Optimal Forest Harvesting with Ordered Site Access*

By Frederic Y. M. Wan and Keith Anderson†

A long-standing problem in forestry management is the optimal harvesting of a growing population of trees to maximize the resulting discounted aggregate net revenue. For an *ongoing forest*, the trees are harvested and replanted repeatedly; for a *once-and-for-all* forest, there is no replanting after a single harvest. In this paper, we outline a new formulation for the optimal-harvest problem which avoids difficulties associated with functional-differential equations or partial differential equations of state in the relevant optimal-control problem encountered in recent studies of the ongoing-forest problem. Our new formulation is based on the observation that tree logging is necessarily ordered by practical and/or regulatory considerations (e.g., it is illegal to cut the younger trees first in some jurisdictions); random access to tree sites does not occur in practice. The new formulation is described here for the simpler problem of a once-and-for-all forest. New results for nonuniform initial age distributions and variable unit harvest costs for this simpler problem are reported herein; results for an ongoing forest will be reported in [10]. The new model is also of interest from a control-theoretic viewpoint, as it exhibits the unique feature of having time as a *state variable*, in contrast to its usual role as an independent variable in conventional control problems.

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1. Introduction

Forestry is a major industry in many regions in the world, and proper forest management is of major concern to the governments of these regions and to the forest industry, albeit for different reasons. Private logging companies are keenly interested in the financial return on their operations. Governments are interested in a healthy forest industry for tax revenues and regular employment for the labor force without damaging the environment. In either case, a good understanding of forestry economics is necessary.

To an economist, a forest of trees is a stock of capital which increases in value with tree age. For example, a typical stand of 1 l0-year-old British Columbia Douglas fir was worth $1000 (after harvesting cost) for timber production in 1967, while a 30-year-old stand had no net commercial value [1]. With the average (or marginal) yield rate and relative growth rate of the trees diminishing with tree age and with a positive real discount rate (net inflation) for future income, the "when to cut a tree" question has long been a fundamental problem in forestry economics. Given the net commercial value V of a tree (before the onset of biological decay) as a monotone increasing concave function of tree age A, and a constant real interest rate δ for discounting future revenue, the optimal cutting age of the tree, A_{IF}, is easily determined by the condition that $e^{-\delta t}V(A) \equiv e^{-\delta t}V(t-T_0)$ attains its maximum value at $t = T_0 + A_{IF}$. (Here, t is the chronological time measured from now, and $T_0 \leqslant 0$ is the germination time of the tree.) This condition yields the so-called Fisher's rule [2–4]:

$$\frac{V'(A_{IF})}{V(A_{IF})} = \delta, \qquad (\ )' \equiv \frac{d(\ )}{dA}. \tag{1.1}$$

The optimal cutting time is when the relative yield rate equals the discount rate.

Fisher's rule applies only to a single cutting (a "once-and-for-all forest"), it must be modified to account for the opportunity cost of not logging sooner for an "ongoing forest" where replanting is assumed to be done immediately after cutting. If all biological and economical factors remain unchanged over time, the optimal cutting age, A_{MF}, for trees in an ongoing forest is the so-called *Faustmann rotation* determined by the condition [3-6]

$$\frac{V'(A_{MF})}{V(A_{MF})} = \frac{\delta}{1-e^{-\delta A_{MF}}}. \tag{1.2}$$

The Faustmann formula (1.2) is obtained by choosing a sequence of cutting times $\{t_1, t_2, \ldots\}$ to maximize the present value of the total future net revenue [3]

$$P \equiv e^{-\delta t_1}V(A_1) + e^{-\delta t_2}V(A_2) + \cdots = \sum_{k=1}^{\infty} e^{-\delta t_k}V(A_k), \tag{1.3}$$

where $A_k = t_k - t_{k-1}$ with $t_0 \equiv T_0$ being the (uniform) germination time of the initial forest. An alternate maximum-rent formulation which leads to the same result is given in [4], where the effect of labor as an input to production as well as non-steady-state considerations can also be found. Other related problems, such as the effect of forest thinning, are discussed in [3].

