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Linear Partial Differential Equations with Random Forcing

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1. Introduction

We are concerned here with the solution of non-self-adjoint initial-boundary value problems in linear partial differential equations when the forcing terms are random functions. To be concrete, we will consider here the dimensionless second order equation

$$Lu \equiv u_{tt} + au_{xx} + bu_{tx} + cu_x + du_t + eu = f(x,\ t),\ (t > 0, 0 < x < 1) \tag{1.1}$$

where the known coefficients, $a, b, \ldots,$ e, are at least continuous functions of x and t, subject to the homogeneous initial and boundary conditions

$$u(x, 0) = u_t(x, 0) = 0, \qquad (0 < x < 1) \tag{1.2}$$

$$u(0, t) = u(1, t) = 0, \qquad (t > 0). \tag{1.3}$$

If f(x, t) is a known function, the solution of this problem can be written down immediately in terms of the associated Green's function,

$$u(x, t) = \int_0^1 \int_0^t G(x, t; x', t')\, f(x', t')\, dt'\, dx'. \tag{1.4}$$

When the excitation, f (x, t), is a random function of known statistics, a solution of the problem consists of obtaining moment functions (or joint probability density functions) of all orders of the response u(x, t). With the help of (1.4), this is in principle straightforward. For example, the expected value (or the first moment function) of u is given by

$$\langle u(x,t)\rangle = \int_0^1 \int_0^t G(x, t; x', t')\, \langle f(x', t')\rangle\, dt'\, dx' \tag{1.5}$$

where $\langle \cdot | \rangle$ is the ensemble-averaging operation. The higher order moments can be determined by forming the ensemble average of different combinations of u(x, t) with the help of (1.4).

Unfortunately, the Green's function, G(x, t; x', t'), of (1.1), (1.2) and (1.3) cannot be obtained in terms of elementary or special functions except for the simplest cases. In general, we will have to obtain G by some numerical method. The desired statistics of the response will still have to be calculated by multiple integration. Even if we are willing to settle for the mean and the mean square value of u and u_t

(which is a bare minimum for the first passage problem), the above procedure is still very inefficient if not impractical.

More efficient methods than working through (1.4) are available for the first and second order statistics (which are all we need for a Gaussian process) of certain special classes of problems. For example, the normal mode approach and the transfer matrix approach [1] have been used successfully for many problems in structural mechanics. To varying degrees, these methods depend on special properties of the relevant differential operator such as constant coefficients, self-adjointness, separability with respect to the independent variables, etc. On the other hand, operators which do not have any of these nice properties do occur in the analysis of technical problems. For instance, the small amplitude flapping motion of the simplest model of a rotating wing (with very small effective bending stiffness) in forward flight is governed by the equation

$$u_{tt} + \left[\beta + \frac{\gamma}{6}|x + \mu \sin t|\right] u_t + \frac{\gamma}{6}\mu \cos t|x + \mu \sin t| u_x - \tfrac{1}{2}[(1 - x^2)u_x]_x = f(x, t) \tag{1.6}$$

where β, γ and μ are known parameters. None of these efficient methods applies to this equation unless we are willing to make some approximation which may or may not be justifiable.

To seek a more effective method for the second order statistics of u for non-self-adjoint, non-separable equations with variable coefficients such as (1.6), we are motivated by a method used recently for linear time-varying dynamical systems subject to random excitations [2,3]. The essential feature of that method is to obtain (deterministic ordinary) differential equations for the desired statistics themselves. More specifically, appropriate equations for the mean, mean square values and the correlation functions were derived from the given ordinary stochastic differential equation(s)*. The two sets of equations for the mean square values and the correlation functions were solved successively with the solution of the first set served as the initial condition for the second set. At the very least, this approach enables us to obtain the desired statistics by numerical integration in a straightforward manner and the necessary calculation is much more manageable than working through the equivalence of (1.4) whenever the impulse response function cannot be found in terms of known functions [2,3].

Our objective here is to extend this method to problems involving partial differential equations and offer an efficient way to calculate at least the mean and mean square value of u and u_t and hopefully the correlation functions as well. By ensemble-averaging (1.1)-(1.3), we find immediately that $\langle u(t) \rangle$ is determined by the original initial-boundary value problem, but now with a known forcing function. We will assume henceforth that f(x, t) is of zero mean so that u is also of zero mean.

In trying to obtain equations for the mean square response, we encounter a certain difficulty illustrated by the case of the simple wave equation

$$u_{tt} - u_{xx} = f(x, t). \tag{1.7}$$

* Though the method can be used for higher order statistics, only first and second order statistics are calculated in practice.

If we multiply (1.7) through by $u(x, t)$ and ensemble-average the resulting equation, we get

$$\langle uu_t\rangle_t - \langle u_t^2\rangle - \langle uu_x\rangle_x + \langle u_x^2\rangle = \langle fu\rangle. \tag{1.8}$$

Evidently, we need more equations. We can of course perform the same operation using u_t and u_x instead of u as the multiplier. We get after some manipulation,

$$\tfrac{1}{2}\langle u_t^2\rangle_t - \langle u_t u_x\rangle_x + \tfrac{1}{2}\langle u_x^2\rangle_t = \langle fu_t\rangle, \tag{1.9}$$

$$\langle u_x u_t\rangle_t - \tfrac{1}{2}\langle u_t^2\rangle_x - \tfrac{1}{2}\langle u_x^2\rangle_x = \langle fu_x\rangle. \tag{1.10}$$

We can get one more equation by noticing that $(uu_t)_x = (uu_x)_t$ so that

$$\langle uu_t\rangle_x - \langle uu_x\rangle_t = 0. \tag{1.11}$$

This seems to have exhausted all the possibilities and we have only four equations for the five unknowns on the left hand side of these equations even if we assume (or can show) that the right hand members are known quantities*.

