LECTURE NOTES ON PROBLEMS IN ELASTICITY:

I. FUNDAMENTALS OF LINEAR THEORY OF ELASTICITY

Frederic Y.M. Wan

Technical Report No. 82-17

November, 1982

Cover photo courtesy of the U.B.C. Museum of Anthropology:

Haida totem pole; main figure, possibly bear, holding wolf between legs, frog in mouth, wolf between ears.

PROLOGUE

It is generally accepted that a body of substance is made of molecular and atomic units. These units are bound together by intermolecular (or interatomic) ties. Under action of external forces (be it gravitational, electromagnetic, contact or other forces), the molecular units will re-distribute themselves to reach a state of equilibrium (which might involve the substance being in motion). This re-distribution generally results in a change of position and shape of the body which we will call deformation. In principle, it is possible to determine this re-distribution of molecular units by way of the laws of mechanics, electromagnetics, thermodynamics, etc., along with the binding laws among the molecules. But the nature of the molecular forces (except for a few situations) is so complicated that the progress along this line toward an understanding (explanation and prediction) of the response of the body to external excitation is very difficult.

If we are only interested in the macroscopic behaviour of the body, much less detailed description and analysis of the substance than those at the molecular level often suffice. The less detailed approach in turn makes progress easier. Such an approach consists of regarding a body of substance as a mathematical continuum S (a closed connected set of points S in the 3-dimensional Euclidean space) and assuming that it remains a continuum S' after deformation. As such, the deformation of a body of substance may be thought of as a mapping of S in E^3 in the case of no crack resulting from the deformation.

For most problems of interest and certainly for those studied in this course, we can in fact require that the map be 1-1.

For a given body acted upon by external excitation, we would like to know what is the deformed configuration S'. In other words, what is the relevant mapping which sends S into S'? In general, this piece of information, together with the material property of the substance, tells us whether the strength requirements of a structural design are met or whether the intermolecular ties are broken by the excitation and some catastrophic or undesirable process is to take place.

Starting from our idealized view of substance, one can use the accepted laws of physics, the macroscopic properties intrinsic to the particular substance along with logical deduction based on known mathematical results to analyze the influence and predict the outcomes of the action of the external forces on the body. Such an endeavour is called <u>Continuum Mechanics</u> or the <u>Mechanics of Deformable Bodies</u>. Occasionally, new mathematics may have to be developed to cope with a particular problem.

The <u>Theory of Elasticity</u> is a branch of Continuum Mechanics dealing with deformable <u>solid</u> bodies having the physical property called <u>elasticity</u>. A body is elastic if, when the external forces are removed, the bodies return to their original (undeformed) shape. In varying degrees, this property is shared by a very wide range of substances, especially commonly used metals.

To understand the behavior of an elastic body under external load, we can attack each problem ab initio and in an ad hoc manner. This approach has been and is still being employed by experienced applied scientists and engineers. However, its success often depends on a deep insight into the physical problem in question. As an alternative, we can also develop a general theory which is applicable to all problems concerning elastic solids. In principle this second approach makes the solution of a specific problem strictly an application of mathematical methods. This approach to the analysis of elastic bodies is generally favored by mathematicians and the first correct general continuum theory of elasticity was in fact developed by A.L. Cauchy who is more widely known in the mathematical community for his contributions to the theory of functions of complex variables. In the history of science, however, his contributions to the continuum mechanics of solid matters are no less monumental. It is the intention of these lectures to present the bare minimum of the fundamentals of the classical linear theory of elasticity as formulated by Cauchy. We then use the basic theory developed here to solve a few two dimensional problems both as applications of the theory and as preparations for the main topics of these lectures, namely, the theories of elastic thin plates and shells and their applications.

CONTENTS

CONTENTS (Cont'd)

I. FUNDAMENTALS OF LINEAR THEORY OF ELASTICITY

1. Why Stress and Strain

1.1 The Notion of Displacement and Strain

When subject to external loads[1], a body of substance will deform. To extract quantitative informations about this deformation, it is not enough to talk abstractly about mappings or transformations of a point set in Euclidean space. Ideally, we would like to know the mapping itself explicitly. As a first step toward the explicit determination of the specific mapping associated with a given set of external loads, let us describe a point in the body V before deformation by a position vector (Figure (1))

$$\vec{r} = \sum_{k=1}^{3} x_k \vec{i}_k$$

Figure (1)

with reference to a set of cartesian axes. After deformation, the same point is now in a new position with a new position vector

$$\vec{r}\,' = \sum_{k=1}^{3} x'_k \vec{i}_k$$

---

(1) We use the word "loads" here to denote all conceivable kinds of external disturbance including forces, moments, pressure, etc. Also, we will be thinking only about statics in this section for the sake of clarity.

For different points in V, the coordinates $\{x_1, x_2, x_3\}$ take on different numerical values. The corresponding coordinates of new position $\{x_1', x_2', x_3'\}$ are evidently scalar functions of the x_k's, i.e., $x_k' = x_k'(x_1, x_2, x_3)$. We now define a <u>displacement</u> vector $\vec{u}$ by

$$\vec{u} = \sum_{k=1}^{3} u_k \vec{i}_k = \vec{r}' - \vec{r}$$

so that

$$u_k = x_k' - x_k$$

Clearly, we know all about the deformed configuration V' if we know $\vec{u}$ (since we know $\vec{r}$).

For the special case $\vec{u} = \vec{u}_o$, where $\vec{u}_o$ is a constant vector, the body has simply undergone a rigid motion and moved from V to V'. It did not deform in the ordinary sense of the word and the elastic property of the body played no role throughout this changing of position.

A body experiences a genuine deformation and exhibits its elasticity only if there is a non-uniform or relative displacement of points in the body. In other words, there exist points in the body which did not experience the same displacement in either magnitude or direction. This leads us to the more useful concept (at least in linear elasticity) of <u>strain</u>. Very crudely, strain may be thought of as a relative displacement per unit distance between any two points in a body. Evidently, the body suffers no strain if it experiences only a rigid motion. We will give a quantitative definition of strain and how to relate it to the displacement vector $\vec{u}$ in section (3).

1.2 Internal Reactions and Stresses

When external loads cause a body to deform, the body resists this deformation by developing internal forces and moments which counter the effect of the external loads. A helical spring under axial tension is a familiar example of this general phenomenon of "resisting internal reactions" in a deformed body. Let us talk a little more about this example.

Figure (2)

Imagine that the spring is (fictitiously) cut into two pieces. The forces, $\vec{F}_1$ and $\vec{F}_2$, and the moments, $\vec{M}_1$ and $\vec{M}_2$, at the two sides of the cut are the internal reactions to the externally applied force $\vec{P}$ at the end of the spring and $\vec{P}_o = -\vec{P}$ at the wall (Figure (2)). If the original (uncut) spring is in a state of equilibrium, we know from elementary mechanics that each of the two pieces (isolated subsystems) resulting from the cut must also be in a state of equilibrium as well. This requires

$$\vec{F}_1 + \vec{P} = \vec{0}, \qquad \vec{F}_2 + \vec{P}_o = \vec{F}_2 - \vec{P} = \vec{0},$$

$$\vec{M}_2 + \vec{r} \times \vec{F}_2 = \vec{0}, \qquad \text{etc.}$$

where $\vec{r}$ is the position vector of the cut end (of the left half) measured from the point 0 at the wall(2).

Just how much will the spring stretch under the action of $\vec{P}$ and $\vec{P}_o$ depends of course on the material and geometrical properties of the spring. If the magnitude of the deformation is not too large, the internal forces developed in the spring is proportional to the axial extension per unit axial length of the spring. So the spring will cease to deform further when the axial extension is such that the corresponding internal reactions are in equilibrium with the external forces.

In general, we start with a deformable body in its natural (undeformed) state with no loads applied to it(3). We apply external loads, e.g., gravity, to the body, thus deform it. The body resists the deformation by developing a system of internal reactions to counter the applied loads. These internal reactions usually increase in magnitude with increasing deformation (strain). When every part of the body, i.e., every isolated subsystem, is in equilibrium under the action of the external loads and internal reactions, the body ceases to deform further.

---

(2) Otherwise, the moment equilibrium equations would be more complicated.

(3) Whether such a natural state exists is a question in thermodynamics. We will not go into that here. Intuitively, a piece of steel can exist for a long time without any (mechanical) change if left undisturbed.

To get a better feel for the internal reactions, we consider a body V in equilibrium with external forces $\vec{P}_1,\ldots,\vec{P}_5$. Let us cut the body arbitrarily and fictitiously into two isolated subsystems, denoting the surface of the left side of the cut S_+ with unit normal $\vec{\nu}$ and the surface of the right side of the cut by S_- with unit normal $-\vec{\nu}$, respectively (Figure (3)). Let $\vec{F}_+$ and $\vec{M}_+$ be the resultant internal force and moment acting on S_+ and $\vec{F}_-$ and $\vec{M}_-$ be those of S_-. By Newton's third law, we have

$$\vec{F}_- = -\vec{F}_+ \quad \text{and} \quad \vec{M}_- = -\vec{M}_+$$

Whatever the external loads may be, it is clear that $\vec{F}_+$ and $\vec{M}_+$ are the resultants of certain force and moment intensity distributed over the entire surface S_+, respectively. Similarly, $\vec{F}_-$ and $\vec{M}_-$ are resultants of some force and moment intensity distributed over S_-. Therefore, the internal reactions of the body are more fundamentally described by (surface) force and moment intensity vectors. They are called (force) stress vector and <u>moment stress</u> vector with the dimension of force/area and moment/area, respectively. We will discuss these fundamental quantities of elasticity theory in more details in the next section.

Figure (3)

2. Analysis of Stress

2.1 Stress Vectors

We spoke freely of force and moment vectors in the last section and will continue to assume that these rather non-trivial and fundamental concepts in mechanics are well understood. We only mention here that forces may be surface forces or body forces. An example of the former is contact pressure and an example of the latter is gravitational pull. Correspondingly, we have also the less familiar concepts of surface and body moments. In addition, forces are associated with their lines of action and give rise to the notion of moment of force with respect to a point.

A. Resultant Forces and Moments

Suppose a body of volume V is divided up (fictitiously) into n small elemental volumes ΔV_i, $i = 1,2,\ldots,n$, each with an elemental surface area ΔS_i. We let $\Delta\vec{F}_i^{(v)}$ be the body force associated with ΔV_i. The portion of the corresponding elemental surface ΔS_i in contact with the surface ΔS_k of another elemental volume ΔV_k is denoted by ΔS_{ik}. The portion ΔS_i (if any) belonging to boundary surface S of the original body is denoted by ΔS_{io}. We let $\Delta\vec{F}_{ik}^{(s)}$ and $\Delta\vec{F}_{io}^{(s)}$ be the surface force associated with ΔS_{ik} and ΔS_{io}, respectively. The resultant force $\vec{F}$ of all these elemental surface and body forces acting on the body is given by the vector sum

$$\vec{F} = \sum_{i=1}^{n} \sum_{k} \Delta\vec{F}_{ik}^{(s)} + \sum_{i=1}^{n} \Delta\vec{F}_i^{(v)}$$

where the index k in the double sum is taken only over those elements in contact with ΔS_i (and 0 if ΔS_i includes a part of the boundary surface)[4].

By the principle of action and reaction (Newton's third law of motion), the only terms which appear in the double sum are those associated with the boundary surface(s) of the entire body. In the limit as all elemental surface areas and elemental volumes tend to zeros, we have

$$\vec{F} = \oiint_S d\vec{F}^{(s)} + \iiint_V d\vec{F}^{(v)} . \tag{2.1}$$

The integrals are to be considered as Stieltjes integrals.

The moment $\vec{M}^{(F)}$ of the forces $\Delta\vec{F}^{(s)}$ and $\Delta\vec{F}^{(v)}$ with respect to the origin of the cartesian coordinate system is given by

$$\vec{M}^{(F)} = \sum_{i=1}^{n} \sum_{k} \vec{r} \times \Delta\vec{F}^{(s)}_{ik} + \sum_{i=1}^{n} \vec{r} \times \Delta\vec{F}^{(v)}_{i}$$

In order to make precise the meaning of the position vector $\vec{r}$, we assume that the forces $\Delta\vec{F}^{(s)}_{ik}$ and $\Delta\vec{F}^{(v)}_{i}$ pass through the centroid of the elements ΔS_{ik} and ΔV_i, respectively, and that $\vec{r}$ is the position vector of the appropriate centroid.

---

(4) We may also define $\Delta\vec{F}^{(s)}_{ik} = \vec{0}$ if k is not associated with an element in contact with $\Delta\vec{S}_i$ and $\Delta\vec{F}_{io} = \vec{0}$ if ΔS_i does not include a part of S. In that case, the index k in the double sum may be taken from 0 to n.

In addition to the moment $\vec{M}^{(F)}$ due the forces, we may have a moment $\vec{M}^{(M)}$ due to surface moment $\Delta\vec{M}_{ik}^{(s)}$ and volume moments $\Delta\vec{M}_{i}^{(v)}$. Presence of these additional moments means that the choice of centroids of ΔS_{ik} and ΔV_i as points of application of $\Delta\vec{F}_{ik}^{(s)}$ and $\Delta\vec{F}_{i}^{(v)}$ needs not be considered a restrictive assumption.

Combination of $\vec{M}^{(F)}$ and $\vec{M}^{(M)}$ leads to a resultant moment $\vec{M}$ of the form

$$\vec{M} = \sum_{i=1}^{n} \sum_{k} (\vec{r} \times \Delta\vec{F}_{ik}^{(s)} + \Delta\vec{M}_{ik}^{(s)}) + \sum_{i=1}^{n} (\vec{r} \times \Delta\vec{F}_{i}^{(v)} + \Delta\vec{M}_{i}^{(v)}).$$

In the limit as ΔS_{ik} and ΔV_i tend to zero, we have

$$\vec{M} = \oint\!\!\oint_S (\vec{r} \times d\vec{F}^{(s)} + d\vec{M}^{(s)}) + \iiint_V (\vec{r} \times d\vec{F}^{(v)} + d\vec{M}^{(v)}). \quad (2.2)$$

B. Stresses and Load Intensities

We define force stress vectors, $\vec{\sigma}$, as the limit (as $\Delta S \to 0$) of the ratio of the surface force vector $\Delta\vec{F}^{(s)}$ to the surface element ΔS on which $\Delta\vec{F}^{(s)}$ is acting,

$$\vec{\sigma} = \lim_{\Delta S \to 0} \frac{\Delta\vec{F}^{(s)}}{\Delta S} \quad (2.3)$$

Analogously, we define moment (or couple) stress vectors, $\vec{\tau}$, by

$$\vec{\tau} = \lim_{\Delta S \to 0} \frac{\Delta\vec{M}^{(s)}}{\Delta S} . \quad (2.4)$$

Body force intensity vectors $\vec{f}$ and body moment intensity vectors $\vec{m}$ are defined by

$$\vec{f} = \lim_{\Delta V \to 0} \frac{\Delta \vec{F}^{(v)}}{\Delta V}, \qquad \vec{m} = \lim_{\Delta V \to 0} \frac{\Delta \vec{M}^{(v)}}{\Delta V}. \tag{2.5}$$

The various limits mentioned above may or may not exist as ΔS and $\Delta V \to 0$. On the other hand, the various integrals defining the resultant force and moment are always meaningful. In the event that the limiting ratios do not exist in the ordinary sense, we may adopt the notion of Dirac's delta function whenever appropriate so that we can continue to talk about these ratios (without having to spend time talking about distributions).