Our research is mainly concerned with recent attempts to improve upon Fisher's result for a once-and-for-all forest and Faustmann's result for an ongoing forest in two directions. When applied to an entire forest, it is implicit in these two classical results that harvesting of the whole forest or any part of it can be instantaneous. In other words, there are sufficient equipment and a sufficiently large labor force to log the entire forest in a short time interval compared to the time scale for a significant loss in interest on the delayed profit. For a large forest, this assumption is not realistic, and an upper limit to the available harvesting effort and therefore the harvesting rate should be imposed in the determination of the optimal solution.

As well, harvesting costs have been either ignored in the Fisher age and the Faustmann rotation, or taken in a form so that V could be regarded as the net commercial value or profit. However, the unit harvesting cost (per tree say) does vary substantially with the rate of harvest. At low harvesting rates, fixed-cost facilities for cutting and transportation are not used to their full capacity, giving a high unit harvesting cost. At high harvesting rates, available facilities and manpower are used beyond their normal capacity, resulting in more frequent equipment breakdown and overtime pay. Therefore, as a function of the harvesting rate, the unit harvesting cost is U-shaped and cannot be absorbed in V(A).

The effect of an upper bound on harvesting rate and a U-shaped dependence of unit harvesting cost on harvesting rate have been investigated in several articles [7-10]. With regard to the effect of a harvest-rate-dependent unit harvesting cost, only some partial results have been obtained, principally because the mathematical problems associated with the improved models are too difficult to yield a complete solution (see Section 2 of this article). With regard to the effect of an upper bound on the harvesting rate, the mathematical problems associated with the improved models are also difficult, but it is possible to deduce their solution by arguments based on insight. In the next section, we indicate the nature of the difficulties in the two extensions with a brief summary of the models developed by Heaps and Neher [7], which are natural extensions of the conventional formulation. In subsequent developments, we show how a different approach to the same problem, which allows for an additional degree of realism (namely, trees are cut in a certain order imposed by law, government regulations, or economic considerations), leads to parallel models which are completely tractable mathematically. Only results pertaining to once-and-for-all forests are reported in this article; results for ongoing forests will be reported in [10].

Aside from the new results generated for the forest harvesting problem, the mathematical model formulated for the problem in this paper exhibits the unique feature of having *time* as a *state variable,* in contrast to its usual role as an independent variable in conventional control problems. Therefore, our model for forest harvesting should also be of interest from a control-theoretic viewpoint.

2. **Constrained harvest rate and harvest-rate-dependent harvest cost**

2.1. *The once-and-for-all Forest*

When there is an upper bound to the harvesting rate $h(t)$, say $h(t) \leqslant m$, the harvesting of an entire forest (or any part of it) has to be spread over a finite time interval, say $t_s \leqslant t \leqslant t_e$. In that case, the present value of the discounted future profit is given by

$$P \equiv \int_{t_s}^{t_e} [p(t) - c(h)] h e^{-\delta t} dt, \tag{2.1}$$

where p is the price function for unit harvest and c is the rate-dependent harvesting cost for unit harvest. (For simplicity, c is usually assumed not to depend explicitly on t in the conventional formulation [7].) If the unit cost function c is independent of h, then $p(t)- c$ is identical to the net commercial value of a unit harvest $V(A)$ of the last section. Again, δ is the real interest rate (apart from inflation, and assumed to be constant over time). The logging company is to choose a starting time t_s and a harvesting rate $h(t)$ (which determines the completion time t_e) to maximize P subject to

$$0 \leqslant h(t) \leqslant m, \tag{2.2}$$

$$\int_{t_s}^{t_e} h(t)\, dt = F_0. \tag{2.3}$$

The second condition is simply a statement of the fact that the logging of the entire forest, F_0, is completed by t_e.

