hausdorff converges to $X \subseteq \mathbb{Q}_p^m$ then $\dim X \leq \limsup_{k \to \infty} \dim X_k$.*

Note that *any* compact subset of $\mathbb{Z}_p^m$ is a Hausdorff limit of a sequence of finite sets so the restriction to definable families of sets is necessary. We will need to use Fact 9.2, an immediate consequence of the equality of naive dimension and the canonical dimension on $\mathbb{Q}_p^{\mathrm{an}}$-definable sets.

Fact 9.2. *Suppose that $(X_a : a \in \mathbb{Q}_p^n)$ is an $\mathbb{Q}_p^{\mathrm{an}}$-definable family of subsets of $\mathbb{Q}_p^m$. Then $\{a \in \mathbb{Q}_p^n : \dim X_a = l\}$ is $\mathbb{Q}_p^{\mathrm{an}}$-definable for any $0 \leq l \leq m$.*

We now proceed to prove Theorem 9.1. Our proof is very similar to that in [32] so we omit some details. We also make a nonessential use of ultrafilter convergence.

Proof. Let $\mathbb{Q}_p^{\mathrm{an}} \prec \mathcal{L}$ be highly saturated. Let $|x| = m$ and $\phi(x; y)$ be a formula such that $\mathcal{X}$ is $(\phi(\mathbb{Q}_p^m; b) : b \in \mathbb{Q}_p^{|y|})$. For each k fix $b_k \in \mathbb{Q}_p^{|y|}$ such that $X_k = \phi(\mathbb{Q}_p^m; b_k)$. Let $\mathfrak{u}$ be a nonprinciple ultrafilter on $\mathbb{N}$. Applying saturation fix $b \in K^{|y|}$ such that $\mathrm{tp}(b_k|\mathbb{Q}_p) \to \mathrm{tp}(b|\mathbb{Q}_p)$ as $k \to \mathfrak{u}$. Let $Y := \phi(L^m; b)$. It is easy to see that $Y \subseteq V^m$ and $X = \mathrm{st}(Y)$. So X is $\mathbb{Q}_p^{\mathrm{an}}$-definable by Fact 8.2. By Fact 9.2 we have

$$\dim_{\mathcal{L}} Y = \lim_{k \to \mathfrak{u}} \dim X_k \leq \limsup_{k \to \infty} \dim X_k.$$

So by Proposition 8.4 we have $\dim X \leq \limsup_{k \to \infty} \dim X_k$. □

Our proof of Theorem 9.1 goes through over $\mathbb{Q}_p$. We give an attractive formulation of the first claim in this setting. For each $k \geq 0$ we let P_k be a unary relation defining the set of kth powers in $\mathbb{Q}_p$. It is a famous theorem of Macintyre [16] that every paremeter free formula in the language of rings is equivalent over $\mathbb{Q}_p$ to a boolean combination of formulas of the form $f = g$, $\mathrm{val_p}(f) \leq \mathrm{val_p}(g)$, or $P_k(f)$ for $f, g \in \mathbb{Z}[x_1, \ldots, x_m]$. Suppose $X \subseteq \mathbb{Q}_p^m$ is $\mathbb{Q}_p$-definable. The **complexity** of X is $\leq n$ if X may be defined using $\leq n$ formulas of the form $f = g, \mathrm{val_p}(f) \leq \mathrm{val_p}(g)$, or $P_k(f)$ where each $k \leq n$ and each $f, g \in \mathbb{Q}_p[x_1, \ldots, x_m]$ has degree $\leq n$. It is easy to see that Theorem 9.3 follows from saturation and the fact that a Hausdorff limit of a sequence of elements of a $\mathbb{Q}_p$-definable family of sets is $\mathbb{Q}_p$-definable.

Theorem 9.3. *For every n, m there is an l such that if $(X_k)_{k \in \mathbb{N}}$ is a Hausdorff converging sequence of bounded $\mathbb{Q}_p$-definable subsets of $\mathbb{Q}_p^m$, each of complexity $\leq n$, then the Hausdorff limit X of $(X_k)_{k \in \mathbb{N}}$ is $\mathbb{Q}_p$-definable of complexity $\leq l$.*

Proposition 9.4 shows that Theorem 9.1 is equivalent to the fact that $\mathrm{st}(X)$ is $\mathbb{Q}_p^{\mathrm{an}}$-definable when $X \subseteq V^m$ and Proposition 8.4.