With $\langle uu_t\rangle = \frac{1}{2}\langle u^2\rangle_t$ and $\langle uu_x\rangle = \frac{1}{2}\langle u^2\rangle_x$, we can obtain the mean square value of u once we have $\langle uu_t\rangle$ and $\langle uu_x\rangle$. Alternately, we can replace the latter two unknowns in (1.8) by the derivatives of the $\langle u^2\rangle$. But in that case, equation (1.11) becomes an identity and we are still left with three equations for (at least) four unknowns.

We can circumvent the impasse by formulating an initial-boundary value problem for the autocorrelation function of u which contains all the mean square quantities as special cases. While this can in fact be accomplished as we will show in the next section, the result is useful only if the operator L is relatively simple. For an equation such as (1.6), this new approach is no more efficient than working directly with the Green's function representation (1.4) (see section (2)).

In the process of analyzing the shortcomings of the autocorrelation function method, a related approach to the problem occurred to this author. This third approach will in fact lead to an efficient solution scheme for the problem defined by (l.l)-(1.3). By this, we mean an improvement at least by a factor n x m in the computing time required for a numerical solution over the two other methods -the Green's function method and the autocorrelation function method- mentioned above. Here m is the number of mesh points used in (0,l) and n is the number of time steps. The factor is larger for the class of temporallyuncorrelated excitations. The method has been extended to higher dimensions, single higher order equations and systems [5]. More general auxiliary conditions than (1.2) and (1.3) are also allowed.

While the method is particularly useful for problems for which available methods are either not applicable or not practical, it is also competitive with other methods for problems solvable by both. This is illustrated by applying the present method to a problem solved recently in [4]. In contrast to the analysis of [4] which involves contour integration in the complex plane, an exact solution of this problem by our method involves only simple algebraic calculations.

* We note that the sum $\langle u_x^2\rangle + \langle u_t^2\rangle$ alone appears in (1.9) and (1.10) so that these become two equations for this sum and $\langle u_x u_t\rangle$. But there does not seem to be any way for us to determine $\langle u_x^2\rangle$ and $\langle u_t^2\rangle$ separately. In any event, the fact that equations (1.9) and (1.10) form a closed system is only a special feature of (1.7) and not true in general for (1.1).

2. The autocorrelation of the response

The difficulty encountered in our effort to formulate an initial-boundary value problem for the variances and covariances suggests that we should try to formulate the same for the correlation function(s) instead. This can be accomplished by ensemble-averaging both sides of (1.1)-(1.3) after multiplying through by $u(y, \tau)$ where $0 < y < 1$ and $\tau > 0$ are fixed parameters. In this way, we get

$$L_{tx}R_{uu} = R_{fu}(x, t; y, \tau) \qquad (t > 0, 0 < x < 1) \tag{2.1}$$

$$R_{uu}(x, 0; y, \tau) = R_{uu,t}(x, 0; y, \tau) = 0 \qquad (0 < x < 1) \tag{2.2}$$

$$R_{uu}(0, t; y, \tau) = R_{uu}(1, t; y, \tau) = 0 \qquad (t > 0) \tag{2.3}$$

where the subscripts t and x in L_{tx} indicate that we are working in the x, t-space and where $R_{gh}(x, t; y, \tau) = \langle g(x, t)h(y, \tau)\rangle$ with

$$R_{gg}(x, t; y, \tau) = R_{gg}(y, \tau; x, t), \qquad R_{gh}(x, t; y, \tau) = R_{hg}(y, \tau; x, t). \tag{2.4}$$

To obtain the cross-correlation R_{fu}, *we* repeat the same operation but now with $f(y, \tau)$ as the multiplier. After we interchange the role of (x, t) and (y, τ) in the result, we get

$$L_{\tau y}R_{fu} = R_{ff}(y, \tau; x, t) \qquad (\tau > 0, 0 < y < 1) \tag{2.5}$$

$$R_{fu}(x, t; y, 0) = R_{fu,\tau}(x, t; y, 0) = 0 \qquad (0 < y < 1) \tag{2.6}$$

$$R_{fu}(x, t; 0, \tau) = R_{fu}(x, t; 1, \tau) = 0 \qquad (\tau > 0). \tag{2.7}$$

Our problem now is to solve (2.5)-(2.7) to get $R_{fu}(x, t; y, \tau)$ for all $\tau > 0$ and all y in (0,l) with x and t as parameters. The solution is then used in (2.1) for the determination of $R_{uu}(x, t; y, \tau)$. In each case, the problem is a well posed initial-boundary value problem in the usual sense and poses no conceptual difficulty. Setting $x = y$ and $t = \tau$ in the solution R_{uu} and $R_{uu\,t\tau}$, *we* have the special cases of mean square "displacement" and mean square "velocity?

But from the point of view of a numerical solution (which is often necessary for non-self-adjoint equations such as (1.6)) this approach is no more practical than working directly with (1.4). Suppose that we want to determine $R_{uu}(x, t; y_1, \tau_1)$ for a fixed y_1 and τ_1 and for all $0 < t \leq \tau_1$ and $0 < x < 1$*. We will have to solve the problem (2.5)-(2.7) numerically once for every (x, t) mesh-points in the region $(0 < t < \tau_1, 0 < x < 1)$ used for the problem (2.1)-(2.3). For a very simple difference scheme with 19 x 19 equally spaced mesh-points in the interior of a unit square $(0 < y < 1, m < \tau < m + 1)$† and with $\tau_1 = 10$ say, the solution for R_{fu} for one set of (x, t) takes about 5 seconds on a CDC 6400. The entire calculation to get R_{fu} for all the mesh points used in problem (2.1)-(2.3) would take about 5 hours ! The time required will be even longer for larger τ_1. Also, unless we can store a large four dimensional array of R_{fu}, *we* will have to repeat the whole calculation for every pair of (y_1, τ_1).