To put the problem in a conventional form of an optimal-control problem, we define an accumulated harvest function $F(t)$ by

$$\frac{dF}{dt} = h(t) \qquad (t_s < t). \tag{2.4}$$

Evidently, we have

$$F(t_s) = 0, \qquad F(t_e) = F_0. \tag{2.5a, b}$$

The methods of dynamic optimization may now be applied to obtain the conditions which determine the optimal harvest policy $\{\bar{t}_s, \bar{h}(t)\}$ [7]. We merely wish to point out that the solution of the relevant *free-end-point* BVP defined by these conditions is not at all straightforward, though it is made less complicated in [7] by a separate treatment of the two effects and by the assumption of a uniform distribution of tree ages.

2.2. *The Ongoing Forest*

For a repeatedly harvested forest, it is natural to attempt an extension of the formulation which yields the Faustmann rotation. This is the approach used in [7], in which the present value of the total future net profit P is maximized with

$$P = \sum_{n=1}^{\infty} \int_{t_{sn}}^{t_{en}} [p(A_n(t)) - c(h_n)] h_n e^{-\delta t} dt. \tag{2.6}$$

In this expression, (t_{sn}, t_{en}) is the time interval of the n th harvest with harvesting rate $h_n(t)$, and $A_n(t)$ is the age of the tree(s) being cut at time t during the n th harvest. Evidently, (2.6) is a straightforward extension of (1.3). The logging company is to choose a sequence of starting times $\{t_{sn}\}$ and a sequence of harvesting rates $\{h_n(t)\}$ to maximize P subject to the constraints

$$0 \leqslant h_n(t) \leqslant m \qquad (n=1,2,\ldots), \tag{2.7}$$

$$\int_{t_{sn}}^{t_{en}} h_n(t)\, dt = F_0 \qquad (n=1,2,\ldots). \tag{2.8}$$

In addition, it is important to note that the tree age function A_n is not an independent quantity. The portion of the forest cut during the nth harvest from t_{sn} to t must be the same as that portion cut during the $(n-1)$th harvest from $t_{s(n-1)}$ to $t - A_n(t)$. We have therefore the additional set of constraints

$$\int_{t_{s(n-1)}}^{t - A_n(t)} h_{n-1}(Z)\, dZ = \int_{t_{sn}}^{t} h_n(Z)\, dZ \qquad (n=2,3,\ldots). \tag{2.9}$$

For the application of the methods of dynamic optimization, we again transform (2.8) and (2.9) into a more suitable form. As in the single-harvest case, we introduce a sequence of accumulated harvest functions $\{F_n(t)\}$ by

$$\frac{dF_n}{dt} = h_n(t), \qquad F_n(t_{sn}) = 0, \;\; F_n(t_{en}) = F_0, \tag{2.10}$$

for $n = 1,2,3,\ldots$. The conditions (2.10) for $\{F_n\}$ replace (2.8). To get a local form of (2.9), we differentiate both sides of that expression with respect to t to get (after some rearrangement)

$$\frac{dA_n}{dt} = \frac{h_{n-1}(t-A_n) - h_n(t)}{h_{n-1}(t-A_n)} \qquad (n=2,3,\ldots) \tag{2.11a}$$

with

$$A_n(t_{sn}) = t_{sn} - t_{s(n-1)}, \;\; A_n(t_{en}) = t_{en} - t_{e(n-1)} \qquad (n=2,3,\ldots). \tag{2.11b}$$

Unfortunately, the conventional maximum principle in optimal-control theory does not apply to problems with state equations in the form of functional-differential equations. By an ad hoc argument suggested by experience and insight, the determination of the optimal harvest policy for the case of rate-independent unit harvest cost has been reduced in [7] to an equivalent once-and-for-all forest problem, so that the results already obtained for that case apply immediately. For more general situations, only some partial results have been obtained in [7] as well as in more recent efforts such as [8] and [9].