Suppose we have two bodies which are in contact over the surface element ΔS with outward unit normal $\vec{\nu}$ and $-\vec{\nu}$, respectively (Figure (4)). The principle of action and reaction says that if $\vec{\sigma}$ and $\vec{\tau}$ are the force and moment stress vectors on ΔS for one body, then the corresponding stress vectors, $\vec{\sigma}_{-\nu}$ and $\vec{\tau}_{-\nu}$, for the other body are of equal magnitude and opposite direction so that[5].

Figure (4)

$$\vec{\sigma}_{-\nu} = -\vec{\sigma}_{\nu}, \quad \vec{\tau}_{-\nu} = -\vec{\tau}_{\nu} \tag{2.6}$$

This is true whether ΔS is a physical surface or one resulting from a fictitious cut of a given body (see Fung, for example, for a proof of this claim).

---

(5) When more than one stress vector are being discussed, we distinguish them by the direction of the normal of the associated surface.

It is generally accepted that the internal moment stress vector $\vec{\tau}$ plays only a negligible role for ordinary subject to mechanical loads[(6)]. We will henceforth omit it from our discussion, and will call $\vec{\sigma}$ the <u>stress vector</u> without any ambiguity. Also, we will consider only problems where there is no body moment intensity $\vec{m}$ and no externally applied moment stress on S[(6)]. All these restrictions leave us with the <u>classical theory of elasticity</u> which has as its cornerstone the following fundamental postulate:

<u>On any imaginary closed surface S in the interior of a continuum, a stress vector field can be defined in such a way that its action on the material occupying the space interior to S is equipollent to the action of the exterior material on the material inside S.</u>

---

(6) In the last two decades, there has been considerable interest in elasticity theories which include the effect of moment stress, $\vec{\tau}$, to analyze phenomena such as dislocations in metals. We will have an opportunity to see the usefulness of allowing $\vec{\tau} \neq \vec{o}$ later in our discussion of plate and shell theories. Also, $\vec{m}$ is known to be present in a magnet (in a magnetic field) and in dialectric material (in an electric field).

2.2 Equilibrium

A. Cauchy's Formula and Component Representation

At any point P of a body there is an infinity of different surface elements with centroid at P. Hence, associated with the point P, there are an infinity of stress vectors. It turns out that not all stress vectors are independent of each other. Given any three elemental surfaces with centroid at P and with unit normals $\vec{\nu}_1$, $\vec{\nu}_2$ and $\vec{\nu}_3$ which are not coplanar, the stress vector associated with a fourth elemental surface with centroid at P and unit normal $\vec{\nu}$ can be expressed in terms of the three stress vectors associated with the first three elemental surfaces. The relation among the four stress vectors is given by Cauchy's formula the proof of which is based on the condition of force equilibrium and can be found in most texts of elasticity (see Fung for example). For the special case where $\vec{\nu}_1$, $\vec{\nu}_2$ and $\vec{\nu}_3$ are simply $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ respectively, Cauchy's formula takes the form

$$\boxed{\vec{\sigma}_\nu = \vec{\sigma}_j \nu_j, \quad \nu_j = i_j = \vec{i}_j \cdot \vec{\nu}} \tag{2.7}$$

The simplest representation of the state of stress at a point in terms of scalar quantities is given by writing the stress vector $\vec{\sigma}_j$ acting on faces with unit normal vectors $\vec{i}_k$, $k = 1,2,3$, in the form

$$\vec{\sigma}_j = \sigma_{jk} \vec{i}_k \tag{2.8}$$

It is customary to designate the components σ_{jj} as normal stress components and the components σ_{jk} with $j \neq k$ as tangential or shearing stress components. A little reflection and some drawings will convince us that these designations are rather connotative. With the help of Cauchy's formula, we see that the nine scalar stress components σ_{ij}, $i,j = 1,2,3$, completely describe the state of stress at any point in the body. We will show presently that the condition of moment equilibrium (with $\vec{\tau} = \vec{0}$ and $\vec{m} = \vec{0}$) requires that

$$\boxed{\sigma_{ij} = \sigma_{ji}, \quad (i,j = i,2,3)} \tag{2.9}$$

so that for the classical theory of elasticity, there are only six distinct stress components.

B. Differential Equations for Stress Components

We may for the present purpose leave aside the problem of motion and limit ourselves to the consideration of bodies in static equilibrium. We postulate as conditions of equilibrium the two equations of vanishing resultant force and vanishing resultant moment:

$$\oiint_{S'} \vec{\sigma}_\nu dS + \iiint_{V'} \vec{f}\, dV = 0, \quad \oiint_{S'} \vec{r} \times \vec{\sigma}_\nu dS + \iiint_{V'} \vec{r} \times \vec{f}\, dV = 0$$

valid for the entire body as well as any of its (fictitiously) isolated parts.

Using Cauchy's formula for $\vec{\sigma}_\nu$ in the surface integral of the force equilibrium condition, we get

$$\oiint_{S'} \vec{\sigma}_j \nu_j \;\; dS + \iiint_{V'} \vec{f} \, dV = \vec{0}.$$

We limit our discussion in these lectures to piecewise smooth stress vectors $\vec{\sigma}_j$ so that we may apply Gauss' theorem (also known as divergence theorem) to the surface integral to get

$$\iiint_{V'} (\vec{\sigma}_{j,j} + \vec{f}) \, dV = \vec{0}, \quad (\;)_{,j} \equiv \frac{\partial(\;)}{\partial x_j}$$

(assuming of course that the hypotheses of the theorem are satisfied). Since the last integral holds for arbitrary V', the integrand must vanish. The proof of this claim will be left as an exercise. This gives as <u>differential equation of force equilibrium</u>

$$\boxed{\vec{\sigma}_{j,j} + \vec{f} = 0} \; . \tag{2.10}$$

A similar consideration of the integral moment equilibrium condition gives us as <u>differential equation of moment equilibrium</u>

$$\boxed{\vec{r}_{,j} \times \vec{\sigma}_j \;\; = \;\; 0} \tag{2.11}$$

which turns out to be simply an algebraic equation.

The two vector differential equations of equilibrium are each equivalent to three scalar equations. Using the component representation for $\vec{\sigma}_j$ and the corresponding component representation for $\vec{f}$,

$$\vec{f} = f_k \vec{i}_k \tag{2.12}$$

the force equilibrium equations implies

$$\boxed{\sigma_{jk,j} + f_k = 0, \quad (k = 1,2,3)} \tag{2.13}$$

The moment equilibrium equation implies

$$\boxed{\sigma_{jk} - \sigma_{kj} = 0, \quad j \neq k} \tag{2.14}$$

which are the symmetry conditions stated earlier. The number of distinct stress components is therefore reduced from nine to six. The symmetry conditions cease to be valid if there are either moment stresses (from $\vec{\tau}_j$) or body moment intensities (from $\vec{m}$) present in the body.

2.3 Transformation Laws

We have defined our stress components, σ_{ij}, with respect to a particular set of cartesian axes, x_1, x_2 and x_3. If we want to use a different set of cartesian axes[7], x_1', x_2' and x_3', the corresponding set of stress components $\sigma_{i'j'} \equiv \sigma_{ij}'$ will in general not be the same as σ_{ij}. On the other hand, Cauchy's formula suggests that the two sets, σ_{ij} and σ_{ij}', must be related somehow. Just why we should want to change axes will become apparent later in our discussion of stress strain relations. Meanwhile, let us establish the transformation law relating the two sets of stress components. We will do this in several steps.

Since $\{x_k\}$ and $\{x_j'\}$ are both cartesian, they can be obtained from each other by a translation of the origin and/or a rotation about the origin (orthogonal transformations). Since the origin of the cartesian coordinate system has no bearing on the definition of σ_{ij}, we assume that the new coordinate system is obtained by rotating the old one about its origin. The directional (unit) vectors of the two sets of axes, denoted by $\vec{i}_k$ and $\vec{i}_j'$ respectively, are then related by

$$\vec{i}_k' = \alpha_{kj}\, \vec{i}_j \,, \quad \alpha_{km} \equiv \vec{i}_k' \cdot \vec{i}_m \tag{2.15}$$

Note that α_{km} are the directional cosines and $\alpha_{kj} \neq \alpha_{jk}$!

We have the following properties of α_{kj}:

---

(7) The prime in σ_{ij}' etc., does not indicate quantities associated with the deformed body; it denotes quantities associated with the new reference frame $\{x_1', x_2', x_3'\}$.

(i) Evidently, there are the inverted relations, $\vec{i}_k = \gamma_{kj}\ \vec{i}_j'$ expressing $\vec{i}_k$ in terms of $\vec{i}_j'$. But we have

$$\gamma_{mn} = \vec{i}_m\ \vec{i}_n' = \vec{i}_m \cdot \alpha_{nk}\ \vec{i}_k = \alpha_{nk}\ \delta_{mk} = \alpha_{nm}$$

and therefore $\gamma_{mn} = \alpha_{nm}$ so that

$$\boxed{\vec{i}_m = \alpha_{km}\ \vec{i}_k'} \tag{2.16}$$

(ii) Since $\vec{i}_k' \cdot \vec{i}_m' = \delta_{km}$ and, at the same time,

we have $\vec{i}_k' \cdot \vec{i}_m' = \alpha_{kp}\ \vec{i}_p \cdot \alpha_{mq} \vec{i}_q = \alpha_{kp} \alpha_{mq}\ \delta_{pq} = \alpha_{kp}\ \alpha_{mp}$

$$\boxed{\alpha_{kp} \alpha_{mp} = \delta_{km}} \tag{2.17}$$

(iii) Similarly, with $\vec{i}_k \cdot \vec{i}_m = \alpha_{pk}\ \vec{i}_p' \cdot \alpha_{qm}\ \vec{i}_q' = \alpha_{pk} \alpha_{qm}\ \delta_{pq} = \alpha_{pk} \alpha_{pm}$

we have also

$$\boxed{\alpha_{pk} \alpha_{pm} = \delta_{km}} \tag{2.18}$$

(iv) With $\vec{r} = x_k \vec{i}_k = x_m' \vec{i}_m'$, we get

$$x_m' = x_k \vec{i}_k \cdot \vec{i}_m' = x_k \vec{i}_k \cdot \alpha_{mn} \vec{i}_n$$

or
$$\boxed{x_m' = \alpha_{mn} x_n} \tag{2.19}$$

Similarly, we can show that

$$\boxed{x_n = \alpha_{mn} x_m'} \tag{2.20}$$

(v) From (2.19) and (2.20), we get

$$\boxed{\frac{\partial x'_k}{\partial x_m} = \alpha_{km}, \quad \frac{\partial x_k}{\partial x'_m} = \alpha_{mk}} \tag{2.21}$$

With these properties, we can bow prove the following <u>transformation law for σ_{ij}</u>:

$$\boxed{\sigma_{ij} = \alpha_{im}\alpha_{jn}\,\sigma_{mn}} \tag{2.22}$$

We begin with Cauchy's formula

$$\vec{\sigma}_\nu = \nu_k\,\vec{\sigma}_k, \quad \nu_k = \vec{\nu}\cdot\vec{i}_k$$

Now, take $\vec{\nu} = \vec{i}'_m$, then

$$\vec{\sigma}_{i'_m} \equiv \vec{\sigma}'_m = (\vec{i}'_m\cdot\vec{i}_k)\,\vec{\sigma}_k \quad \text{or} \quad \vec{\sigma}'_m = \alpha_{mk}\vec{\sigma}_k.$$

But by definition $\vec{\sigma}'_m = \sigma'_{mn}\,\vec{i}'_n$, therefore we have

$$\sigma'_{mn} = \vec{\sigma}'_m\cdot\vec{i}'_n = \alpha_{mk}\,\vec{\sigma}_k\cdot\vec{i}'_n = \alpha_{mk}(\sigma_{kj}\vec{i}_j)\cdot\vec{i}'_n$$

and the transformation law (2.22) follows. With (2.21), the transformation law may be written as

$$\sigma'_{mn} = \frac{\partial x'_m}{\partial x_k}\,\frac{\partial x'_n}{\partial x_j}\,\sigma_{kj} \tag{2.23}$$

With these two results, we have effectively shown that σ_{ij}, i, j = 1,2,3 are components of a <u>cartesian tensor</u>. When we do not specify the value of i and j, σ_{ij} itself is called the (cartesian) <u>stress tensor</u>.

2.4 Principal Directions and Invariants of the Stress Tensor

Recall that $\vec{\sigma}_\nu$ is the stress vector at a point P in the deformed body and is associated with an elemental surface passing through P (and having P as its centroid) with unit normal $\vec{\nu}$. In general $\vec{\sigma}_\nu$ has a component in all three directions of the cartesian reference frame. Superficially, it would seem that there is at least one elemental surface, among all those passing through P, for which $\vec{\sigma}_\nu$ is in the same direction as $\vec{\nu}$, i.e. $\vec{\sigma}_\nu = \sigma\vec{\nu}$ where σ is a scalar . Such an elemental surface (if it exists) is called a <u>principal plane</u> at P, the corresponding $\vec{\nu}$ and σ are called a <u>principal direction</u> (or axis) and a <u>principal stress</u> at P, respectively. To the extent that $\vec{\sigma}_\nu$ is not a constant vector but varies with $\vec{\nu}$, the existence of a principal direction is far from certain.