Evidently, the straightforward method for the autocorrelation function R_{uu} as outlined above is not a viable scheme whenever a numerical solution of the relevant initial-boundary value problems is necessary.

* For $t > \tau_1$, $R_{uu}(x, t; y_1, \tau_1)$ can be obtained by the symmetry condition (2.4).

† We assume here $b \leq 1$ so that we can take $\Delta\tau = \Delta y$.

In an effort to search for a more efficient method of solution, we note that we can always write the forcing function f(x, t) as

$$f(x, t) = E(x, t)N(x, t) \tag{2.8}$$

where $E(x, t)$ is a known envelope function and N(x, t) is a random process of zero mean (since **f** is assumed to be of zero mean) as we can always set $E = 1$. For many problems including the problem of a rotating wing in advance flight in atmospheric turbulence, $E(x, t)$ is a nonconstant continuous function of x and t. If N(x, t) has an autocorrelation function of the form

$$R_N(x_1, t_1; x_2, t_2) = R_s(x_1, x_2)\delta(t_2 - t_1) \tag{2.9}$$

it is not difficult to see (by way of (1.4)) that

$$R_{fu}(x, t; y, \tau) = 0 \qquad (t > \tau). \tag{2.10}$$

While this result does not simplify the calculation necessary for a numerical solution of R_{uu} since only R_{fu} with $t < \tau$ enters into (2.1), it does suggest another approach to the problem.

If we have somehow obtained the *spatial correlation functions* $R_{uu}(x, \tau; y, \tau) = U(x, y, \tau)$ and $R_{uu,t}(x, \tau; y, \tau) = T(x, y, \tau)$ for $\tau > 0$, then for $t > \tau$, $R_{uu}(x, t; y, \tau)$ is the solution of the simple initial-boundary value problem

$$L_{tx}R_{uu} = 0 \qquad (t > \tau) \tag{2.11}$$

with

$$\begin{cases} R_{uu}(x, \tau; y, \tau) = U(x, y, \tau), \\ R_{uu,t}(x, \tau; y, \tau) = T(x, y, \tau) \end{cases} \qquad (0 < x < 1) \tag{2.12}$$

$$R_{uu}(0, t; y, \tau) = R_{uu}(1, t; y, \tau) = 0 \qquad (t > \tau) \tag{2.13}$$

where use has been made of (2.10). An exact solution of this problem can be expressed in terms of a one-dimensional integral in x of a combination of the Green's function and the initial conditions. From the point of view of a numerical solution of the initial-boundary value problem with the previously chosen mesh, the solution for a fixed y and τ and $t \leq \tau + M$ requires less than M seconds of machine calculation on a CDC 6400. This is a tremendous improvement over the Green's function method or the autocorrelation function method.

The question now is whether we can determine the spatial correlation functions efficiently. This question will be answered affirmatively in the next section.-How we can remove or at least substantially weaken the restriction (2.9) will be discussed in section (5).

Before leaving the subject of autocorrelation, it should be pointed out that the use of (2.12) as initial conditions for (2.11) implicitly assumes the continuity of R_{uu} and $R_{uu\,t}$ in t for all $t > 0$. That this is so follows from the fact that, for arbitrary (positive) values of t and τ,

$$R_{fu}(x, t; y, \tau) = E(x, t) \int_0^1 G(y, \tau; x', t)E(x', t)R_s(x, x')\, dx' \tag{2.14}$$

where $G(y, \tau; x', t) = 0$, for $\tau \leq t$. With R_{fu} continuous across $t = \tau$, we see from the properties of the associated Green's function (see appendix) that the solution R_{uu} of (2.11) and its time derivative $R_{uu,t}$ are continuous in t for all $t > 0$.

3. Spatial correlation functions

With the help of (2.9), we have from (1.4) and (2.8)

$$\langle f(x_j, t)u(x_k, t)\rangle = \tfrac{1}{2}E(x_j, t)\int_0^1 G(x_k, t; x', t)E(x', t)R_s(x_j, x')\,dx' = 0 \tag{3.1}$$

since G(x, t; x', t) = 0 (see appendix). Similarly, we have

$$\langle f(x_j, t)u_t(x_k, t)\rangle = \tfrac{1}{2}E(x_j, t)\int_0^1 G_t(x_k, t; x', t)E(x', t)R_s(x_j, x')\,dx'. \tag{3.2}$$

But $G_t(x_k, t; x', t) = \delta(x_k - x')$ (see appendix), we get

$$\langle f(x_j, t)u_t(x_k, t)\rangle = \tfrac{1}{2}g(x_j, x_k, t) \tag{3.3}$$

where $g(x_j, x_k, t) = E(x_j, t)E(x_k, t)R_s(x_j, x_k) = g(x_k, x_j, t)$.

We are now in a position to derive a set of four equations for the four spatial correlation functions

$$\begin{aligned} U(x, y, t) &= \langle u(x, t)u(y, t)\rangle, & S(x, y, t) &= \langle u(x, t)u_t(y, t)\rangle \\ T(x, y, t) &= \langle u_t(x, t)u(y, t)\rangle, & V(x, y, t) &= \langle u_t(x, t)u_t(y, t)\rangle. \end{aligned} \tag{3.4}$$

Note that U and V are symmetric in x and y and S(y, x, t) = T(x, y, t). Together, they form a spatial correlation matrix function.