Necessary and sufficient conditions for optimal-control problems with functional-differential equations of state relevant to the above problem have been investigated in the literature, e.g., [11]. However, we will take a different approach to overcome the difficulties. More specifically, we will reformulate the problem so that our alternative mathematical model does not involve functional-differential equations and at the same time includes another realistic feature of forest harvesting.

3. The once-and-for-all forest

3.1. *Ordered site access*

Inherent in all conventional forest logging models (cf. Section 2) is an assumption that loggers may cut trees from any part of the forest. This is not a serious restriction if the initial tree age distribution of the forest is uniform. In forests with a nonuniform initial age distribution, there may be requirements or regulations which dictate the order in which trees are to be cut. For example, some jurisdictions require that trees in a forest be cut in the order of their age, the oldest one to be cut first. In the absence of such laws, it may be physically necessary and economically prudent to cut trees in the order of their distance from one or more logging camp sites. In short, tree logging is necessarily ordered by practical and regulatory considerations; random access to tree sites does not occur in reality.

In this article, we consider only the situation where a single logging crew is to cut trees in a single array along a prescribed path (of uniform width w) winding through the entire forest. In that case, the position of any tree site can be described by the arc length s along the path to the site. We use here a continuous model of the forest with the discrete tree stands of the forest smeared out as follows. The entire forest is divided up into F_0 locations of (small) land area, each occupied by a single tree. The entire stumpage of the tree is distributed over its assigned area. Except for cases of sharp discontinuities in the actual initial distribution of tree ages, the initial tree age distribution over the logging path for the continuous model is taken to be a continuous (usually piecewise smooth) approximation to the actual distribution. The commercial value of the stumpage at different sites along the logging path may be different because of a nonuniform age distribution, different growth conditions, market-price differences at different cutting times, etc.

Let $T(s) \geqslant 0$ be the time at which the tree site at location s along the logging path (of width w) is harvested in the future, $T = 0$ being now. The initial age distribution of the trees in the forest is denoted by $-T_0(s)$, with $T_0(s) \leqslant 0$ being the germination-time distribution of the current trees. At cutting time, the tree stand at s will be $A(s) \equiv T(s) - T_0(s)$ years old. By construction, $T' \equiv dT/ds$ is nonnegative along the path, $1/T'$ being effectively the harvest rate h of References [7] and [9]. Thus, we have $T' = 0$ only if instantaneous harvesting is possible (with an unbounded harvest rate), as T' is a measure of the time consumed in logging a particular tree site.

Let $p(s, T, A, T')\,ds$ and $c(s, T, A, T')\,ds$ be the commercial price and the harvesting (cutting, shipping, etc.) cost of the timber from the incremental strip

$(s, s+ds)$ of the logging path.[1] That the price per unit area harvested at location s and the harvesting cost per unit area at that location may depend on tree age A and absolute time has been discussed in the literature of forestry economics; the dependence of absolute time reflects in part the fluctuation of the lumber and labor market. The possible dependence of p and c on location is not unexpected; trees may grow faster at one site than another, and they may be more difficult to log at some locations because of the geography and topography. We have already discussed how the harvesting cost may change with the harvesting rate. If the logging company has any degree of monopolistic power, the lumber price may also be affected by the rate of harvest.

The present value of the discounted future net revenue for the tree stumpage along a path-length increment $(s, s+ds)$ is $e^{-\delta T(s)}(p-c)\,ds$. The present value of the discounted future net revenue for the entire forest is therefore

$$P \equiv \int_0^1 (p-c)e^{-\delta T}\,ds, \tag{3.1}$$

where we have taken the logging path to be of *unit* total length. (Note that a change of the independent variable from s to T transforms (3.1) back to the expression for P [*see* (2.1)] used in the conventional model.) The management problem for the logging company is to choose a harvest schedule $T(s)$ for the forest so that this present value is a maximum. The maximization is subject to the constraint on the harvest rate, which takes the form of $T' \geqslant \tau\ (\geqslant 0)$ in our model, and $T(0) \geqslant 0$. At this point, we have effectively completed our model formulation. The solution of the optimization problem will offer some insight into the real phenomenon.