To show that there will always be such a principal direction $\vec{\nu}$, let us write

$$\vec{\sigma}_\nu = \sigma\vec{\nu} = \sigma\nu_k\vec{i}_k$$

On the other hand, the Cauchy formula says

$$\vec{\sigma}_\nu = \nu_k\vec{\sigma}_k = \nu_k\sigma_{kj}\vec{i}_j$$

Therefore,

$$(\sigma_{kj} - \delta_{kj}\sigma)\nu_k\vec{i}_j = 0$$

or

$$(\sigma_{kj} - \sigma\delta_{kj})\nu_k = 0 , \qquad j = 1,2,3$$

This is a set of three homogeneous linear algebraic equations for ν_k subject to the condition $\nu_k\nu_k = 1$ and that σ_{ij} are symmetric (in its subscripts) so that $\sigma_{ij} = \sigma_{ji}$. (Thus we have an eigenvalue problem with σ being the eigenvalue parameter). For these equations to have a non-trivial solution, we need

$$| \sigma_{jk} - \sigma\delta_{jk} | = 0$$

Writing it out in full, the equation determining the eigenvalue σ becomes

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

Let $\sigma^{(k)} = 1,2,3$, be the three roots of the cubic equation for σ. Then $(\sigma - \sigma^{(1)})\ (\sigma - \sigma^{(2)})\ (\sigma - \sigma^{(3)}) = 0$ and

$$I_1 = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}$$

$$I_2 = \sigma^{(1)}\sigma^{(2)} + \sigma^{(2)}\sigma^{(3)} + \sigma^{(3)}\sigma^{(1)}$$

$$I_3 = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}$$

One of the three roots must be real; so there always exists one principal direction which depends on the state of stress of the body at P (not too surprisingly). But since the matrix $[\sigma_{ij}]$ is symmetric, we have the following results:

(i) All (eigenvalues) $\sigma^{(k)}$, $k = 1,2,3$, are real.

(ii) In the case of three distinct eigenvalues, $\vec{\nu}^{(k)}$, $k = 1,2,3$, are mutually orthogonal. (See texts such as Fung for a discussion of the special cases $\sigma^{(1)} = \sigma^{(2)}$, and $\sigma^{(1)} = \sigma^{(2)} = \sigma^{(3)}$).

With the existence of three distinct principal directions, we call the quantity $\sigma_o = \frac{1}{3}(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}) = \frac{1}{3} I_1 = \frac{1}{3} (\sigma_{kk})$ the <u>mean stress</u> at P, and the quantity $d_{ij} = \sigma_{ij} - \sigma_o \delta_{ij}$ the <u>stress deviation tensor</u> or the <u>deviator</u>.

3. Analysis of Strain

3.1 Components of Finite Strain

The general discussion of section (1) suggests that the fundamental notion of deformation is the relative displacement of any two points in a body. To describe this relative displacement quantitatively, we consider two points P_1 and P_2 in the undeformed body with position vector $\vec{r}$ and $\vec{r} + d\vec{r}$, respectively. After deformation, P_1 and P_2 become P_1' and P_2' with new position vector $\vec{r}\,'$ and $\vec{r}\,' + d\vec{r}\,'$, respectively (Figure (5)).

Figure (5)

The distance between P_1 and P_2 is $ds = |d\vec{r}| = (d\vec{r} \cdot d\vec{r})^{1/2}$ and the distance between P_1' and P_2' is $ds' = |d\vec{r}\,'| = (d\vec{r}\,' \cdot d\vec{r}\,')^{1/2}$. The difference $ds' - ds$ is clearly the change in distance between the points due to deformation. But the relative displacement of these two points involves more than just this change of distance between them. For instance, it involves the orientation of one point relative to the other before and after deformation. It turns out that a complete description of the relative displacement is contained in the quantity

$$(ds')^2 - (ds)^2 = d\vec{r}\,' \cdot d\vec{r}\,' - d\vec{r} \cdot d\vec{r} \; .$$

This quantity is used to define the <u>components of finite strain</u> E_{jk}

$$(ds')^2 - (ds)^2 = 2E_{jk}dx_j dx_k \qquad (3.1)$$

where repeated indices are to be summed over the range of the indices. In this way, we may omit the usual summation sign Σ.

We will now relate the strain components to the displacement components, u_i, introduced in section (1.1). We limit our discussion to external loads which give rise only to continuous and piecewise twice differentiable displacement fields. In that case, we have

$$d\vec{r} = \vec{r},_j dx_j = x_{k,j} dx_j \vec{i}_k = \delta_{kj} dx_j \vec{i}_k = dx_k \vec{i}_k$$

and

$$d\vec{r}\,' = \vec{r}\,',_j dx_j = (\vec{r}+\vec{u}),_j dx_j = (\delta_{kj}+u_{k,j}) dx_j \vec{i}_k \ .$$

It then follows from (3.1)

$$\boxed{2E_{jk} = u_{k,j} + u_{j,k} + u_{m,j}u_{m,k}, \qquad (j,k = 1,2,3)} \qquad (3.2)$$

Note that E_{jk} is symmetric in subscripts, i.e., $E_{jk} = E_{kj}$.

3.2 Small, Linear, Infinitesimal, Normal and Shear Strain

The strain in a body is small if $|E_{jk}| << 1$ for all j and k. That portion of the expression for the components of finite strain which is linear in the derivatives of the displacement components is called the linear strain components, e_{jk},

$$\boxed{e_{jk} = \frac{1}{2}(u_{k,j} + u_{j,k}), \quad (j,k = 1,2,3).}$$

It should be emphasized that E_{jk} can not always be approximated by e_{jk} even if we have small strain. The strain state of a body is infinitesimal if $E_{jk} \cong e_{jk}$.

An infinitesimal strain theory of elasticity has been found adequate for a large class of technical problems in engineering. In engineering literatures, e_{kk}'s (no sum in k) are referred to as normal strain components while $2e_{jk}$ $(j \neq k)$ are referred to as shear strain components. We will presently attempt to indicate the connotative nature of these expressions. Consider the following two dimensional picture (Figures (6) and (7)). With

$$\lim_{\Delta x_1 \to 0} \frac{u_1(x_1+\Delta x_1) - u_1(x_1)}{\Delta x_1} = u_{1,1} = e_{11} \text{ ,}$$

Figure (6)

e_{11} is evidently the change of length per unit length in a direction normal to the edge x_1 = constant. From the following two figures

Figure (7a)

Figure (7b)

with the composite figure

Figure (7c)

and the relations

$$\lim_{\Delta x_2 \to 0} \frac{u_1(x_2+\Delta_2) - u_1(x_2)}{\Delta x_2} = u_{1,2}$$

$$\lim_{\Delta x_1 \to 0} \frac{u_2(x_1+\Delta x_1) - u_2(x_1)}{\Delta x_1} = u_{2,1},$$

we see that $e_{12} = e_{21}$ is evidently one half of the change of the right angle $\angle\ x_2 0 x_1$. It has a <u>shearing</u> effect on the sheet.

3.3 Transformation Laws

We have defined the strain components E_{ij} with respect to a particular set of cartesian axes, x_1, x_2 and x_3. If we want to use a different set of cartesian axes, x_1', x_2' and x_3', the corresponding set of strain components E_{ij}' will generally not be the same as E_{ij}. We will presently establish the transformation law for E_{ij} under an orthogonal transformations of axes. We do this in a number of steps, each giving an independently useful result. In all cases, primed-quantities are associated with the $\{x_k'\}$ coordinate system.

(a) With

$$\vec{u} = u_k \vec{i}_k = u_j' \vec{i}_j',$$

we have

$$u_j' = u_k \vec{i}_k \cdot \vec{i}_j' = \alpha_{jk} u_k \quad \text{or} \quad \boxed{u_j' = \alpha_{jk} u_k = \frac{\partial x_k}{\partial x_j'} u_k}$$

where

$$\alpha_{jk} = \vec{i}_j' \cdot \vec{i}_k$$

(b) With

$$u_{k,j}' \equiv \frac{\partial u_k'}{\partial x_j'} = \frac{\partial u_k'}{\partial x_m} \frac{\partial x_m}{\partial x_j'} = \alpha_{jm} \frac{\partial}{\partial x_m} (\alpha_{kn} u_n),$$

we have

$$\boxed{u_{k,j}' = \alpha_{jm} \alpha_{km} u_{n,m} = \frac{\partial x_m}{\partial x_j'} \frac{\partial x_n}{\partial x_k'} u_{n,m}}$$

(c) With

$$2e_{ij} = u_{i,j} + u_{j,i},$$

we have

$$2e'_{ij} = u'_{i,j} + u'_{j,i} = \alpha_{im}\alpha_{jn}u_{m,n} + \alpha_{jp}\alpha_{iq}u_{p,q}$$

$$= \alpha_{im}\alpha_{jn}(u_{m,m} + u_{n,m}) = 2\alpha_{im}\alpha_{jn}e_{mn}$$

or

$$\boxed{e'_{ij} = \alpha_{im}\alpha_{jn}e_{mn}}$$

(d) Finally, with

$$u'_{m,j}u'_{m,k} = \frac{\partial u'_m}{\partial x'_j}\frac{\partial u'_m}{\partial x'_k} = \alpha_{mp}\alpha_{jq}u_{p,q} \cdot \alpha_{ms}\alpha_{kt}u_{s,t}$$

$$= \delta_{ps}\alpha_{jq}\alpha_{kt}u_{p,q}u_{s,t} = \alpha_{jq}\alpha_{kt}u_{p,q}u_{p,t},$$

we get

$$\boxed{E'_{ij} = \alpha_{im}\alpha_{jn}E_{mn}}$$

The transformation laws obtained above establish the fact that both e_{ij} and E_{ij} are cartesian tensors. It then follows that the quantities, $I_k(E_{ij})$ and $I_k(e_{ij})$, $k = 1,2,3$, similar to those defined for σ_{ij}, are invariant with respect to an orthogonal transformation.

4. Stress Strain Relations

4.1 Elasticity

In the absence of thermal, electromagnetic and chemical effects, experimental evidence indicates that, within a certain allowable limit of deformation, most materials encountered in our daily life exhibit the following properties:

(1) If it is not under the influence of any external disturbance, a body of material is free of any internal stress and can remain in this "unstressed" or "natural" state indefinitely.

(2) When subject to external loads, the state of stress at each point in the body depends only on the state of strain at the same point and conversely.

(3) The body returns to the unstressed state once the external loads are removed.

We call such a body of material an elastic body and the properties (1)-(3) elasticity.

To indicate other possible modes of behaviour, we note for instance that the stress state at a point of a body may depend on the time history of the strain state at that same point (visco-elasticity) or on the strain state of all points in some neighborhood of the given point (nonlocal theory). We are interested only in elastic bodies in this course.

Since the state of strain and stress at each point of a body is described by the six distinct components of the strain tensor E_{ij} and the stress tensor σ_{ij}, respectively, it appears that the elasticity of a body should be adequately described by six functional relations of the form

$$\sigma_{ij} = f_{ij}(E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}) \quad , \quad (i,j = 1,2,3) \qquad (4.1)$$

with $f_{ij} = f_{ji}$ and $f_{ij}(0,0,\ldots,0) = 0$, or by the even more general implicity relations (with suitable restriction to ensure elasticity)

$$g_k(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{33}; E_{11}, \ldots, E_{33}) = 0 \quad , \quad (k = 1,2,\ldots,6) . \qquad (4.2)$$

Either set of relations is called the (elastic) stress strain relations. The fact that both σ_{ij} and E_{ij} obey the same transformation law makes these postulated relations for the elastic property of the material even more appropriate.

4.2 Generalized Hooke's Law

For sufficiently "small" external loads, experimental results indicate that strain components of an elastic medium will be small compared to unity and that the elastic stress strain relations are effectively linear. We will be interested in these notes only in this particular range of elasticity. In this range, the stress strain relations are essentially a generalization of Hooke's original observation and are often referred to as generalized Hooke's law, and the body is said to be linearly elastic. Furthermore, we will be concerned only with infinitesimal strains so that $E_{ij} \cong e_{ij}$. The most general form of the linear stress strain relations may then be written as

$$\boxed{\sigma_{ij} = C_{ijk\ell}e_{k\ell} \quad , \quad (i,j = 1,2,3)} \qquad (4.3)$$

where $C_{ijk\ell}$ are known functions of position and are determined by suitable experiments. An elastic body is homogeneous if the elastic moduli $C_{ijk\ell}$ are constants throughout the body. It is inhomogeneous otherwise. Incidentally, the relations (4.3) may be thought of as the leading terms of the Taylor series expansion of the relations (4.1) for infinitesimal strains. It should be mentioned that there is enough content in this linear theory, both from the theoretical and the practical point of view, to justify its development without first attempting to cope with the more difficult nonlinear theory.

A few observations on the linear stress-strain relation (4.3) are now in order.

(i) The eighty-one elastic moduli $C_{ijk\ell}$ are not completely independent of each other. Since $\sigma_{ij} = \sigma_{ji}$ and $e_{ij} = e_{ji}$, we must have

$$\boxed{C_{ijk\ell} = C_{jik\ell} = C_{ij\ell k}} \tag{4.4}$$

These symmetry conditions reduce the number of distinct $C_{ijk\ell}$ to thirty-six.

(ii) In terms of previously defined notations the transformation law for $C_{ijk\ell}$ is

$$\boxed{C'_{ijk\ell} = \alpha_{im}\alpha_{jn}\alpha_{kp}\alpha_{\ell q}C_{mnpq}} \quad . \tag{4.5}$$

This is evident from

$$C'_{ijk\ell}e'_{k\ell} = \sigma'_{ij} = \alpha_{im}\alpha_{jn}\sigma_{mn} = \alpha_{im}\alpha_{jn}C_{mnpq}e_{pq}$$

$$= \alpha_{im}\alpha_{jn}C_{mnpq}\alpha_{kp}\alpha_{\ell q}e'_{k\ell}$$

$C_{ijk\ell}$ is evidently a fourth degree cartesian tensor, the <u>Hookean tensor</u>.

(iii) For $e_{k\ell}$ to be completely described by σ_{ij}, we must be able to invert these relations to get

$$\boxed{e_{k\ell} = c_{k\ell ij}\sigma_{ij}} \qquad (4.6)$$

with $c_{k\ell ij} = c_{\ell kij} = c_{k\ell ji}$.

(iv) The transformation law for $c_{ijk\ell}$ is exactly the same as $C_{ijk\ell}$.

4.3 The Inconsistency of the Proposed Stress Strain Relations

The finite strain components E_{ij} (and the linear strain components e_{ij}) characterize the distortion of an orginally rectangular box and give rise to internal stress $\vec{\sigma}_k^*$. When in equilibrium, $\vec{\sigma}_k^*$ are associated with the faces of the deformed box. The deformed box will not be rectangular in general and the normals of its faces are not necessarily in the direction $\vec{i}_k$. On the other hand, the components of the stress vectors $\vec{\sigma}_k$ were defined with respect to an infinitesimal rectangular box with edge vectors $\vec{i}_j dx_j$. So, we should be consistent and relate E_{ij} (or e_{ij}) to the components of $\vec{\sigma}_k^*$ instead of $\vec{\sigma}_k$. Alternatively, we can introduce a new strain tensor which characterizes the deformation of a body from whatever (unknown) shape into a rectangular box (see the Cauchy-Almansi strain tensor described in various references) and relate the components of this new strain tensor to σ_{ij} .