If we multiply (1.1) through by u(y, t) and ensemble-average the resulting equation, we get with the help of (3.1)

$$T_t - V + a(x, t)U_{xx} + b(x, t)T_x + c(x, t)U_x + d(x, t)T + e(x, t)U = 0. \tag{3.5}$$

Interchanging the role of x and y and repeating the same operation give us another equation

$$S_t - V + a(y, t)U_{yy} + b(y, t)S_y + c(y, t)U_y + d(y, t)S + e(y, t)U = 0. \tag{3.6}$$

Next, we multiply (1.1) through by $u_t(y, t)$ and add the result to a corresponding result with the role of x and y interchanged. We then ensemble average the sum and use (3.3) to get

$$\begin{aligned} &V_t + a(x, t)S_{xx} + a(y, t)T_{yy} + b(x, t)V_x + b(y, t)V_y + c(x, t)S_x \\ &\quad + c(y, t)T_y + [d(x, t) + d(y, t)]V + e(x, t)S + e(y, t)T = g(x, y, t). \end{aligned} \tag{3.7}$$

A fourth equation is obtained from the fact that $[u(y, t)u(x, t)]_t = u_t(y, t)u(x, t) + u(y, t)u_t(x, t)$. Ensemble-average this and we get

$$U_t = S + T \tag{3.8}$$

Equations (3.5)-(3.8) are four equations for the four unknown elements of the spatial correlation matrix. The domain of these equations is the interior of the semi-infinite square column, $0 < x, y < 1$ and $t > 0$, in the (x, y, t) space. To supplement these equations, we have from (1.2) the initial conditions

$$U(x, y, 0) = S(x, y, 0) = T(x, y, 0) = V(x, y, 0) = 0 \tag{3.9}$$

in the interior of the base square, and from (1.3) the boundary conditions

$$U = S = T = V = 0 \qquad (t > 0) \tag{3.10}$$

on the four walls of the column.

The solution of the initial-boundary value problem (3.5)-(3.10) can be obtained at least by numerical integration in a straightforward manner. With a forward difference scheme in t, we can start with (3.9) and calculate U, S, T and V for successive values of t, $(0 <) t_1 < t_2 < t_3 <$, step by step without any difficulty. The boundary conditions (3.10) are exactly what we need at the boundary walls for a central difference scheme in both x and y. With 19 interior mesh points in both x and y, it takes only fourteen minutes on the CDC 6400 to calculate the solution for $t \leq 10$ (with $\Delta t = \Delta x$).

If $E(x, t)$ is continuous in x and $R_s(x, y)$ is continuous in both x and y, g(x, y, t) is continuous in both x and y. The solution U, S, T and V are therefore at least continuous in x and y, in particular across the line x = y. U(x, x, t) is therefore the well-defined mean square response $\langle u^2(x, t)\rangle$ and V(x, x, t) is the well defined mean square "velocity".

If g(x, y, t) is discontinuous or singular for some x and y, whether U, S, T and V are continuous in x and y would depend on the particular problem. If they are **continuous,** then the mean square value of u and u_t can still be obtained from U and V (see section (4)).

Note that in the present formulation, we no longer have to calculate the auto-correlation function if we are only interested in the mean square value of u and u_t. With the above efficient scheme for the numerical solution of the spatial correlation functions and with the observation made in the last part of section (2) on the general correlation function, the numerical solution for $R_{uu}(x, t; y, \tau)$ itself requires only a few minutes of machine time provided of course τ and $t - \tau$ are not extremely large. The same problem using the naive autocorrelation function approach developed in the first part of section (2) would take many many hours on the same machine!

If $a = b = c = 0$ and d, e and f are independent of x, equations (3.5)-(3.8) reduce to the three distinct equations

$$\begin{aligned} &\langle uu_t\rangle_t - \langle u_t^2\rangle + d(t)\langle uu_t\rangle + e(t)\langle u^2\rangle = 0 \\ &\langle u_t^2\rangle_t + 2d(t)\langle u_t^2\rangle + 2e(t)\langle uu_t\rangle = E^2(t) \\ &\langle u^2\rangle_t = 2\langle uu_t\rangle \end{aligned} \tag{3.11}$$

which are what we have for a dynamical system characterized by the second order ordinary differential equation [2]

$$\ddot{u} + d(t)\dot{u} + e(t)u = E(t)N(t). \tag{3.12}$$

Our experience with dynamical systems shows that once we have formulated the initial-boundary value problem for the desired statistics, we can often take advantage of the special properties of the problem and obtain an approximate solution in terms of known functions by the conventional methods of applied mathematics.

4. A point-driven string

In spite of the rather severe restriction imposed by (2.9), the formulation of sections (2) and (3) is already applicable to a number of technical problems including the rotating wing problem mentioned earlier. Also, its usefulness is not restricted to such an analytically untractable problem for which other methods are either inapplicable or impractical. To demonstrate this point, we will apply our method to the problem of a finite string fixed at both ends and driven at a point $x = x_f$ by a purely random process (ideal white noise). The solution for a band-limited white noise excitation may be obtained by deleting from the normal mode solution for the white noise problem terms associated with a natural frequency higher than the cut-off frequency. An approximate solution for the band-limited white noise problem was obtained in [4] by the method of transfer function. Unlike the analysis of [4] which involves integration in the complex plane, an exact (formal) solution in our case involves only simple algebraic calculations

In dimensionless form, the equation of motion of a uniform string with internal damping driven at a point by a zero mean white noise process may be written as

$$u_{tt} + \beta u_t - u_{xx} = f(x, t) = \delta(x - x_f)N(t) \tag{4.1}$$

where β is a positive constant and $R_N(t, \tau) = \delta(t - \tau)$. The initial and boundary conditions for a string fixed at both ends and at rest at $t = 0$ are given by (1.2) and (1.3).