3.2. *The optimal harvest policy*

In order to bring the available tools in modern control theory to bear on the solution of the optimization problem described in the previous section, we introduce a new control variable u by the defining equation (of state)

$$\frac{dT}{ds} \equiv u \tag{3.2}$$

and write the present value of future net revenue P as

$$P = \int_0^1 e^{-\delta T} V(s, T, A, u)\,ds, \qquad V \equiv p - c, \tag{3.3}$$

where $A = T - T_0$ and V is twice continuously differentiable, say. The maximum

[1] We will omit the uniform path-width parameter w in the subsequent development.

principle [3] now requires that $u(s)$ be chosen to maximize the Hamiltonian

$$\mathcal{H} \equiv V(s, T, A, u)e^{-\delta T} + \lambda u \tag{3.4}$$

subject to the equation of state (3.2), the equation for the adjoint variable $\lambda(s)$

$$\frac{d\lambda}{ds} = -\frac{\partial \mathcal{H}}{\partial T} = -[V_T + V_A - \delta V]e^{-\delta T}, \quad (\ )_y \equiv \frac{\partial (\ )}{\partial y}, \tag{3.5}$$

the transversality conditions

$$\lambda(0) = \lambda(1) = 0 \tag{3.6}$$

and the inequality constraint

$$u \geqslant \tau\ (\geqslant 0). \tag{3.7}$$

In addition, we must have $T(s) \geqslant 0$. When this constraint on the state variable is binding, the condition $\lambda(0) = 0$ of (3.6) is replaced by $T(O) \equiv 0$ [which, along with $T'(s) \geqslant 0$, implies $T(s) \geqslant 0$]. By allowing τ to be positive, we include the possibility of an imposed maximum feasible harvesting rate. If there is no such imposed upper limit, then u $\geqslant$ 0 simply reflects the fact that, in our model, the tree sites are ordered for the purpose of harvesting, as is usually the case in reality.

(A) *Interior solution.* When the inequality constraint in (3.7) is not binding, we have an interior solution for the optimal-control problem given by

$$\frac{\partial \mathcal{H}}{\partial u} = e^{-\delta T}\frac{\partial V}{\partial u} + \lambda = 0. \tag{3.8}$$

Equations (3.8) and (3.2) may be used to eliminate λ and u from (3.5) and (3.6) to get a second-order differenial equation for T in the interval (0,l) and one boundary condition for T' (or T) at each end of the interval. This two-point boundary value may then be solved to get the "optimal"[2] harvest time for different tree sites.

'Strictly speaking, T(s) so determined satisfies only the necessary condition for a maximum P. However, the differential equation of state for our problem is linear in the control variabl$u$, and the integrand of (3.3) is concave downward as a function of u for all problems considered herein. We know from [12] that an optimal control exists. The unique solution of the necessary conditions for op timali ty is therefore the optimal policy.

(B) *Corner solution.* When the constraint $u \geqslant \tau$ is binding, we have then a corner solution with

$$u(s) \equiv \frac{dT}{ds} = 7 \tag{3.9}$$

which can be integrated to give

$$T(s) = \tau s + t_0, \tag{3.10}$$

where the constant of integration t_0 will be determined presently. Upon substituting (3.9) and (3.10) into (3.5), we get

$$\lambda(s) = -\int_0^s \left[V_T + V_A - \delta V\right) e^{-\delta T}\Big]_{T=\tau s + t_0} ds, \tag{3.11}$$

where the condition $\lambda(0) = 0$ has been used to eliminate a new constant of integration. The remaining transversality condition $\lambda(1) = 0$ becomes

$$\int_0^1 \left[(V_T + V_A - \delta V) e^{-\delta T}\right]_{T=\tau s + t_0} ds = 0 \tag{3.12}$$

and serves as a condition for the determination of the constant t_0.