If either of these two approach to a consistent set of stress strain relations is carried out and then subsequently linearized with respect to all the unknowns, we will find that, in the first approach, the equilibrium equations are formally the same as those for the stress tensors defined with respect to the rectangular box, and in the second approach, the strain displacement relations are formally the same as our linear strain displacement relations. Thus, as long as we are talking about a linear theory of elasticity, we do not need to distinguish between the deformed and the undeformed configuration insofar as the equilibrium equations and the strain displacement relations are concerned. Therefore, we do not need to distinguish $\vec{\sigma}_k$ and $\vec{\sigma}_k^*$ in the linear stress strain relations. In other words, we can use the relations (4.3) inspite of their conceptual inconsistency as long as we are working with a linear theory.

4.4 Hyperelasticity

The stress strain relations for a certain class of (linear or nonlinear) elastic bodies can be expressed in terms of a scalar function (or more appropriately, functional) of the strain components. For infinitesimal strains, we have $S = S(e_{11},\ldots,e_{33})$ and the stress strain relations are in the form

$$\boxed{\sigma_{ij} = \frac{\partial S}{\partial e_{ij}} \quad , \qquad (i,j = 1,2,3) \quad .} \tag{4.7}$$

(For finite strains, e_{ij} should be replaced by E_{ij} and σ_{ij} should be defined in a way consistent with E_{ij} .) Such an elastic body is said to be hyperelastic. As long as there is no dissipative mechanism within the body, it can be shown through a thermodynamical consideration that such a scalar functional S does exist and is called the strain energy (density) function in engineering literature. It seems more appropriate to call S a stress potential (in strain space); we will occasionally do so.

The existence of a stress potential imposes certain additional restrictions on $C_{ijk\ell}$ of the general linear stress strain relations considered earlier. To see this, let us write out the following:

$$\sigma_{11} = C_{1111}e_{11} + C_{1112}e_{12} + C_{1113}e_{13} + \cdots + C_{1133}e_{33}$$

$$= \frac{\partial}{\partial e_{11}} \left[\frac{1}{2} C_{1111}e_{11}^2 + C_{1112}e_{12}e_{11} + C_{1113}e_{13}e_{11} + \cdots \right.$$

$$\left. + C_{1133}e_{33}e_{11} + S_1(e_{12},\ldots,e_{33}) \right] = \frac{\partial S}{\partial e_{11}}$$

where $S_1(e_{12},\ldots,e_{33})$ is independent of e_{11} . But

$$\sigma_{12} = \frac{\partial S}{\partial e_{12}} = C_{1112}e_{11} + \frac{\partial S_1}{\partial e_{12}} = C_{1211}e_{11} + C_{1212}e_{12} + \ldots + C_{1233}e_{33}$$

where $\dfrac{\partial S_1}{\partial e_{12}}$ is independent of e_{11} . So we must have

$$C_{1112} = C_{1211} \quad .$$

Similar arguments lead eventually to the additional symmetry condition

$$\boxed{C_{ijk\ell} = C_{k\ell ij}} \tag{4.8}$$

and

$$\boxed{S = \frac{1}{2}C_{ijk\ell}e_{ij}e_{k\ell}} \tag{4.9}$$

Thus the number of distinct $C_{ijk\ell}$ is further reduced to twenty-one.

Another way of seeing how the 36 coefficients get reduced to 21 is to rename the stress and strain components as one dimensional arrays. For instance, let $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{33} = \sigma_3$, $\sigma_{12} = \sigma_4$, $\sigma_{13} = \sigma_5$ and $\sigma_{23} = \sigma_6$. Make the analogous change for e_{ij} . The linear stress strain relations (4.3) may now be written as

$$\sigma_i = D_{ij}e_j \quad .$$

The existence of the strain energy function requires that the 6 x 6 coefficient matrix D_{ij} be symmetric and thereby reducing the distinct coefficients from 36 to 21 (see also p. 106 of Love).

4.5 Isotropy

In a crystalline body, the elastic coefficients $C_{ijk\ell}$ at a given point in the body depend on the orientation of the body. For this reason, the elastic coefficients are in general different for different choices of the cartesian reference frame subject only to the symmentry conditions (4.4). However, many metals have a sufficiently small and randomly oriented crystal grain structure that the elastic properties as averaged over a few grains are essentially independent of direction. An elastic medium is called (elastically) isotropic if there are no preferred directions insofar as relations between stress and strain are concerned.

For a linearly elastic body, this lack of preferred directions means that the coefficients C_{ijkm} must be the same as C'_{ijkm} for an arbitrary rotation of the reference frame about the origin. From the transformation law (4.5), we have

$$C'_{ijkm} = \alpha_{in}\alpha_{jp}\alpha_{kq}\alpha_{mr}C_{npqr} = C_{ijkm} \quad . \qquad (4.10)$$

In other words, the elastic coefficients of an linearly elastic isotropic material must satisfy the thirty-six simultaneous linear homogeneous equations given by the last equality of (4.10). This in turn requires that the symmetric fourth order (cartesian) Hookean tensor C_{ijkm} must be an (invariant or) isotropic tensor.

A strictly tensorial consideration[8] shows that the most general isotropic tensor of fourth order satisfying the symmetry conditions (4.4) is of the form

---

[8] See for example, C.E. Pearson, "Theoretical Elasticity", Harvard Univ. Press, Cambridge, Mass., 1959.

$$C_{ijkm} = \lambda\delta_{ij}\delta_{km} + 2\mu\delta_{ik}\delta_{jm} \tag{4.11}$$

where the arbitrary parameters λ and μ are the so-called Lamé parameters. They may be functions of position in space.

Correspondingly, the stress strain relations (4.3) become

$$\boxed{\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}} \tag{4.12}$$

The strain energy density function S corresponding to (4.12) may be constructed without additional constraints imposed on the elastic coefficients λ and μ. As such, isotropic media form a subclass of hyperelastic media at least in the linear range. It is not difficult to verify that

$$\boxed{S = \tfrac{1}{2}\lambda(e_{kk})^2 + \mu e_{ij}e_{ij} \quad .} \tag{4.13}$$

It follows that the quadratic form

$\boxed{S \text{ is positive definite}^{(9)} \text{ if } \mu \text{ and } \lambda \text{ are positive.}}$

We note that S is a sum of two invariants of e_{ij}, with e_{ij} and e_{ji} $(i \neq j)$ treated as distinct quantities.

From (4.12), we have

$$\sigma_{ii} = 2\mu e_{ii} + \lambda\delta_{ii}e_{kk} = (2\mu + 3\lambda)e_{kk} \tag{4.14}$$

so that

$$2\mu e_{ij} = \sigma_{ij} - \lambda\delta_{ij}\sigma_{kk}/(2\mu + 3\lambda) \ , \quad (i,j = 1,2,3) \ . \tag{4.15}$$

---

(9) A quadratic form $f(y_1,\ldots,y_n)$ is positive definite if and only if (i) $f(y_1,\ldots,y_n) \geq 0$ for all $(y_1,\ldots,y_n)$, (ii) $f(0,\ldots,0) = 0$, and (iii) $f(y_1,\ldots,y_n) = 0$ implies $y_i = 0$ for all $i = 1,\ldots,n$.

Equation (4.15) is the inverted relations corresponding to (4.13) provided that μ and $2\mu + 3\lambda$ are not zero. The relations (4.15) or (4.12) effectively prove the following theorem:

The principal axes for stress and strain coincide for an (linearly) isotropic medium.

4.6 Elastic Moduli

A. Young's Modulus and Poisson's Ratio

For a given material, the values of the two Lamé parameters, μ and λ must be determined experimentally. To do this, we recall from an exercise that when a cylindrical bar (Figure (8)) with a rectangular cross-section is subject to only equal and opposite uniform axial tension at the two ends (whose unit surface normals are $\vec{i}_1$ and $-\vec{i}_1$, respectively), the stress state in the bar is $\sigma_{11} = T_o$ and $\sigma_{ij} = 0$ for $ij \neq 1$. If the bar material is isotropic and homogeneous, the strain components are related to the stress components by

$$2\mu e_{ij} = \sigma_{ij} - \frac{\lambda}{2\mu + 3\lambda}\delta_{ij}\sigma_{kk} \quad , \qquad (i,j = 1,2,3) \quad .$$

For the particular loading, we have

$$2\mu e_{11} = \frac{2(\mu + \lambda)}{2\mu + 3\lambda}\sigma_{11} = \frac{2(\mu + \lambda)}{2\mu + 3\lambda}T_o$$

or

$$\frac{T_o}{e_{11}} = \boxed{\frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \equiv E} \quad . \qquad (4.16)$$

The quantity E defined by (4.16) is called Young's modulus or modulus of elasticity and can be determined by measuring $e_{11} = \frac{\Delta L}{L}$ and T_o .

Figure (8)

Also, we have

$$2\mu e_{22} = -\frac{\lambda}{2\mu + 3\lambda} \quad , \qquad 2\mu e_{33} = -\frac{\lambda}{2\mu + 3\lambda} T_o$$

so that

$$\frac{e_{33}}{T_o} = \frac{e_{22}}{T_o} = -\frac{\lambda}{2\mu(2\mu + 3\lambda)}$$

or

$$-\frac{e_{33}}{e_{11}} = -\frac{e_{22}}{e_{11}} = \boxed{\frac{\lambda}{2(\mu + \lambda)} \equiv \nu} \,. \tag{4.17}$$

The quantity ν defined by (4.17) is called Poisson's ratio. If $\nu > 0$, then ν is a measure of the lateral contraction (in the $\vec{i}_2$ and $\vec{i}_3$ directions) accompanied the axial extension. Evidently ν can also be determined from measurements.

To get a feel for the magnitude of the two important elastic parameters, we note that $E \cong 10^7$ lb/sq.in. and $\nu \cong 0.3$ for most alloys.

B. Lamé Parameters

From the definition of E and ν, we have

$$E = 2\mu + \frac{\lambda\mu}{\mu + \lambda} = 2\mu(1 + \nu)$$

or

$$\boxed{\mu = \frac{E}{2(1 + \nu)}} \quad . \tag{4.18}$$

From the definition of ν, we have $\lambda = 2\nu(\mu + \lambda)$ so that

$$\boxed{\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}} \quad . \tag{4.19}$$

Evidently, λ and μ are also determined once E and ν are, provided that $\nu \neq -1$ and $1/2$. In fact, we see from the expression (4.13) for the stress potential $S(e_{ij})$ of an linearly elastic and isotropic material that μ must be positive (and therefore $\nu > -1$ and $E > 0$) for S to be positive definite. Otherwise, S would be negative for some pure shear strain state.

C. Bulk Modulus

Among other experiments which can be performed to obtain the Lamé parameters or to check the results of (A) and (B), we have the case of a body subject to uniform normal pressure. Recall from an exercise that the stress state of such a body is given by $\sigma_{ij} = -p_o\delta_{ij}$ where p_o is the uniform normal pressure. Since the change of volume per unit volume (of the undeformed body) is

$$e_{jj} = -\frac{3p_o}{2\mu + 3\lambda} \equiv \frac{p_o}{K} \quad ,$$

we can determine the bulk modulus K by measuring the change of volume of the body after deformation. K is related to E and ν by

$$K = \frac{2}{3}\mu + \lambda = \frac{E}{3(1 - 2\nu)} \quad .$$

Note that we have $Ee_{jj} = (1 - 2\nu)\sigma_{kk}$. Therefore, a material with $\nu = \frac{1}{2}$ is called an incompressible elastic medium (no change in volume).

D. The Range of E and ν

It may be verified directly that the expression (4.13) for the stress potential is equivalent to

$$S(e_{ij}) = \frac{1}{2}K\,(e_{kk})^2 + \mu(\hat{e}_{ij}\hat{e}_{ij})$$

where

$$\hat{e}_{ij} \equiv e_{ij} - \frac{1}{3}e_{kk}\delta_{ij} \quad .$$

is the strain deviation tensor (analogous to the stress deviation tensor of section (2.4)). Note that if $e_{ij}=0$ for $i\neq j$ and $e_{11}=e_{22}=e_{33}=\alpha$, then

$$\hat{e}_{ij}\hat{e}_{ij} = e_{ij}e_{ij} - \frac{1}{3}(e_{kk})^2 = 3\alpha^2 - \frac{1}{3}(3\alpha)^2 = 0 \quad .$$

It is therefore necessary and sufficient to have $K > 0$ and $\mu = G > 0$ for S to be positive definite. From the expressions for μ and K in terms of E and ν , we have equivalently the follow theorem

The stress potential of a linearly elastic and isotropic material is positive definite if and only if $E > 0$ and $-1 < \nu < \frac{1}{2}$.

4.7 Complementary Energy (Density) Function

The inverted stress strain relations

$$2\mu e_{ij} = \sigma_{ij} - \frac{\lambda}{2\mu + 3\lambda}\delta_{ij}\sigma_{kk} \tag{4.20}$$

for an isotropic medium can be written in terms of E and ν as

$$e_{11} = \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] \quad , \quad e_{22} = \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] \quad ,$$

$$e_{33} = \ldots \tag{4.21}$$

$$e_{ij} = \frac{1}{2G}\sigma_{ij} \qquad (i \neq j)$$

where

$$\boxed{G = \frac{E}{2(1 + \nu)}} \tag{4.22}$$

is called the shear modulus. Evidently, such a system of stress strain relations can be expressed in terms of a scalar potential C by

$$e_{ij} = \frac{\partial C}{\partial \sigma_{ij}} \tag{4.23}$$

with

$$C = \frac{1}{2}\left[\frac{I_1^2}{E} - \frac{I_2}{G}\right] \tag{4.24}$$

where I_1 and I_2 are the first and second degree invariant of σ_{ij}. The quantity C is called the complementary energy (density) function for the (linearly elastic) medium. We will occasionally refer to it by the more descriptive term of strain potential in stress space. For the two relations (4.23) and (4.24) to be equivalent to (4.21) and (4.22), σ_{ij} and σ_{ji} should be treated as distinct quantities in (4.24). Alternatively, we may replace e_{ij} in (4.23) by engineering strain components γ_{ij} (and write $\sigma_{12}\sigma_{21}$ type terms in I_2 as σ_{12}^2 etc.).