With $a = -1$, $d = \beta$ and $b = c = e = 0$, the equations for the spatial correlation functions, (3.5)-(3.8), become

$$\begin{aligned} T_t - V + \beta T - U_{xx} = 0, \qquad S_t - V + \beta S - U_{yy} = 0, \\ V_t + 2\beta V - S_{xx} - T_{yy} = \delta(x - x_f)\delta(y - x_f), \qquad U_t - S - T = 0. \end{aligned} \tag{4.2}$$

The initial conditions at $t = 0$ and the boundary conditions on the four walls of the semi-infinite unit square column remain as given by (3.9) and (13.10).

The exact complementary solution of the four equations (4.2) can be taken in the form

$$W_c = \sum_{m,n=1}^{\infty} \sum_{k=1}^{4} C_{kmn} W_{kmn} e^{-a_{kmn} t} \phi_m(x)\phi_n(y)\phi_m(x_f)\phi_n(x_f) \tag{4.3}$$

where

$$W = (U, S, T, V), \qquad \phi_j(z) = \sqrt{2}\ \sin(j\pi z)$$

$$a_{kmn} = \beta \pm \frac{i}{2}\sqrt{4m^2\pi^2 - \beta^2} \pm \frac{i}{2}\sqrt{4n^2\pi^2 - \beta^2} \qquad (k = 1, 2, 3, 4)$$

$$U_{kmn} = 1, \qquad V_{kmn} = \tfrac{1}{2}[a_{kmn}(a_{kmn} + \beta) + (m^2 + n^2)\pi^2] \tag{4.4}$$

$$S_{kmn} = \frac{a_{kmn}(a_{kmn} + \beta) - (m^2 - n^2)\pi^2}{2(a_{kmn} + \beta)}, \qquad T_{kmn} = \frac{a_{kmn}(a_{mn} + \beta) + (m^2 - n^2)\pi^2}{2(a_{kmn} + \beta)}.$$

A particular solution of the equations may be obtained by writing $\delta(x - x_f)$ $\delta(y - x_f)$ as

$$\delta(y - x_f)\delta(x - x_f) = \left[\sum_{n=1}^{\infty} \phi_n(y)\phi_n(x_f)\right]\left[\sum_{m=1}^{\infty} \phi_m(x)\phi_m(x_f)\right] \tag{4.5}$$

and the particular solution W_p itself as

$$W_p = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{pmn}\phi_m(x)\phi_m(x_f)\phi_n(y)\phi_n(x_f). \tag{4.6}$$

For (4.6) and (4.5) to satisfy the differential equations (4.2), we must have

$$\pi^2 DU_{pmn} = 2\beta, \qquad DS_{pmn} = -DT_{pmn} = m^2 - n^2, \qquad DV_{pmn} = \beta(m^2 + n^2) \tag{4.7}$$

where

$$D = (m^2 - n^2)^2\pi^2 + 2\beta^2(m^2 + n^2). \tag{4.8}$$

The complete solution of (4.2) is therefore

$$W = \sum_{m,n=1}^{\infty}\left[\sum_{k=1}^{4} C_{kmn} W_{kmn} e^{-a_{kmn}t} + W_{pmn}\right]\phi_m(x)\phi_m(x_f)\phi_n(y)\phi_n(x_f). \tag{4.9}$$

The boundary conditions on the four wall are satisfied by (4.9) because of our particular choice of $\phi_m(x)\phi_n(y)$. The four homogeneous initial conditions (3.9) can be satisfied by an appropriate choice of the still arbitrary constants C_{kmn}. Though they can be easily determined by way of the orthogonality condition among the ϕ_j's, we will not be concerned here with the explicit expressions for these constants. Instead, we consider only the steady state solution for the mean square quantities for the purpose of comparison with the results obtained in [4].

For $\beta > 0$, the real part of a_{kmn} is positive. For large t, the solution W tends to the particular solution W_p which is independent of t. If we set $y = x$ in this steady state solution, we get

$$W(x, x, \infty) = \sum_{m,n=1}^{\infty} W_{pmn}\phi_m(x)\phi_n(x)\phi_m(x_f)\phi_n(x_f). \tag{4.10}$$

From the expression for the components of W_{pmn} given in (4.7), we see that the series solution for the steady state mean square "displacement" and "power flow" converges uniformly and absolutely. But the steady state mean square "velocity" given by

$$\langle u_t^2\rangle_{\text{s.s.}} = \sum_{m\neq n}^{\infty} \frac{\beta(m^2 + n^2)}{(m^2 - n^2)^2\pi^2 + 2\beta^2(m^2 + n^2)}\phi_m(x)\phi_n(x)\phi_m(x_f)\phi_n(x_f) + \frac{1}{2\beta}\sum_{n=1}^{\infty}\phi_n^2(x)\phi_n^2(x_f) \tag{4.11}$$

is dominated by a divergent "diagonal" subseries. This is not surprising since we used an ideal white noise process in our forcing function. Had we started with a band-limited white noise excitation, the corresponding series solution would have terminated at the Mth term where M is determined by the cut-off frequency. No

problem with convergence arises in this case. For a lightly damped string so that $2\beta^2 \ll \pi^2$, we have then

$$\langle u_t^2 \rangle_{\text{s.s.}} \cong \frac{1}{2\beta} \sum_{n=1}^{M} \phi_n^2(x)\phi_n^2(x_f) \tag{4.12}$$

which is exactly what was obtained in [4] with the help of some plausible arguments based on physical consideration.