By way of (A) or (B), we have in effect found a (and usually the only) candidate $T(s)$ for the optimal policy. That it is optimal may be verified by checking the appropriate concavity conditions or using arguments similar to those in [7]. The conceptual and computational simplicity of our solution procedure can be attributed to the fact that the end points in our optimal-control problem are *fixed.* Solutions for problems with various types of nonuniform and possibly discontinuous initial age distributions and unit harvesting costs have been obtained by the procedure outlined above and will be described in the remaining sections of this article.

It is possible that a combination of interior and corner solutions is appropriate for the problem. We will see examples of this situation in later sections. Also, if $T(s) \geqslant 0$ is binding, then [with $T'(s) \geqslant \tau \geqslant 0$] *we* must replace $\lambda(0) = 0$ by $T(0) = 0$ and rework the solution.

4. Fisher's rule and nonuniform initial age distributions

4.1. *Uniform initial age distribution*

Throughout this section, we consider only the case of no limit to the harvesting rate, so that $\tau = 0$. Suppose the price (per unit area) is a function of tree age only, $p = p(A)$, and the cost (per unit area) is a constant, $c = c_0$. Since $\partial V/\partial u = \partial(p -$

$c)/\partial u \equiv 0$, we have from (3.8)

$$\lambda \equiv 0 \tag{4.1}$$

for an interior solution. The solution (4.1) satisfies the two transversality conditions (3.6), and reduces (3.5) (with $V_T \equiv 0$) to

$$\frac{\dot{V}(A)}{V(A)} \equiv \frac{\dot{p}(A)}{p(A)-c_0} = \delta, \tag{4.2}$$

where a dot indicates differentiation with respect to the argument of the function. With $A = T - T_0$, the condition (4.2) is Fisher's rule (1.1) in a slightly different notation.

For a monotone increasing, concave function $p(A)$ (with $p - c_0 > 0$ for sufficiently large A), *we* let $\bar{A}(\delta)$ denote the root of (4.2). When the initial age distribution of the forest is *uniform* $[T_0(s) = T_0 =$ constant], then the "optimal" policy is $T(s) = T_0 + \bar{A}(\delta)$. It requires the tree site at s to be logged when the tree stump there reaches the Fisher age $\bar{A}(\delta) \equiv A_{IF}$. If $T_0 + \bar{A}(\delta) < 0$, then $T(0) = 0$ and $T'(s) = 0$ require $T(s) \equiv 0$. Logging should have been done before now; therefore we log immediately.

When the initial age distribution of the forest $-T_0(s)$ is *nonuniform,* the situation is more complicated. We must distinguish and treat separately the three cases where (1) $T_0'(s) \geqslant 0$, $0 \leqslant s \leqslant 1$, (2) $T_0'(s) \leqslant 0$, $0 \leqslant s \leqslant 1$, and (3) T_0' changes sign over the interval $0 \leqslant s \leqslant 1$.

4.2. *Nondecreasing* $T_0(s)$

The inequality constraint $u \equiv T'(s) \geqslant 0$ is not binding in this "oldest tree first" case, so that the interior solution (4.2) is applicable. From (4.2), we get as before

$$T(s) = \bar{A}(\delta) + T_0(s). \tag{4.3}$$

The optimal harvest policy is to spread out the logging process and cut a tree only when it reaches Fisher's age, no sooner. Also, it should not be later unless $T_0(s) + \bar{A}(\delta) < 0$. If $T_0(s) + \bar{A}(\delta) < 0$ for $0 \leqslant s < \bar{s}$, we do the best we can and log the path segment $0 \leqslant s \leqslant \bar{s}$ immediately, i.e., $T(s) \equiv 0$ for $0 \leqslant s < \bar{s}$. [See Figure 1(a) and (b).]