5. Linear Theory of Elasticity

5.1 Navier's Reduction of Equations for the Infinitesimal Strain Theory

Recall that the governing equations for the infinitesimal strain theory of linearly elastic solids are:

Strain-Displacement Relations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.1a}$$

Stress-Strain Relations:

$$\sigma_{ij} = C_{ijk\ell}e_{k\ell} \tag{5.1b}$$

Equilibrium Equations:

$$\sigma_{ij,i} + f_j = 0 \quad (\text{and} \quad \sigma_{ij} = \sigma_{ji}) \tag{5.1c}$$

where all subscripts range from 1 to 3. For an isotropic medium, we have

$$\sigma_{ij} = 2\mu e_{ij} + \lambda\delta_{ij}e_{kk} \quad . \tag{5.2}$$

In terms of E and ν, the corresponding inverted relations are

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \quad . \tag{5.3}$$

The fifteen equations in (5.1) may be reduced to three second order linear patial differential equations for u_1 , u_2 and u_3 . (We assume here that the body force intensities, f_k , are either known functions or dependent only on u_i and their derivatives.) For an isotropic and homogeneous medium, we accomplish this reduction by combining the strain-displacement relations and the stress strain relations to get

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda\delta_{ij}u_{k,k} \tag{5.4}$$

and then by substituting (5.4) into the equilibrium equations to get

$$\boxed{(\mu + \lambda)u_{k,kj} + \mu u_{j,ii} + f_j = 0 \ , \quad (j = 1,2,3)} \tag{5.5}$$

with

$$u_{j,ii} = \nabla^2 u_j \quad .$$

5.2 Boundary Conditions

From our knowledge of partial differential equations, we do not expect the equations of linear elasticity theory to completely determine the stress, strain and displacement field of an elastic body. They must be supplemented by appropriate conditions on the boundary surface(s) of the body. For example, $\vec{\sigma}_\nu$ should be equal to $p_o\vec{\nu}$ on the boundary in the case of a body subject to uniform (normal) surface pressure.

Among many realizable sets of boundary conditions on a bounding surface S, defined by $g_s(x_1 , x_2 , x_3) = 0$, we single out here the following two:

A. Prescribed Displacement Field

$$\vec{u} = \vec{w}(\vec{x}) \qquad \text{on} \quad S \tag{5.6}$$

where $\vec{w}(\vec{x})$ is a given vector function of position on S.

If only the displacement field is prescribed on the boundary, the problem of determining the solution of the equations of elasticity theory (subject to these prescribed displacement conditions) is called the first fundamental problem.

B. Prescribed Surface Traction

$$\vec{\sigma}_\nu = \vec{T}(\vec{x}) \qquad \text{on} \quad S \tag{5.7}$$

where $\vec{T}(\vec{x})$ is a prescribed vector function of position on S.

Note that this is equivalent to prescribing combinations of the derivatives of u_i. Such a boundary value problem is called the <u>second fundamental problem</u> of elasticity theory.

Other kinds of boundary conditions are also admissable. For example, the displacement field is prescribed on a part of the closed surface S while the stress field is prescribed on the rest of the surface.

A different kind of mixed conditions occur when the body is bounded by two or more closed surface (e.g., two concentric spherical surfaces). For example, we may have the displacement field prescribed on one of these surfaces while the stress field prescribed on another.

Still another kind of mixed conditions is to have a combination of stress and displacement for every point on the surface. A typical example of this is when the boundary experiences an elastic support with $\beta u_i + \alpha\sigma_{\nu i} = 0$, $i = 1,2,3$, where α and β are known quantities.

Evidently, the first fundamental problem is analogous to the Dirichlet problem for Laplace's equation (since only the unknowns u_i are prescribed) while the second fundamental problem is akin to the Neumann problem. From these analogies, it is not difficult to imagine that prescribing both the stress and displacement field at points of the same boundary is not permissable in the sense that the problem would become over-determined. More precise statements about admissable boundary conditions will be made later in conjunction with existence and uniqueness theorems for the solution of boundary value problems in elasto-statics of solid bodies.

5.3 Dynamical Problems and Initial Conditions

Within the framework of our linear theory, we can get the equations for the elasto-dynamic problems by way of d'Alembert's principle. More specifically, the equations of motion are obtained by setting

$$f_k = -\rho u_{k,tt} + g_k(\vec{x},t) \quad , \quad (\ )_{,t} \equiv \frac{\partial(\ )}{\partial t}$$

where ρ is the mass density of the undeformed body and $g_k(\vec{x},t)$ are components of some known body force intensity vector. The equations of equilibrium in this case become

$$\sigma_{ij,i} + g_j = \rho u_{j,tt} \tag{5.8}$$

while the other equations remain unchanged.

Evidently, the differential equations for time dependent problems will have to be supplemented by initial conditions which, not surprisingly, usually appear in the form of prescribed displacement and velocity field at some initial time (say $t = 0$):

$$\left\{\begin{array}{l} \vec{u}(\vec{x},0) = \vec{U}(\vec{x}) \\ \\ \vec{u}_{,t}(\vec{x},0) = \vec{V}(\vec{x}) \end{array}\right. \qquad \text{in} \quad V \quad . \tag{5.9}$$

Again, more precise statements on admissable initial conditions must await existence and uniqueness theorems for the solution of initial-boundary value problems in elasto-dynamics of solid bodies.

5.4 Well-Posed Problems

Having formulated a theory which is expected to describe the behaviour of any elastic body under the influence of external (mechanical) excitations (which give rise to only a infinitesimally small strain field), we may ask whether this is a reasonably sound theory before using it to make prediction about elastic bodies (and to set up design criteria). A reasonable degree of confidence would be established if we can show that as a (initial-) boundary value problem in linear partial differential equations, it is a well-posed problem. By that, we mean

1) A solution for the problem exists, since we expect something to happen if we load the body;
2) The solution is unique, since we expect the same response from the body if we apply the same loading under the same circumstances.
3) The solution is stable, since, under normal situations, we expect the response to be nearly the same if we change the loading and boundary (and initial) conditions only slightly.

The task of deciding whether the various classes of boundary value problems in elasto-statics and elasto-dynamics are well-posed problems is an important and difficult area of research in mathematics and theoretical mechanics. The difficulty is especially severe in the nonlinear range, whether it is geometrical nonlinearity (through the strain-displacement relations) or material nonlinearity (through the stress-strain relations). For example, the phenomenon of (elastic) buckling makes non-uniqueness and instability the expected reality under certain circumstances. Neither the objective of, nor the available time for these lectures permit a thorough report of results on well-posed and ill-posed problems in the Theory of Elasticity. To indicate the nature of results for well-posed problems, we will establish a uniqueness theorem for the first and second fundamental problem in section (5.5). We will remark on the question of existence and stability later.

5.5 Two Uniqueness Theorems

For the purpose stated above, we will not aspire for the utmost generality in our discussion of uniquessness theorems. We begin with the following result in linear elasticity:

> If a body is linearly hyperelastic and its stress potential is positive definite, then the solution of the second fundamental problem is unique except for a rigid body displacement.

For a positive definite $S(e_{ij})$, we have (i) $S(e_{ij}) \geq 0$ and (ii) $S \equiv 0$ if and only if $e_{ij} \equiv 0$ $(i,j = 1,2,3)$.

To prove the above result, suppose that there are two different solutions denoted by the superscripts 1 and 2, respectively. Let $\sigma_{ij} \equiv \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, etc. Then, we have in the interior V of the surface of the elastic body S_o,

$$\sigma_{ij,i} = 0 \quad , \quad \sigma_{ij} = \frac{\partial S}{\partial e_{ij}}$$

(where the linearity of stress strain relations is needed for the second) and

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where

$$S = S(e_{11}, \ldots, e_{33}) \quad .$$

Also, we have $\vec{\sigma}_\nu = \vec{0}$ on S_o. Now $\sigma_{ij,i} = 0$ implies

$$0 = \iiint_V \sigma_{ij,i} u_j dV = \iiint_V \left[(\sigma_{ij}u_j)_{,i} - \sigma_{ij}u_{j,i}\right] dV$$

$$= \oiint_{S_o} (\sigma_{ij}\nu_i)u_j dS - \frac{1}{2}\iiint_V \left[\sigma_{ij}(u_{i,j} + u_{j,i})\right] dV \ .$$

With $\vec{\sigma}_\nu = \vec{0}$, we get

$$0 = -\iiint_V e_{ij} \frac{\partial S}{\partial e_{ij}} \, dV = -2 \iiint_V S \, dV$$

where we have made use of the fact that S is homogeneous quadratic. Since S is non-negative, the vanishing of the volume integral implies $S \equiv 0$. Since S is positive definite, we must have $e_{ij} \equiv 0$ and therefore $\sigma_{ij} \equiv 0$ in V . Thus the two different solutions can only differ by a rigid body displacement field. (Note that if V is bounded by more than one closed surfaces, interpret the surface integral over S_o as a sum of surface integrals each over one of the bounding surfaces.)

The above uniqueness theorem gives the following corollary as an immediate consequence:

Under the same hypothesis, the solution of the first fundamental problem is unique.

By the theorem, we have $\vec{u} = \vec{u}_o + \vec{\omega}_o \times \vec{r}$ where $\vec{u}_o$ and $\vec{\omega}_o$ are constant vectors. But $\vec{u} = \vec{0}$ on S_o implies $\vec{u}_o = \vec{\omega}_o = \vec{0}$ and therewith $\vec{u} = \vec{0}$ throughout the body.

Before leaving the subject of uniqueness, we mention once more that the stress potential (i.e., strain energy function) of an initially unstressed linear isotropic medium is a homogeneous quadratic functional of e_{ij} and is positive definite if $-1 < \nu < \frac{1}{2}$ and $E > 0$.

6. Energy and Work

6.1 Stress Potential and Internal Energy

For our infinitesimal strain theory of elasto-dynamics, the incremental change of work done by the external loads acting on the body ($\vec{g}(\vec{r},t)$ in V and $\vec{T}(\vec{r},t)$ on S) is given by

$$\Delta W = [\iiint_V \vec{g}\cdot\dot{\vec{u}}\,dV + \iint \vec{T}\cdot\dot{\vec{u}}\,dS]\Delta t \ , \ (\ )^{\bullet} \equiv \frac{\partial(\ )}{\partial t} \equiv (\ )_{,t} .$$

In the limit as $t \to 0$, we have as the time rate of work done by the external loads,

$$\frac{dW}{dt} = \iiint_V \vec{g}\cdot\dot{\vec{u}}\,dV + \iint_S \vec{T}\cdot\dot{\vec{u}}\,dS \tag{6.1}$$

If it should happen that displacement components are prescribed on a part S, we take $\vec{T}$ to be the <u>unknown</u> external traction required to produce the prescribed displacement field on that portion of the surface. Since elastic deformation is a reversible process by definition and since heat exchange does not take place between the body and its surrounding in our theory, the laws of thermodynamics require that W be equal to the internal energy of the body. We will show that W is a sum of the kinetic energy of the body and a part which is just the volume integral of the stress potential $S(e_{ij})$ for a hyperelastic material.

By the equilibrium equation $\vec{\sigma}_{i,i} + \vec{g} = \rho\ddot{\vec{u}}$, we have

$$\iiint_V \vec{g}\,\dot{\vec{u}}\,dV = \iiint_V (\rho\ddot{\vec{u}}\cdot\dot{\vec{u}} - \vec{\sigma}_{i,i}\cdot\dot{\vec{u}})dV$$

$$= \iiint_V [\tfrac{1}{2}\rho|\dot{\vec{u}}|^2]^{\bullet} dV - \iiint_V (\vec{\sigma}_i\cdot\dot{\vec{u}})_{,i} dV + \iiint_V \sigma_{ij}\dot{u}_{j,i} dV \ .$$

where $|\dot{\vec{u}}|^2 = \dot{\vec{u}} \cdot \dot{\vec{u}}$. It follows from this and (6.1)

$$\frac{dW}{dt} = \frac{dK}{dt} + \iiint_V \frac{\partial S}{\partial e_{ij}} \dot{e}_{ij} \, dV = \frac{dK}{dt} + \frac{dU}{dt} \tag{6.2}$$

where

$$U = \iiint_V S \, dV \tag{6.3}$$

The quantity U is evidently a new type of energy associated with the elastic-straining of the body. Hence U is called the strain energy of the body and S the strain energy density function (or more precisely the volume density function of the strain energy) in the literature.

For a body initially at rest in the unstrained state, we have then

$$W = K + U \tag{6.4}$$

which summarizes the following fact:

> For a hyper-elastic body initially at rest in its natural state and undergoing (reversible) infinitesimal-strain elastic deformations without heat exchange with its surrounding, the work done by the external loads up to time t is equal to the sum of kinetic energy and strain energy at time t.

Since the natural state of a solid body at rest is taking to be in a "stable" equilibrium state (see section (4.1)), it follows from Gibbs' theorem in thermodynamics that its strain energy must be a minimum, say zero (at constant entropy). The internal energy of the deformed body must therefore be positive. For a deformed body in static equilibrium, we have $K = 0$ and therefore $U \geq 0$. (The equality sign holds only if the deformed state is the natural state itself.) It follows that S must be positive definite. For a linearly elastic and isotropic material, this requires $E > 0$ and $-1 < \nu < 1/2$ as we saw previously in section (4.6). Unfortunately, the validity of these statements cannot be verified as they depend on Gibbs' theorem as well as the first and second laws of thermodynamics; the latter were used in arriving at the conclusion that $U + K$ is the internal energy of the body. These topics in thermodynamics are beyond the scope of these lectures.

In the case of a deformed body in static equilibrium, we have $K = 0$ and

$$W = U = \iiint_V S(e_{ij})dV$$

For a linearly elastic material, $S(e_{ij})$ is homogeneous quadratic in e_{ij} with

$$\frac{\partial S}{\partial e_{ij}} e_{ij} = 2S'(e_{ij}).$$

Therefore, we may write

$$W = U = \frac{1}{2} \iiint_V \frac{\partial S}{\partial e_{ij}} e_{ij}\, dV = \frac{1}{2} \iiint_V \sigma_{ij} e_{ij}\, dV$$

$$= \frac{1}{2}\left[\iiint_V f(\vec{r}) \cdot \vec{u}(r)\, dV + \iint_S \vec{T} \cdot \vec{u}\, dS\right] \tag{6.5}$$

Thus, we have the following observation:

> In the infinitesimal-strain theory of Hookean elastostatics, the (unspecified) rate of load application and rate of elastic deformation toward the final equilibrium state are taken to be uniform.

The observation is an alternative statement of the original theorem by B.P.E. Clapeyron (1799-1864) on the following portion of (6.5)

$$U = \frac{1}{2}\left[\iiint_V \vec{f}\cdot\vec{u}\,dV + \iint_S \vec{T}\cdot\vec{u}\,dS\right]$$

Clapeyron's theorem states that the strain energy U of the deformed body is equal to one half the work that would be done by the external loads (of the equilibrium state) acting through the displacements U_i from the unstressed state to the state of equilibrium.