With all the. interesting observations concerning the physical problem already made in [4], the results obtained here only provide additional facts in support of the accuracy of the approximate solution of [4]. They do demonstrate that, even if a problem can be solved by available methods, the solution scheme developed in section (3) with the extension in section (5) is still a viable alternative.

5. Time-correlated excitations

If N(x, t) is temporally correlated so that (2.9) is not satisfied, the solution scheme developed in section (3) does not apply. The problem with a general random forcing is best discussed within the framework of systems of linear partial differential equations with general auxiliary conditions. This will be done in a future publication [5].

We will confine ourselves here to the case $N(x, t) \equiv N(t)$ to indicate the basic technique used for the general case. Many technical problems, including the rotating blade problem, belong to this class. Note that the excitation f(x, t) is still a function of x through the envelope function E(x, t). For this class of problems, we formulate a solution scheme similar to the one in section (3). This is accomplished by associating the time-correlated N(t) with the response of a (fictitious) linear dynamical system to white noise n(t). By this, we mean that N(t) and the output of the supplementary linear system have the same first and second order statistics. By setting up stationary or nonstationary shaping filters of first or higher order with white noise input, almost any correlation function for N(t) can be approximated well enough for practical purpose.

Spatial correlation-an example

To illustrate the details of the formulation for a correlated excitation *N(t), we* consider first a specific N(t) with autocorrelation

$$\langle N(t_1)N(t_2)\rangle = e^{-\alpha|t_1 - t_2|} \qquad (a > 0). \tag{5.1}$$

For the purpose of obtaining second order statistics for u, we may think of such an *N(t)* as the stationary output of a dynamical system characterized by the equation

$$N_t + \alpha N = \sqrt{2\alpha}n(t) \qquad (-\infty < t < \infty) \tag{5.2}$$

where *n(t)* is a white noise process with zero mean and normalized spectral density. We can show by way of the representation

$$N(t) = \sqrt{2\alpha} \int_{-\infty}^{t} e^{-\alpha(t-z)}n(z)\,dz \tag{5.3}$$

that the autocorrelation of N is given by (5.1) and that

$$\langle n(\tau)N(t)\rangle = H(t-\tau)\sqrt{2\alpha}\, e^{-\alpha(t-\tau)} \tag{5.4}$$

where the value $\frac{1}{2}$ is assigned to $H(0)$. From (5.4) and (1.4), we have furthermore

$$\langle u(x, t)n(t)\rangle = \langle u_t(x, t)n(t)\rangle = 0. \tag{5.5}$$

The solution scheme developed in section (3) hinges on the fact that $\langle f(x_j, t)u(x_k, t)\rangle$ and $\langle f(x_j, t)u_t(x_k, t)\rangle$ are completely known for a temporally uncorrelated excitation. For a temporally correlated excitation, the important relations (3.1) and (3.3) no longer hold. Our first task then is to determine the quantities $\langle u(x, t)N(t)\rangle$ and $\langle u_t(x, t)N(t)\rangle$ for the present problem. We do this by formulating an initial-boundary value problem for these quantities.

With the help of (5.5), we get

$$\langle uN_t\rangle + \alpha\langle uN\rangle = 0, \qquad \langle u_tN_t\rangle + \alpha\langle u_tN\rangle = 0 \tag{5.6}$$

by ensemble-averaging (5.2) after multiplying through by u(x, t) and $u_t(x, t)$, respectively. We get our first equation for

$$P(x, t) = \langle N(t)u(x, t)\rangle \qquad Q(x, t) = \langle N(t)u_t(x, t)\rangle \tag{5.7}$$

by ensemble-averaging the relation $(Nu)_t = N_tu + Nu_t$ and then eliminating $\langle uN_t\rangle$ by the first relation of (5.6). The result is

$$P_t = Q - \alpha P. \tag{5.8}$$

To get a second equation for P and Q, we multiply (1.1) by $N(t)$ and ensemble-average both sides. The result is

$$Q_t + aP_{xx} + bQ_x + cP_x + (a + d)Q + eP = E(x, t) \tag{5.9}$$

where use has been made of $\langle N^2\rangle = 1$ and the second relation of (5.6).

Equations (5.8) and (5.9) are supplemented by the initial conditions

$$P(x, 0) = Q(x, 0) = 0 \qquad (0 < x < 1) \tag{5.10}$$

and the boundary conditions

$$P(0, t) = P(1, t) = 0 \qquad t \geq 0) \tag{5.11}$$

which follow from (1.2) and (1.3), respectively.

We can of course use (5.8) to eliminate Q from (5.9) to get a single second order equation for P alone:

$$LP + \alpha[2P_t + bP_x + (d + \alpha)P] = E(x, t). \tag{5.12}$$

Whether we work with (5.12) or with (5.8) and (5.9), it is clear that P and Q are completelydetermined.