Proposition 9.4. *Let $\mathbb{Q}_p^{\mathrm{an}} \prec \mathcal{L}$ be highly saturated. Suppose that $\phi(x;y)$ is a formula in the language of $\mathbb{Q}_p^{\mathrm{an}}$. Fix $b \in L^{|y|}$ and suppose $X := \phi(L^{|x|}; b) \subseteq V^{|x|}$. Then $\mathrm{st}(X)$ is a Hausdorff limit of $\mathcal{X} := (\phi(\mathbb{Q}_p^{|x|}; a) : a \in \mathbb{Q}_p^{|y|})$.*

Let $|x| = m$ and $|y| = n$. Given subsets X, X' of V^m and $t \in \mathbb{R}_{>0}$ we say that $d_H(X, X') < t$ if for every $a \in X$ there is $a \in X'$ such that $\|a - a'\| < t$ and vice versa (we do not define $d_H(X, X')$ in this case).

Proof. By saturation $\mathrm{st}(X)$ is compact so it suffices to show that for every $t \in \mathbb{R}_{>0}$ there is a bounded $Y \in \mathcal{X}$ such that $d_H(\mathrm{st}(X), Y) \le t$. Fix $t \in \mathbb{R}_{>0}$. As $X \subseteq V^m$ it is easy to see there is a finite $A \subseteq \mathbb{Q}_p^m$ such that $d_H(A, X) < t/2$, observe that $d_H(A, \mathrm{st}(X)) \le t/2$. As $\mathbb{Q}_p^{\mathrm{an}}$ is an elementary submodel of $\mathcal{K}$ we obtain $a \in \mathbb{Q}_p^n$ such that $d_H(\phi(\mathbb{Q}_p^m; a), A) < t/2$. Let $Y := \phi(\mathbb{Q}_p^m; a)$. Note that Y is bounded. The triangle inequality for d_H yields $d_H(\mathrm{st}(X), Y) \le t$. $\square$

It should be possible to give a geometric proof of Theorem 9.1 and thereby obtain a geometric proof of Fact 8.2. We are aware of two geometric proofs of the o-minimal analogue of Theorem 9.1, Lion and Speissegger [15] and Kocel-Cynk, Pawlucki, and Valette [13]. The main tool of [13] is the o-minimal Lipschitz cell decomposition, see [24], and there is now a Lipschitz cell decomposition for $\mathbb{Q}_p^{\mathrm{an}}$-definable sets [4]. We believe that there is a purely geometric proof of Shelah completeness for $\mathbb{Q}_p$ and $\mathbb{Q}_p^{\mathrm{an}}$ along these lines, but we have not seriously pursued this.

10. A question

Fix an o-minimal expansion $\mathcal{R}$ of $(\mathbb{R}, +, \times)$ such that the function $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by $t \mapsto t^r$ is only definable when $r \in \mathbb{Q}$, e.g. $\mathbb{R}_{\mathrm{an}}$. Fix $\lambda \in \mathbb{R}_{>0}$ and let $\lambda^{\mathbb{Z}} := \{\lambda^m : m \in \mathbb{Z}\}$. Following [30] Miller and Speissegger show that $(\mathcal{R}, \lambda^{\mathbb{Z}})$ is tame [19, Section 8.6]. It follows by [29, Theorem 4.1.2, Corollary 4.1.7] and [3, Corollary 2.6] that $(\mathcal{R}, \lambda^{\mathbb{Z}})$ is NIP. There is a canonical notion of dimension d for $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable sets which agrees with naive, topological, and Assouad dimension.

Let $(\mathcal{R}, \lambda^{\mathbb{Z}}) \prec \mathcal{N}$, V be the convex hull of $\mathbb{R}$ in H, st be the standard part map $V^m \to \mathbb{R}^m$, and $X \subseteq N^m$ be $\mathcal{N}$-definable. We believe $(\mathcal{R}, \lambda^{\mathbb{Z}})$ is Shelah complete but we have not carefully checked this. Assuming that this is true, it is possible to give a geometric proof that $d(\mathrm{st}(V^m \cap X)) \le d(X)$. There should be a NIP-theoretic proof. More specifically there should be a combinatorical invariant $\mathbf{I}_{\mathcal{M}} Y$ of a definable set Y in an NIP structure $\mathcal{M}$ which satisfies at least the following:

(1) $\mathbf{I}_{(\mathcal{R}, \lambda^{\mathbb{Z}})}$ agrees with d,
(2) $\mathbf{I}_{\mathcal{M}^{\mathrm{Sh}}} Y = \mathbf{I}_{\mathcal{M}} Y$ (this should be immediate from the definition and Fact 4.1),
(3) $\mathbf{I}_{\mathcal{M}} Y \cup Y' = \max\{\mathbf{I}_{\mathcal{M}} Y, \mathbf{I}_{\mathcal{M}} Y'\}$
(4) $\mathbf{I}_{\mathcal{M}} Y \times Y' = \mathbf{I}_{\mathcal{M}} Y + \mathbf{I}_{\mathcal{M}} Y'$,
(5) $\mathbf{I}_{\mathcal{M}} f(Y) \le \mathbf{I}_{\mathcal{M}} Y$ for any $\mathcal{M}$-definable function $f : Y \to M^n$.