The Betti-Rayleigh Reciprocal Theorem

One of the important and useful results involving work and energy in elasticity theory relates the equilibrium states of the same body under different applied loads. Suppose two different sets of body force intensities $\vec{f}^{(1)}$ and $\vec{f}^{(2)}$ and surface tractions $\vec{T}^{(1)}$ and $\vec{T}^{(2)}$ are applied to the same (hyper-) elastic body (on different occasions, of course) resulting in two different static equilibrium states $\{\vec{\sigma}_i^{(1)}, e_{ij}^{(1)}, \vec{u}^{(1)}\}$ and $\{\vec{\sigma}_i^{(2)}, e_{ij}^{(2)}, \vec{u}^{(2)}\}$, respectively. Form the physically meaningless combinations

$$W_{jk} = \frac{1}{2}\iiint_V \vec{f}^{(j)}\cdot\vec{u}^{(k)}dV + \frac{1}{2}\iint_S \vec{T}^{(j)}\cdot\vec{u}^{(k)}dS \qquad (j \neq k,\ j.k = 1,2) \qquad (6.5)$$

where $\vec{T}^{(j)}$ is the applied surface traction over the entire boundary surface (on a portion of which $\vec{T}^{(k)}$ may be determined indirectly by the prescribed displacement). In words, W_{jk} is the hypothetical "work" done by load system j on the response to the load system k. The Reciprocal Theorem of Betti and Rayleigh concludes that we must have for a linearly hyper-elastic body

$$\boxed{W_{12} = W_{21}} \qquad (6.6)$$

Furthermore, if we let

$$\hat{U}_{jk} \equiv \frac{1}{2}\iiint_V \sigma_{mn}^{(j)}\ e_{mn}^{(k)}\ dV\ , \quad (j \neq k,\ j,k = 1,2) \qquad (6.7)$$

The same reciprocal theorem may also be stated as

$$\boxed{W_{jk} = \hat{U}_{jk}} \qquad \text{or} \qquad \boxed{\hat{U}_{12} = \hat{U}_{21}} \qquad (6.8,9)$$

The conclusion $W_{jk} = \hat{U}_{jk}$ in (6.8) is established by using the differential equation of equilibrium to eliminate $\vec{f}^{(j)}$ in (6.5) and then applying the divergence theorem in the usual way to eliminate all surface integrals. The remaining volume integral may be written as $\hat{U}_{jk}$.

The conclusion $\hat{U}_{12} = \hat{U}_{21}$ in (6.9) is obtained by observing the symmetry condition $C_{ijk\ell} = C_{k\ell ij}$ for a linearly hyper-elastic body so that

$$\hat{U}_{12} = \frac{1}{2}\iiint_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \frac{1}{2}\iiint_V C_{ijk\ell} e_{k\ell}^{(1)} e_{ij}^{(2)} dV = \frac{1}{2}\iiint_V C_{k\ell ij} e_{ij}^{(2)} e_{k\ell}^{(1)} dV$$

$$= \frac{1}{2}\iiint_V \sigma_{k\ell}^{(2)} e_{k\ell}^{(1)} \, dV = \hat{U}_{21}$$

Evidently, $W_{12} = W_{21}$ follows from (6.8) and (6.9). Note that the requirement of linear elasticity is needed only in the derivation of $\hat{U}_{12} = \hat{U}_{21}$ and therefore $W_{12} = W_{21}$. It is not needed for $W_{12} = \hat{U}_{12}$ and $W_{21} = \hat{U}_{21}$.

The above reciprocal theorem has many important applications. We will only discuss one of these later in connection with boundary conditions for plate and shell theories.

6.3 Virtual Work and Complementary Virtual Work

There is a well known and much used result in elasto-statics related to work and energy called the principle of virtual work. We will deduce this principle here as a mathematical consequence of the reciprocal theorem. We will not dwell into the physical interpretation of the principle in terms of work and energy as such an interpretation tends to be confusing, given that the principle is really not concerned with the actual work done by a particular set of external loads or the actual strain energy induced by the corresponding elastic deformation.

Consider two external load systems denoted by $\{\vec{f},\vec{T}\}$ and $\{\vec{f} + \delta\vec{f},\ \vec{T} + \delta\vec{T}\}$, respectively (corresponding to $\{\vec{f}^{(1)},\vec{T}^{(1)}\}$ and $\{\vec{f}^{(2)},\vec{T}^{(2)}\}$, respectively.) The two load systems have the same portion of S, denoted by S_d, on which the same displacement field is prescribed. The surface traction on the remaining portion of the boundary, denoted by S_σ, may not be identical for the two systems. Two static equilibrium states $\{\sigma_{ij},\ e_{ij},\ u_i\}$ and $\{\sigma_{ij} + \delta\sigma_{ij},\ e_{ij} + \delta e_{ij},\ u_i + \delta u_i\}$ develop in the body in responses to the two load systems. The reciprocal theorem in the form of $W_{12} = \hat{U}_{12}$ (see equ.(6.8)), takes the form

$$\iiint_V \vec{f}\cdot\delta\vec{u}\ dV + \iiint_{S_\sigma} \vec{T}\cdot\delta\vec{u}\ dS = \iiint_V \sigma_{ij}\delta e_{ij}\ dV \tag{6.10}$$

after some cancellations and observing $\delta\vec{u} = \vec{0}$ on S_d. The relation (6.10) is known as the principle of virtual work with $\{\delta u_j\}$ and $\{\delta e_{ij}\}$ designated as virtual displacement components and virtual strain components. We note only that the reciprocal theorem is used here in a form which does not require the stress strain relation to be linear;

in fact, it does not involve any stress-strain relations at all.

A less well known but nevertheless useful counterpart of (6.10) is the <u>principle of complementary virtual work</u>. It is obtained from the other part of the reciprocal theorem, $W_{21} = \hat{U}_{21}$. With $\delta e_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, we get from $W_{21} = \hat{U}_{21}$

$$\iiint_V \vec{u}\cdot\delta\vec{f}\, dV + \iint_S \vec{u}\cdot\delta\vec{T}\, dS = \iiint_V e_{ij}\delta\sigma_{ij}\, dV \tag{6.11}$$

after some cancellations. Again, the form of the reciprocal theorem used to get (6.11) does <u>not</u> require the body to be linearly elastic. Nowhere are the constitutive relations involved in the derivation of (6.11) (and (6.10) for that matter.)

The right side of (6.10) has been interpreted as the (fictitious) "work" done by the actual loads on the (fictitious or virtual) displacement increments δu_j. Hence, the result (6.10) is called the principle of virtual work. The result (6.11) is evidently complementary to (6.10); hence it may be called the principle of complementary virtual work. However it is also known as the <u>principle of virtual stress</u>.

6.4 The Principle of Minimum Potential Energy

For a hyperelastic medium, we have $\sigma_{ij} = \partial S/\sigma e_{ij}$ for some stress potential $S(e_{ij})$. If we think of $\delta\vec{u}$ as a small variation of $\vec{u}$ and δe_{ij} as small variations of e_{ij}, then we have in the terminology of calculus of variations

$$\frac{\partial}{\partial e_{ij}}[S(e_{ij})]\delta e_{ij} = \delta S(e_{ij})$$

as the (first) variation of $S(e_{ij})$ corresponding to the variations δe_{ij}. Furthermore, we have also

$$\iiint_V \sigma_{ij}\delta e_{ij}\ dV = \iiint_V \frac{\partial S}{\partial e_{ij}}\delta e_{ij}\ dV = \iiint_V \delta S\ dV = \delta\iiint S(e_{ij})dV$$

with the right hand side being the first variation of the strain energy corresponding to δe_{ij}.

The fixed body force intensities f_i and surface tractions T_i, may be expressed in terms of a body load potential $G_b(\vec{u})$ and surface load potential $G_s(\vec{u})$, respectively, by

$$f_i = -\frac{\partial G_b}{\partial u_i}\quad,\quad T_i = -\frac{\partial G_s}{\partial u_i} \tag{6.12}$$

with

$$G_b = -\vec{u}\cdot\vec{f} \qquad G_s = -\vec{T}\cdot\vec{u} \tag{6.13}$$

In general, external loads which can be expressed in terms of load potentials are said to be <u>conservative</u>. For conservative loading on a hyperelastic medium, the principle of virtual work may be written in variational notations as

$$\delta P = 0$$

where

$$P \equiv \iiint_V \{S(e_{ij}) + G_b(\vec{u})\}dV + \iint_S G_s(\vec{u})dS \tag{6.14}$$

keeping in mind $\delta\vec{u} = \vec{0}$ on S_u so that

$$\iint_{S_\sigma} \vec{T}\cdot\delta\vec{u}\ dS = \iint_S \vec{T}\ \delta\vec{u}\ dS \quad .$$

The quantity P, defined in (6.14), is called the <u>potential energy</u> of the elastic body. Thus under the assumption of hyperelasticity and conservative loading, the principle of virtual work asserts that, among the admissable comparison sets of displacement functions, the set that satisfies equilibrium and the displacement boundary conditions on S_d renders the potential energy $P(\vec{u})$ a stationary value.

For a linearly hyperelastic medium with a positive definite quadratic stress potential $S(e_{ij})$, the stationery value of P is in fact a minimum. To see this we compare the potential energy of the equilibrium displacement field $\vec{u}$ and another admissable displacement field $\vec{u} + \delta\vec{u}$ with $\delta\vec{u} = \vec{0}$ on S_d:

$$P(\vec{u} + \delta\vec{u}) - P(\vec{u}) = \iiint_V [S(e_{ij} + \delta e_{ij}) - S(e_{ij})]dV + \iiint_V \frac{\partial G_b}{\partial u_j}\ \delta u_j dV$$

$$+ \iint_S \frac{\partial G_s}{\partial u}\ \delta u_j dS$$

$$= \iiint_V [\frac{\partial S}{\partial e_{ij}}\delta e_{ij} + \frac{\partial G_b}{\partial u_j}\delta u_j]dV + \iint_S \frac{\partial G_s}{\partial u_j}\ \delta u_j\ dS$$

$$+ \iiint_V \frac{\partial^2 S}{\partial e_{ij}\partial e_{k\ell}}\ \delta e_{ij}\delta e_{k\ell}\ dV$$

where $\partial^2 S/\partial e_{ij}\partial e_{k\ell}$ is evaluated at $e_{ij} + \theta\delta e_{ij}$, $0 < |\theta| < 1$. Since $\vec{u}$ renders P stationary, we have

$$P(\vec{u} + \vec{u}) - P(\vec{u}) = \iiint_V \frac{\partial^2 S}{e_{ij}\partial e_{k\ell}} \delta e_{ij}\delta e_{k\ell} dV .$$

For a linearly elastic medium, the integrand on the right side is $2S(\delta e_{ij})$ so that

$$P(\vec{u} + \delta\vec{u}) \geq P(\vec{u})$$

with equality holding only if $\delta e_{ij} = 0$ for $i,j = 1,2,3$. If the displacement field is prescribed on any portion of the boundary (say, more than two isolated points), then the equilibrium displacement field is completely determined and we have the following theorem of minimum potential energy:

> Of all displacements satisfying the given displacement boundary conditions on S_d, the actual displacement field of the deformed body (which satisfies the three scalar force equilibrium equations) makes the potential energy $P(\vec{u})$ a local minimum in the neighborhood of the stable natural (unstrained) state.

Since the solution of the relevant boundary value problem of elastostatics is unique up to a rigid body motion, the local minimum is also the global minimum. On the other hand, the linear theory of elastostatics is usually not adequate except in a small neighborhood of the unstrained state. Therefore, the conclusion of a global minimum (and therefore global stability) on the basis of our linear theory is usually not meaningful.

Conversely to the above theorem of minimum potential energy, we can show that the displacement field, which render $P(u)$ a local minimum, necessarily satisfies the differential equations of equilibrium*. It is this converse that is used in practice especially for approximate solutions of the BVP of elasto-statics by the direct or semi-direct method of calculus of variations.

Stress Functions, Compatibility Equations and the Principle of Minimum Complementary Energy

If the stress-strain relations of the elastic body can be expressed in terms of a strain potential $C(\sigma_{ij})$ by

$$e_{ij} = \frac{\partial C}{\partial \sigma_{ij}} \quad , \quad (i,j = 1,2,3),$$

then we have

$$e_{ij}\delta\sigma_{ij} = \frac{\partial C}{\partial \sigma_{ij}}\ \delta\sigma_{ij} = \delta C$$

so that the right side of the principle of complementary virtual work (6.11) is just the first variation of the complementary strain energy of the deformed elastic body. We now compare stress states in an elastic body with the same prescribed body force intensities (so that $\delta\vec{f} = \vec{0}$), the same prescribed surface fraction on S_σ (so that $\delta\vec{T} = \vec{0}$ there) and the same prescribed displacement fields $\vec{u}^{(o)}$ on S_d. In that case, the principle of complementary virtual work (6.11) is simplified to

$$\iint\limits_{S_d} \vec{u}^{(o)}\cdot\delta\vec{T}\, dS = \delta \iint\limits_{S_d} \vec{u}^{(o)}\cdot\vec{T}\, dS = \delta \iiint\limits_{V} C(\sigma_{ij})dV$$

or

$$\delta P^* = 0$$

with

$$P^* \equiv \iiint\limits_{V} C(\sigma_{ij})dV - \iint\limits_{S_d} \vec{u}^{(o)}\cdot\vec{T}\, dS \qquad (6.15)$$

In terms of the complementary energy P^* of the deformed elastic body in equilibrium, the principle of complementary virtual work under the restrictions specified above may be stated as follows:

> Of all stress states that satisfy the equations of force equilibrium in V and the conditions of prescribed traction on S_σ, the actual stress state of the deformed body makes the complementary energy $P^*(\sigma_{ij})$ a stationary value in the neighborhood of the stable natural (unstrained) state.

Similar to the case of potential energy, we can show also that for a linearly hyper-elastic medium,

> This stationary value of P^* is actually a local minimum if the quadratic functional $C(\sigma_{ij})$ is positive definite.

The proof of this theorem of minimum complementary energy is similar to that for the theorem of minimum potential energy and will not be given here.

In the case of the principle of minimum potential energy, the actual displacement field of the deformed body (that makes $P(\vec{u})$ a minimum) is singled out by the fact it satisfies the differential equations of force equilibrium. For the case of complementary energy principle, all comparison stress states are required to satisfy force equilibrium. The actual stress state (that makes $P^*(\sigma_{ij})$ a minimum) is singled out by the fact that it satisfies the so-called differential equations of compatibility (to be described below). This converse of the theorem of minimum complementary energy is what is actually used in practice, especially for approximate solutions of the BVP of elasto-statics by the direct or semi-direct method of calculus of variations.