Now, we repeat the derivation of section (3) to get the four equations

$$
\begin{aligned}
&U = S + T \\
&S_t - V + a(y,t)U_{yy} + b(y,t)S_y + c(y,t)U_y + d(y,t)S + e(y,t)U \\
&\qquad = E(y,t)P(x,t) \\
&T_t - V + a(x,t)U_{xx} + b(x,t)T_x + c(x,t)U_x + d(x,t)T + e(x,t)U \\
&\qquad = E(x,t)P(y,t) \\
&V_t + a(x,t)S_{xx} + a(y,t)T_{yy} + b(x,t)V_x + b(y,t)V_y + c(x,t)S_x + c(y,t)T_y \\
&\qquad + [d(x,t) + d(y,t)]V + e(x,t)S + e(y,t)T \\
&\qquad = E(x,t)Q(y,t) + E(y,t)Q(x,t)
\end{aligned}
\tag{5.13}
$$

for the four spatial correlation functions. The domain of these equations is the semi-infinite unit square column. The initial conditions at the base square and the boundary conditions on the four walls of the column remain as given by (3.9) and (3.10).

The solution scheme formulated here is more complicated than the special case where *P* and Q are known functions. But from the point of view of a numerical solution, it is still far more efficient and practical than the Green's function approach or the autocorrelation function approach. In this connection, we emphasize here that it is not necessary to store the solution of*P* and Q to be used for the solution of the spatial correlation functions. At any particular time step t_n, *we* can use P_n and Q_n to generate P_{n+1} and Q_{n+1} as well as $U_{n+1}, \ldots, V_{n+1}$. There is no need to store P_n and Q_n in core after that.

Spatial correlation-the general case

To complete the analysis, we sketch presently the corresponding development for an excitation N(t) associated with the mth order differential equation with variablecoefficients

$$
\sum_{k=0}^{m} r_k(t) N_{,k} = n(t), \qquad N_{,k} \equiv \frac{d^k N}{dt^k}, \qquad (-\infty < t < \infty) \tag{5.14}
$$

where r_m does not vanish identically. The solution of (5.14) can be written as

$$
N(t) = \int_{-\infty}^{t} h(t,z)n(z)\,dz \tag{5.15}
$$

where $h(t, z)$ is the impulse response function of (5.14). We note that for $k \leq m-1$,

$$
h_{,k}(t,z) = 0 \qquad (t < z), \qquad h_{,k}(z+,z) = \delta_{k(m-1)} \tag{5.16}
$$

where δ_{ij} is the Kronecker delta. From (5.15) and (5.16) follows

$$
\langle n(\tau)N(t)\rangle = h(t,\tau) \tag{5.17}
$$

With (5.16) and (5.17), we have again from (1.4)

$$
\langle u(x,t)n(t)\rangle = \langle u_t(x,t)n(t)\rangle = 0. \tag{5.18}
$$

Our main task is to determine $P = \langle u(x,t)N(t)\rangle$ and $Q = \langle u_t(x,t)N(t)\rangle$ to be used in (5.13). The two algebraic equations corresponding to (5.6) are obtained from (5.14) with the help of (5.18):

$$\sum_{k=0}^{m} r_k(t)P_k(x,t) = 0, \qquad \sum_{k=0}^{m} r_k(x,t)Q_k(x,t) = 0 \tag{5.19}$$

where

$$P_k(x,t) = \langle u(x,t)N_{,k}(t)\rangle, \qquad Q_k(x,t) = \langle u_t(x,t)N_{,k}(t)\rangle \tag{5.20}$$

with $P_0 \equiv P$ and $Q_0 \equiv Q$. The two equations (5.19) contain $2(m+1)$ unknowns; so we need $2m$ more equations. m of these can be obtained from (1.1) by forming

$$\langle N_{,k}(t)Lu(x,t)\rangle = E(x,t)\langle N_{,k}(t)N(t)\rangle \qquad (k = 0, 1, \ldots, m-1) \tag{5.21}$$

which can be rewritten as

$$Q_{k,t} - Q_{k+1} + aP_{k,xx} + bQ_{k,x} + cP_{k,x} + dQ_k + eP_k = E(x,t)C_k(t) \tag{5.22}$$

where the variance and the covariances, $C_k(t) = \langle N(t)N_{,k}(t)\rangle, k = 0, 1, \ldots, m-1$, are known (since they can be calculated from (5.14) by the algorithm used in [2, 3]).

The other m equations are the counterparts of (5.8) obtained by ensemble-averaging the simple relation $(uN_{,k})_t = u_tN_{,k} + uN_{,k+1}$ for $k = 0, 1, \ldots, m-1$. The results are

$$P_{k,t} = Q_k + P_{k+1} \qquad (k = 0, 1, 2, \ldots, m-1). \tag{5.23}$$

The system (5.19), (5.22) and (5.23) are supplemented by the initial conditions

$$P_k(x,0) = Q_k(x,0) = 0, \qquad (0 < x < 1) \qquad (k = 0, 1, \ldots, m) \tag{5.24}$$

and the boundary conditions

$$P_k(0,t) = P_k(1,t) = 0 \qquad (t > 0) \qquad (k = 0, 1, \ldots, m) \tag{5.25}$$

which follow from (1.2) and (1.3), respectively.

Once $P = P_0$ and $Q = Q_0$ are found, they can be used in (5.13) for the determination of the spatial correlation functions. It is not surprising that the algorithm for P and Q gets more complicated with increasing m, since the excitation is also getting more complicated.

The autocorrelation function

The initial-boundary value problem (2.11)-(2.13) for the autocorrelation function $R_{uu}(x,t;y,\tau)$ is not applicable to a temporally correlated excitation since the important relation (2.10) no longer holds. One the other hand, the equation

$$L_{tx}R_{uu} = E(x,t)R_{Nu}(x,t;y,\tau) \tag{5.26}$$

remains valid. We now show that $R_{Nu}(x,t;y,\tau) = R_{Nu}(t;y,\tau)$ (since N is independent of x) can be determined efficiently once we relate $N(t)$ to white noise by (5.14).