If $\mathbf{I}$ satisfies (1) – (5) and $(\mathcal{R}, \lambda^{\mathbb{Z}})$ is Shelah complete then we have

$$d(\mathrm{st}(V^m \cap X)) = \mathbf{I}_{(\mathcal{R}, \lambda^{\mathbb{Z}})} \mathrm{st}(V^m \cap X) \le \mathbf{I}_{\mathcal{N}^{\mathrm{Sh}}} X = \mathbf{I}_{\mathcal{N}} X = d(X).$$

Dp-rank does not satisfy (1). Tychonievich [29] has shown that every countable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable set is internal to $\lambda^{\mathbb{Z}}$ and the induced structure on $\lambda^{\mathbb{Z}}$ is interdefinable with $(\lambda^{\mathbb{Z}}, \times, <)$. So the dp-rank agrees with the canonical Presburger dimension on countable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable sets. Furthermore the dp-rank of any uncountable $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable set is $\aleph_0 - 1$. So in this setting dp-rank is not very useful.

Suppose $Z \subseteq \mathbb{R}^m$ is $(\mathcal{R}, \lambda^{\mathbb{Z}})$-definable. If $d(Z) = 0$ then Z is internal to $\lambda^{\mathbb{Z}}$ and if $d(Z) > 0$ then there is a definable surjection $Z \to \mathbb{R}$. As $(\mathbb{Z}, +, <)$ does not interpret an infinite field (see [35]) we have $d(Z) > 0$ if and only if the induced structure on Z does not interpret an infinite field. We should have $d(Z) = 0$ if and only if Z is "modular". So $\mathbf{I}_{\mathcal{M}} X$ should be the "non-modular dimension" of X.

We give two other examples of NIP structures to which this should apply. The first example is $(\mathcal{S}, \{\lambda, \lambda^{\lambda}, \lambda^{\lambda^{\lambda}}, \ldots\})$ where $\lambda > 1$ and $\mathcal{S}$ is an o-minimal expansion of $(\mathbb{R}, +, \times)$ such that every $\mathcal{S}$-definable function $\mathbb{R} \to \mathbb{R}$ is eventually bounded above by some compositional iterate of the exponential (all known o-minimal expansions of $(\mathbb{R}, +, \times)$ satisfy this condition). Miller and Tyne [20] show that this structure is tame, in particular naive dimension is well behaved. The induced structure on $D := \{\lambda, \lambda^{\lambda}, \lambda^{\lambda^{\lambda}}, \ldots\}$ should be interdefinable with $(D, <)$ and any zero-dimensional definable set should be internal to D. So $\mathbf{I}_{(\mathcal{S}, D)}$ should agree with naive dimension. Second, let Log be the Iwasawa logarithm $\mathbb{Q}_p^{\times} \to \mathbb{Q}_p$. Mariaule [17] shows that $(\mathbb{Q}_p, \mathrm{Log})$ is NIP. We have $\mathrm{Log}(a) = 0$ if and only if $a = bp^k$ for some k and root of unity $b \in \mathbb{Q}_p$. It follows that $p^{\mathbb{Z}}$ is $(\mathbb{Q}_p, \mathrm{Log})$-definable. It is also shown in [17] that the induced structure on $p^{\mathbb{Z}}$ is interdefinable with $(p^{\mathbb{Z}}, \times, \lhd)$ where $p^k \lhd p^l$ if and only if $k < l$. It should follow from [17] that naive dimension is well behaved in $(\mathbb{Q}_p, \mathrm{Log})$ and a zero-dimensional definable set should be internal to $p^{\mathbb{Z}}$. So we expect $\mathbf{I}_{(\mathbb{Q}_p, \mathrm{Log})}$ to coincide with naive dimension.

The vague question here is: What is the right combinatorial definition of the canonical dimension in structures such as $(\mathcal{R}, \lambda^{\mathbb{Z}})$, $(\mathcal{S}, \{\lambda, \lambda^{\lambda}, \lambda^{\lambda^{\lambda}}, \ldots\})$, or $(\mathbb{Q}_p, \mathrm{Log})$?

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