The six strain components e_{ij} are defined in terms of three displacement components by the six strain-displacement relations; they are therefore not independent quantities. A set of e_{ij} induced by an equilibrium state of stress (by the stress-strain relations) is generally not compatible in the sense that (through the strain-displacement relations) they may over-determine the three displacement functions. To put it in another way, there is generally no displacement field $\vec{u}$ which gives rise to these six induced strain components. The consistency conditions, which guarantee the integrability of the six strain-displacement relations for given strain components and thereby determine $\vec{u}$ uniquely up to a rigid motion (at most), are called <u>compatibility equations</u>. It can be verified by direct substitution that the expressions for strain components satisfy the following six Saint Venant compatibility equations:

$$
\begin{aligned}
&e_{22,33} + e_{33,22} - 2e_{23,23} = 0\ , \quad e_{11,23} + e_{23,11} - e_{13,12} - e_{12,13} = 0 \\
&e_{11,22} + e_{22,11} - 2e_{12,12} = 0\ , \quad e_{22,13} + e_{13,22} - e_{23,21} - e_{21,23} = 0 \qquad (6.16) \\
&e_{11,33} + e_{33,11} - 2e_{13,13} = 0\ , \quad e_{33,12} + e_{12,33} - e_{31,32} - e_{32,31} = 0
\end{aligned}
$$

It can also be shown that the strain-displacement relations can be solved as six partial differential equations to determine the three displacement components (up to a rigid motion at most) if the six strain components satisfy these six compatibility conditions. (See Solkonikoff for example.)

In the theorem of minimum potential energy, the strain components are expressed in terms of displacement components and the compatibility equations are automatically satisfied. In the theorem of minimum complementary energy, the stress components are required to satisfy equilibrium. It would be of interest to have some way to single out those stress states which satisfy equilibrium automatically. This is done by way of the so-called Maxwell-Morera stress function representation. It suffices to consider the case of no body force intensities so that $\vec{f} \equiv \vec{0}$. In that case, the (homogeneous) equilibrium equations are satisfied identically by setting

$$\begin{aligned}
\sigma_{11} &= \phi_{22,33} + \phi_{33,22} - 2\phi_{23,23}\,, & \sigma_{23} &= \phi_{13,12} + \phi_{12,13} - \phi_{11,23} - \phi_{23,11} \\
\sigma_{22} &= \phi_{11,33} + \phi_{33,11} - 2\phi_{13,13}\,, & \sigma_{13} &= \phi_{23,21} + \phi_{21,23} - \phi_{22,13} - \phi_{13,22} \qquad (6.17) \\
\sigma_{33} &= \phi_{11,22} + \phi_{22,11} - 2\phi_{12,12}\,, & \sigma_{12} &= \phi_{31,32} + \phi_{32,31} - \phi_{33,12} - \phi_{12,33}
\end{aligned}$$

where $\phi_{ij} = \phi_{ji}$ are six independent arbitrary four times differentiable functions of positions (x_1, x_2, x_3). The verification is by direct substitution of these expressions into the equilibrium equations.

With the help of the (6.17), the following converse of the theorem of minimum complementary energy can now be verified. The equilibrium stress field which renders P^* a local minimum necessarily satisfies the compatibility equations. In practice, it is this converse which is useful for approximate solutions of BVP's in elasto-statics. For these BVP's, the use of stress function representation dispenses with any further consideration of equilibrium. The exact solution of a BVP would then consist of finding a set of six stress functions $\phi_{ij}(\vec{x})$ which satisfy the six compatibility equations. In this way, the stress functions are uniquely determined up to a stress-free part, the analog of a rigid body displacement field. An approximate solution of the same problem would consist of seeking a local minimum (or simply the stationary value) relative to a restricted set of (admissable) stress functions which satisfies the stress boundary conditions.

6.6 Reissner's Variational Principle

For a linearly elastic isotropic medium, it is not difficult to verify that

$$S(e_{ij}) = C(\sigma_{ij})$$

either by substituting the stress-strain relations (4.12) for σ_{ij} in C or the inverted relations (4.20) for e_{ij} in S. The same type of calculations also shows $S(e_{ij}) + C(\sigma_{ij}) = \sigma_{ij}e_{ij}$. Written in the form

$$C(\sigma_{ij}) = \sigma_{ij}e_{ij} - S(e_{ij}) \quad , \tag{6.18}$$

the relation is known as the Legendre transformation as it is analogous to the same transformation relating the Lagrangian to the Hamiltonian in dynamical systems. The transformation is a mathematical consequence of $S(e_{ij}) = C(\sigma_{ij})$ in the case of a linearly hyperelastic material. For a nonlinearly elastic material, (6.18) may be taken as the definition of $C(\sigma_{ij})$ in terms of $S(e_{ij})$. With $S(e_{ij})$ being the strain energy density induced by the work done by the external load as it increases from zero to its final magnitude, and with $\sigma_{ij}e_{ij}$ being the strain energy induced by the work done by the external load acting at its full magnitude from the beginning, we have the following two schematic diagrams which distinguish linearly and nonlinearly elastic material properties

Figure (9a)	Figure (9b)
Linearly Elastic Material	Nonlinearly Elastic Material

If we use (6.18) to express S in terms of C in the expression for the potential energy P and append to the resulting expression the constraint $\vec{u} = \vec{u}^{(o)}$ on S_d with the help of a vector Lagrange multiplier, $\vec{\sigma}'_\nu$ (which we expect to have the dimension of stress), we get

$$I = \iiint_V \left[\sigma_{ij}e_{ij} - C(\sigma_{ij}) + G_b(\vec{u})\right] dV + \iint_{S_\sigma} G_s(\vec{u})dS + \iint_{S_d} \vec{\sigma}'_\nu \cdot (\vec{u}^{(o)} - \vec{u})dS \quad . \tag{6.19}$$

With $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and with $\delta\sigma_{ij}$ and δu_j , $i,j = 1,2,3$, varying independently, it is not difficult to verify that

(1) The Euler differential equations for a stationary value of I are just the three differential equations of equilibrium for σ_{ij} and the six stress-strain relation for e_{ij} in terms of σ_{ij} .

(2) The Euler boundary conditions are the stress boundary conditions on S_σ and the auxiliary conditions $\vec{\sigma}'_\nu = \vec{\sigma}_\nu$

(assigning a physical meaning to the vector Lagrange multiplier) as well as the (constraining) displacement boundary conditions on S_d .

Conversely, actual stress and displacement fields of the deformed body (which satisfy equilibrium equations, stress-strain relation, prescribed stress boundary conditions on S_σ , and prescribed displacement boundary conditions on S_d) render the functional I (with $\vec{\sigma}'_\nu$ replaced by $\vec{\sigma}_\nu$) a stationary value. As a vehicle for approximate solutions of BVP in elasto-statics, by direct or semi-direct methods of calculus of variations, this variational principle (by E. Reissner, 1950) does not require the assumed approximate expressions for the stress and displacement fields to satisfy any of the field equations or boundary conditions as long as we understand that e_{ij} is an abbreviation for $\frac{1}{2}(u_{i,j} + u_{j,i})$. As such, it does not favor one set of field variables over another (stress components over displacement components or vice versa) as in the case of the principle of minimum potential energy and the principle of minimum complementary energy.

The functional I given by (6.19) is related to the potential energy of the deformed elastic body as it was derived from P with the help of the Legendre transformation. It is also seen to be related to the complementary energy P^* if we use divergence theorem to transform the volume integral of $\sigma_{ij}e_{ij}$:

$$\iiint_V \sigma_{ij} e_{ij} dV = \iiint_V \frac{1}{2} \sigma_{ij} (u_{i,j} + u_{j,i}) dV = \ldots.$$

$$= \iiint_V \left[(\sigma_{ij} u_j)_{,i} - \sigma_{ij,i} u_j \right] dV$$

$$= \iint_S \sigma_{\nu j} u_j ds - \iiint_V \sigma_{ij,i} u_j dV$$

so that I may be written as

$$I = -\iiint_V \left[\sigma_{ij,i} u_j + C(\sigma_{ij}) - G_b(u_j) \right] dV$$
$$+ \iint_{S_\sigma} \left[\vec{\sigma}_\nu \cdot \vec{u} + G_s(\vec{u}) \right] dS + \iint_{S_d} \vec{\sigma}_\nu \cdot \vec{u}^{(o)} dS \quad . \tag{6.20}$$

Recall that for the case of prescribed force intensity $\vec{f}$ and surface traction $\vec{T}$ which do not involve the unknowns, we have

$$G_b = -\vec{f} \cdot \vec{u} \qquad\qquad G_s = -\vec{T} \cdot \vec{u} \quad .$$

If the admissable comparison states are required to satisfy force equilibrium and to take on prescribed values of S_σ, then we have

$$\sigma_{ij,i} u_j - [-f_j u_j] = (\sigma_{ij,i} + f_j) u_j = 0 \qquad \text{in } V$$

and

$$\vec{\sigma}_\nu \cdot \vec{u} - \vec{T} \cdot \vec{u} = (\vec{\sigma}_\nu - \vec{T}) \cdot \vec{u} = 0 \qquad \text{on } S_\sigma$$

so that I becomes

$$I_s \equiv -\left\{ \iiint_V C(\sigma_{ij})dV - \iint_{S_d} \vec{\sigma}_\nu \cdot \vec{u}^{(o)} dS \right\} = -P^* \quad . \tag{6.21}$$

We denote the actual stress and displacement states of the deformed body by $\hat{\sigma}_{ij}$ and $\hat{u}_j$, $i,j = 1,2,3$. For these actual states, we must have

$$\hat{I} \equiv I(\hat{\sigma}_{ij}, \hat{u}_j) = P^*(\hat{\sigma}_{ij}) \equiv \hat{P}^* \quad . \tag{6.22}$$

With $P^*(\sigma_{ij}) \geq P^*(\hat{\sigma}_{ij})$, it follows that

$$I_s(\sigma_{ij}) \leq \hat{I} \quad . \tag{6.23}$$

Similarly, if we take the six stress-strain relations $e_{ij} = \partial C/\partial\sigma_{ij}$ as known and solve them for σ_{ij} in terms of u_i , we transform I back to

$$I_d \equiv \iiint_V \{S(e_{ij}) + G_b(\vec{u})\}dV + \iint_{S_\sigma} G_s(\vec{u})d\vec{S} = P(\vec{u}) \quad .$$

With

$$\hat{I} = P(\hat{u}) \equiv \hat{P} \quad ,$$

we have also

$$\hat{I} \leq P(\vec{u}) \quad . \tag{6.24}$$

Combining (6.23) and (6.24), we get

$$P^*(\sigma_{ij}) \le \hat{I} \le P(u_j) \quad . \tag{6.25}$$

In other words, the two minimum principles provide an upper and lower bound for $\hat{I}$. They are useful for estimating the accuracy of an approximate solution of the BVP.

6.7 Other Variational Principles

Reissner's variational principle is more flexible than the two classical minimum principles because it allows the independent variation of both stresses and displacements, while the minimum energy principles allow the independent variation of only one of these two sets of field quantities. It seems natural to ask whether we could allow the strain components to vary independently as well. It is not surprising that the answer to this question is affirmative as we can always use Lagrange multipliers to bring the strain-displacement relations into the functional to be made stationary.

We start with the expression for the potential energy and attempt to minimize it with the strain-displacement relations as six equality constraints, in addition to the previously discussed contraints from the displacement boundary conditions on S_d. We know from calculus of variations that these constraints may be incorporated into a new functional J by way of six Lagrange multipliers in V and three others on S_d. These multipliers would eventually be identified as the stress components. If we take them to be $\vec{\sigma}_j$ (with $\sigma_{ij} = \sigma_{ji}$) and $\vec{\sigma}_\nu$ from the start, we get for J the expression:

$$J = \iiint_V \{S + G_b(\vec{u}) - \sigma_{ij}[e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})]\}dV$$

$$+ \iint_{S_\sigma} G_s(\vec{u})dS + \iint_{S_d} \vec{\sigma}_\nu \cdot (\vec{u}^{(o)} - \vec{u})dS \quad .$$

It is a straightforward calculation to verify that the actual stress, strain and displacement fields of the deformed body render J a stationary value. In other words, the Euler differential equations of J are the equilibrium equations, stress-strain relations and the strain-displacement relations of linear elasticity theory, and the Euler boundary conditions are the stress boundary conditions on S_σ and the displacement boundary conditions on S_d .

Before leaving the subject of variational formulation of linear elasticity, we should mention that there are still other variational principles beyond those described in the last few sections. For example, variational principles exist for the stress function-compatibility approach to linear elasticity. For such variational principles, stress functions, stress components and strain components (all or some of them) are allowed to vary independently; the corresponding Euler differential equations may include the stress function representation, stress-strain relations and compatibility equations. A discussion of these variational principles and the physical meaning of the corresponding Euler boundary conditions may be found in "A note on Gunther's analysis of couple stress," by E. Reissner and F.Y.M. Wan, Mechanics of Generalized Continua (IUTAM Symp., Stuttgart, Germany, 1967), Springer-Verlag, 83-86 (1968).