With (5.17), we get from (5.14)

$$\sum_{k=0}^{m} r_k(t)R_{Nu,k} = 0 \qquad (\tau < t). \tag{5.27}$$

By definition and continuity, we have also

$$R_{Nu,k}(\tau; y, \tau) = P_k(y, \tau) \qquad (k = 0, 1, \ldots, m-1). \tag{5.28}$$

With $P_k(y, \tau)$ already determined, equations (5.27) and (5.28) define an initial value problem. Its solution will be used in (5.26) with auxiliary conditions

$$R_{uu}(x, \tau; y, \tau) = U(x, y, \tau), \qquad R_{uu,t}(x, \tau; y, \tau) = T(x, y, \tau) \tag{5.29}$$

$$R_{uu}(0, t; y, \tau) = R_{uu}(1, t; y, \tau) = 0 \qquad (t > \tau) \tag{5.30}$$

for the determination of R_{uu} for $t > \tau$.

Again, for a numerical solution for a fixed pair of (y, τ), the two problems for R_{Nu} and R_{uu} could be solved simultaneously so that we do not have to store R_{Nu} in core.

6. Concluding remarks

We have presented in this paper the basic ideas behind an efficient solution scheme for the second order statistics of the solution of initial-boundary value problems involving linear partial differential equations with random forcing. While the method is particularly useful for problems for which available methods are either inapplicable or impractical, it is also competitive with other methods for problems solvable by both.

The essential feature of the method is to formulate an initial-boundary value problem for the spatial correlation functions of the response which contain the mean square values as a special case. This can always be done for a temporally uncorrelated input. For a temporally correlated excitation, the problem is reduced to one with a temporally uncorrelated input by associating the correlated excitation with the response of a (fictitious) linear dynamical system to a temporally uncorrelated input. If desired, the general autocorrelation function is also to be obtained from another initial-boundary value problem with the spatial correlation functions as initial conditions.

In a sense, the results presented here are only some partial results. In order to bring out the basic ideas, we have concentrated on a single second order equation in x and t with a loading which is either temporally uncorrelated or of the form $f(x, t) = E(x, t)N(t)$ where $E(x, t)$ is a known envelope function. The result for a temporally uncorrelated excitation represents the fundamental result while the development for the special class of correlated excitations indicates how one can reduce the more general problem to a problem to which the fundamental result applies. More complete results for the general problem involving a system of linear partial differential equations in higher dimensions with general random excitations and auxiliary conditions will be reported in [5]. It should be emphasized however, that even the partial results obtained here are already useful for technical problems such as the rotating wing problem mentioned earlier.

Appendix

The Green's function $G(x, t; x', t')$ associated with the initial-boundary value problem (1.1)–(1.3) is the solution of

$$L_{tx}G = \delta(t - t')\delta(x - x') \qquad (0 < x' < 1) \tag{A.1}$$

subject to the causality conditions

$$G(x, t; x', t') = G_t(x, t; x', t') = 0 \qquad (t < t') \tag{A2}$$

and the boundary conditions

$$G(0, t; x', t') = G(1, t; x', t') = 0 \qquad (t > t') \tag{A3}$$

where $\delta(z)$ is the Dirac delta function. We are interested here in the behavior of G and G_t at $t = t'$. In particular, we will show that

$$G(x, t'+; x', t') = 0, \qquad G_t(x, t'+; x', t') = \delta(x - x'). \tag{A4}$$

To prove the above results, we consider a function $w(x,t)$ which is the solution of the equation

$$Lw = 0 \qquad (t > t') \tag{A5}$$

subject to the initial conditions

$$w(x, t'+) = 0, \qquad w_t(x, t'+) = \delta(x - x') \quad (0 < x < 1) \tag{A.6}$$

and the boundary conditions

$$w(0, t) = w(1, t) = 0 \qquad (t > t'). \tag{A.7}$$

We now claim that we can write G in terms of w by the relation

$$G(x, t; x', t') = H(t - t')w(x, t) \tag{A.8}$$

where $H(z)$ is the unit step function.

As given by (A.8), G evidently satisfies the causality conditions (A.2) and the homogeneous boundary conditions (A.3). It remains to show that G satisfies the differential equation (A. 1). With

$$\begin{aligned} G_t &= \delta(t - t')w(x, t) + H(t - t')w_t(x, t) \\ G_{tx} &= \delta(t - t')w_x(x, t) + H(t - t')w_{tx}(x, t) \\ G_{tt} &= \delta(t - t')w(x, t) + 2\delta(t - t')w_t(x, t) + H(t - t')w_{tt}(x, t) \end{aligned} \tag{A9}$$

and

$$\begin{aligned} \delta(t - t')F(x, t) &= \delta(t - t')F(x, t'+) \\ \delta_t(t - t')F(x, t) &= \delta_t(t - t')F(x, t'+) - \delta(t - t')F_t(x, t'+) \end{aligned} \tag{A. 10}$$

we have

$$\begin{aligned} LG = {} & H(t - t')Lw + \delta_t(t - t')w(x, t'+) + \delta(t - t')w_t(x, t'+) \\ & + b(x, t')\delta(t - t')w_x(x, t'+) + d(x, t')\delta(t - t')w(x, t'+). \end{aligned} \tag{A.11}$$

With the help of (A.5) and (A.6), (A.1 1) becomes (A.1). The useful results (A.4) now follow from (A.6), (A.8) and the first equation of (A.9).

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