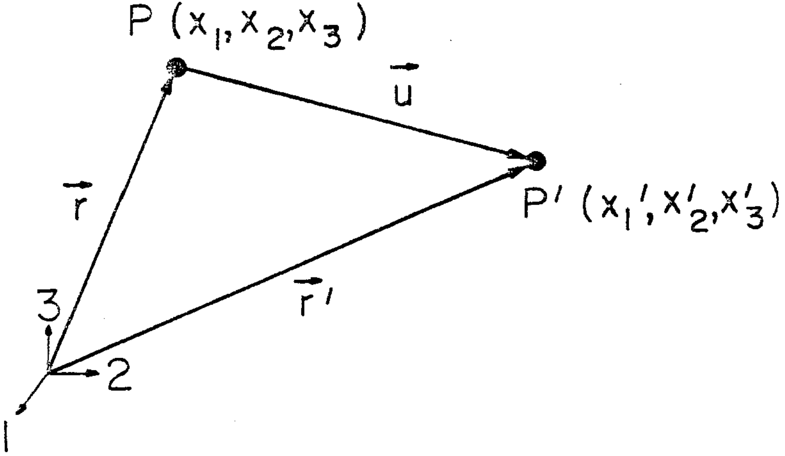

Figure 1

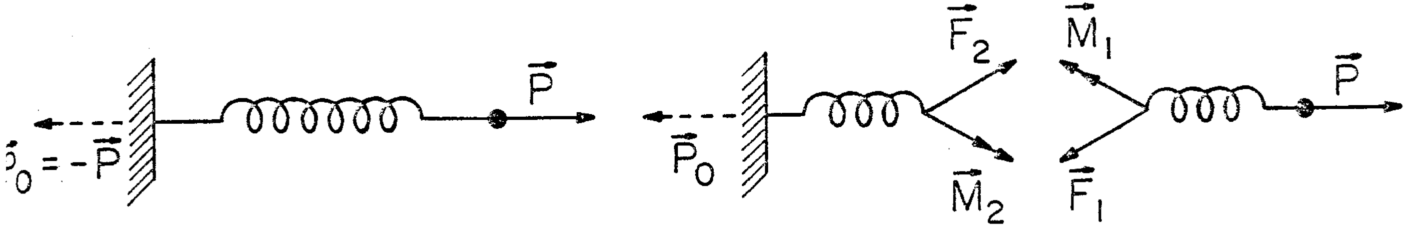

Figure 2

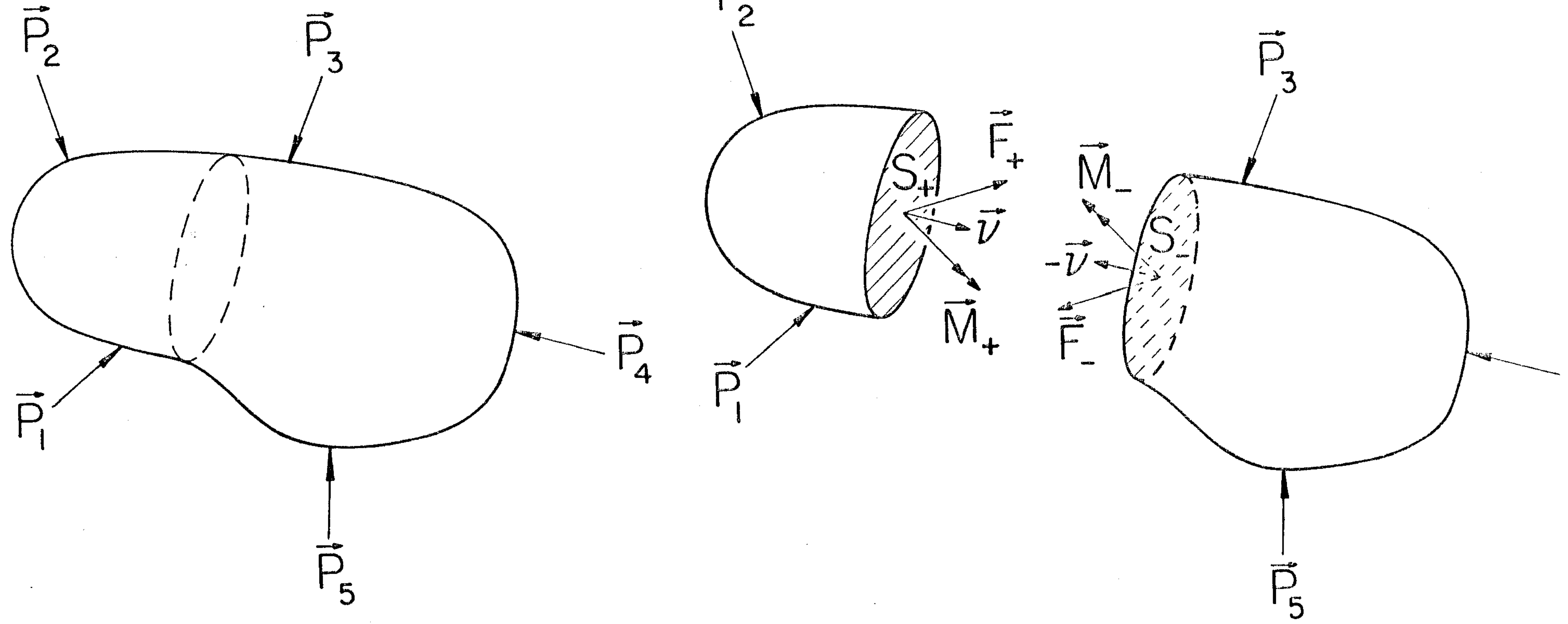
$\vec{P}_2$
$\vec{P}_3$
$\vec{P}_1$
$\vec{P}_4$
$\vec{P}_5$
$\vec{P}_1$
S_+
$\vec{F}_+$
$\vec{\nu}$
$\vec{M}_+$
$\vec{M}_-$
S_-
$-\vec{\nu}$
$\vec{F}_-$
$\vec{P}_3$
$\vec{P}_5$

Figure 3

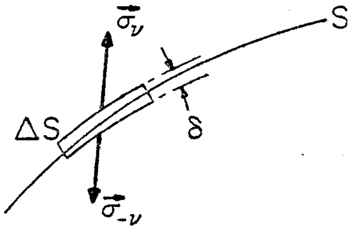

Figure 4

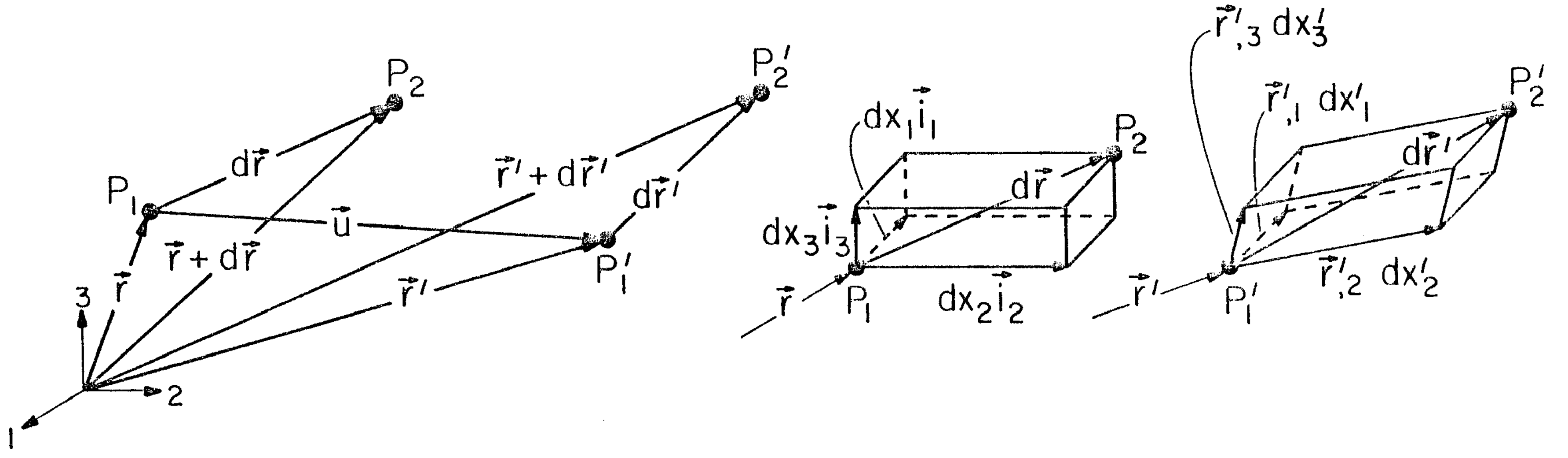
P₂
P₁
$d\vec{r}$
$\vec{u}$
$\vec{r}$
$\vec{r}+d\vec{r}$
$\vec{r}'+d\vec{r}'$
$\vec{r}'$
P_2'
P_1'
$d\vec{r}'$
3
2
1
$dx_1\vec{i}_1$
$dx_3\vec{i}_3$
$dx_2\vec{i}_2$
P_1
P_2
$d\vec{r}$
$\vec{r}$
$\vec{r}'_{,3}\,dx_3'$
$\vec{r}'_{,1}\,dx_1'$
$\vec{r}'_{,2}\,dx_2'$
$d\vec{r}'$
$\vec{r}'$
P_1'
P_2'

Figure 5

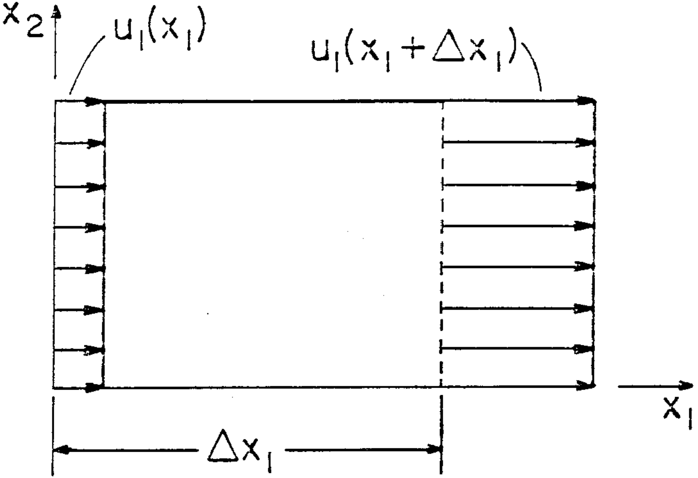

Figure 6

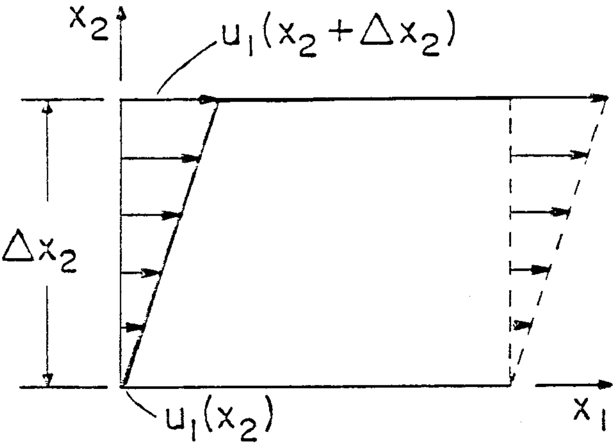

Figure 7(a)

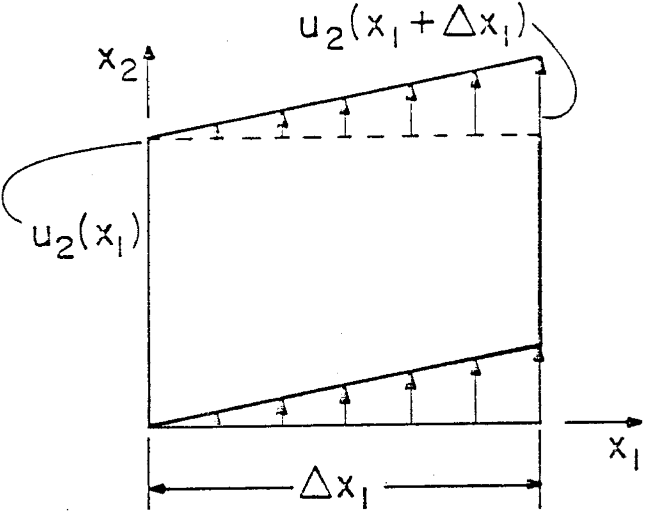

Figure 7(b)

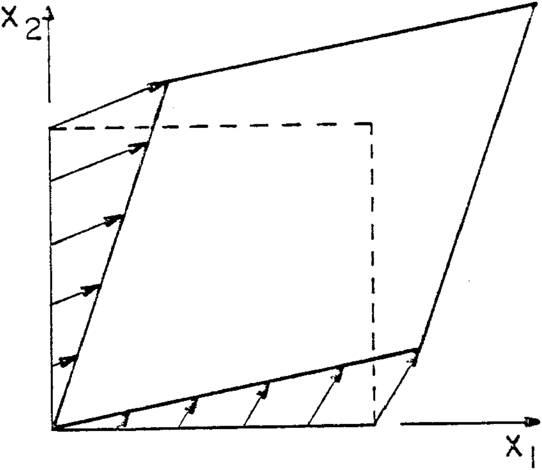

Figure 17 (c)

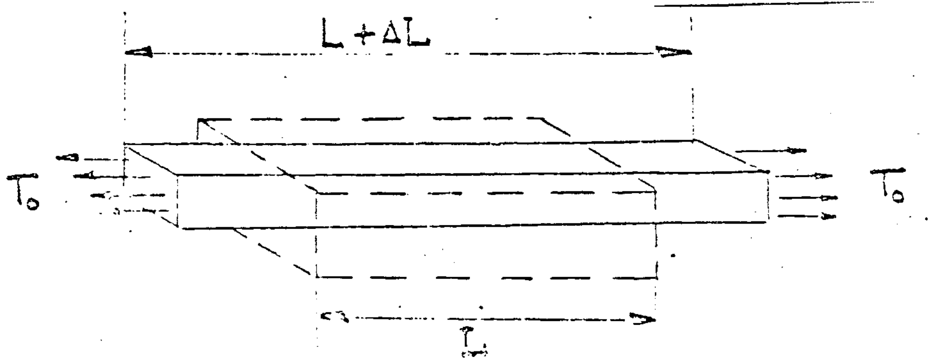

Figure 8

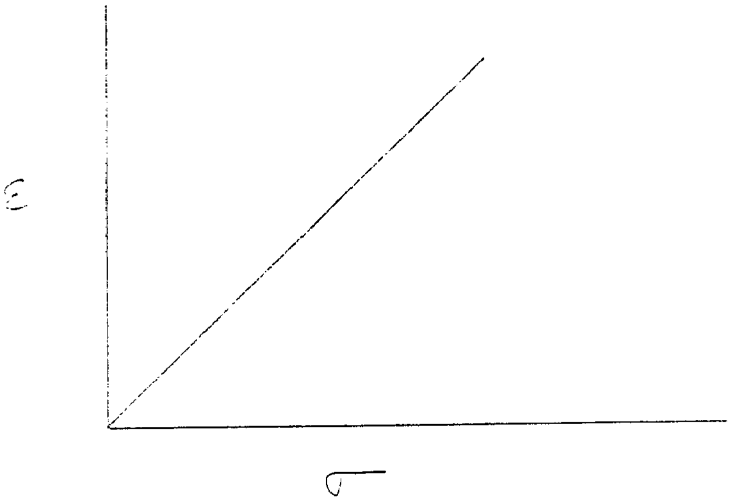

Figure 9(a)

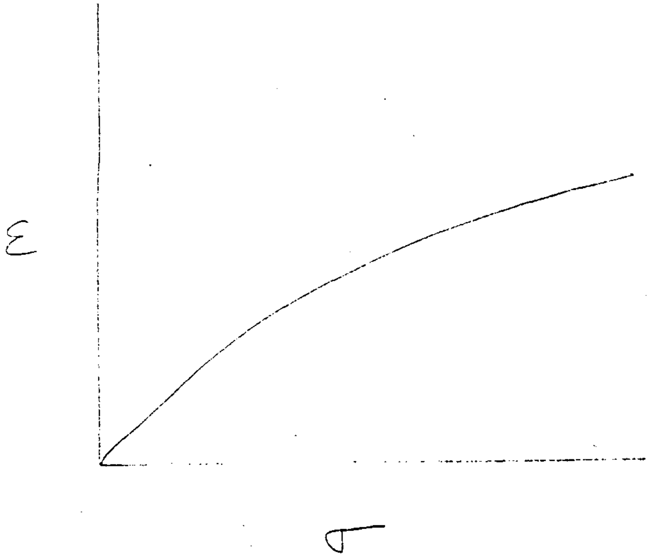

Figure 9(b)