4.3. *Nonincreasing* $T_0(s)$

Consider first the case $T_0'(s) < 0$ (except possibly at isolated points), so that $T'(s) \equiv 0$ or $T(s) \equiv t_0$ for $0 \leqslant s \leqslant 1$. Clearly, we should take $t_0 = 0$ if $T_0(s) + \bar{A}(\delta) \leqslant 0$ for the entire forest. If $T_0(s) + \bar{A}(\delta) > 0$ for a range of $s > 0$, the constant t_0 is to be determined by (3.12) which, for the class of p and c considered in this

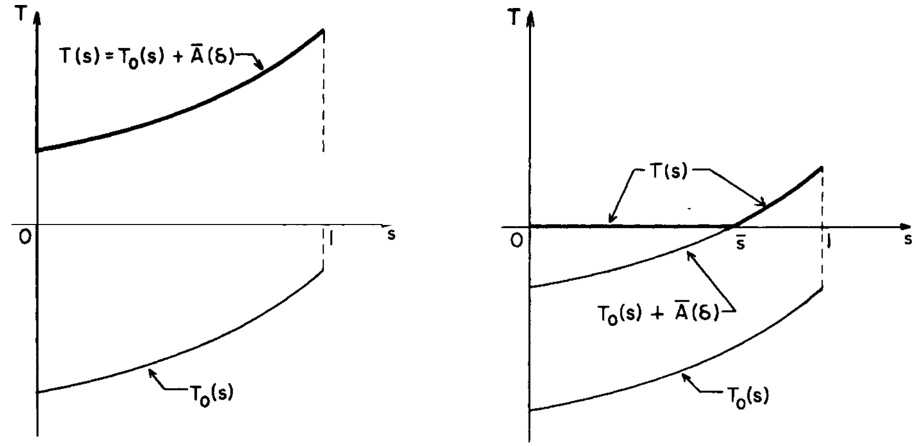


Figure 1. Optimal logging schedule for $T_0'(s) \geqslant 0$.

section, may be written as

$$\frac{\int_0^1 e^{-\delta t_0}\dot{p}(t_0 - T_0(s))\,ds}{\int_0^1 e^{-\delta t_0}[p(t_0 - T_0(s)) - c_0]\,ds} = \delta. \tag{4.4}$$

Thus, the optimal harvest policy requires the logging to begin at the instant t_0 when the discounted marginal net yield of all tree sites as a fraction of the present value of the net revenue from the whole forest is equal to the discount rate δ. The entire forest is to be cut instantaneously at t_0, no sooner [Figure 2(a)]. It should also not be later unless $t_0 < 0$, in which case we do the best we can and take $T(s) \equiv 0$.

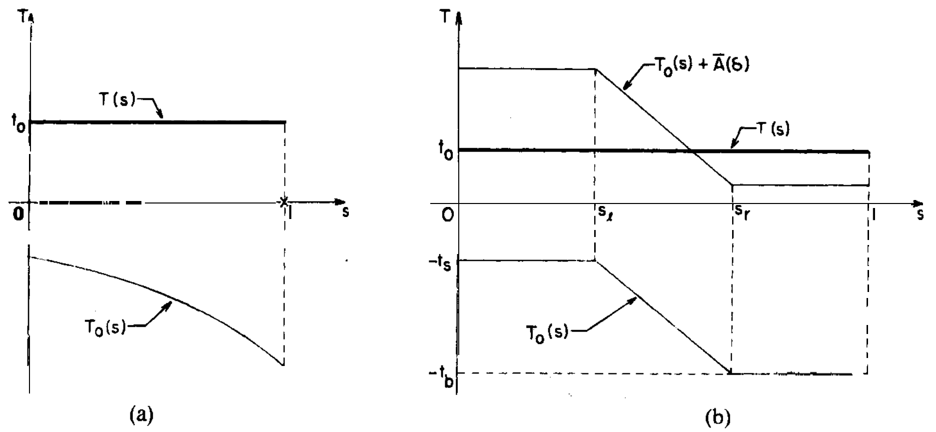


Figure 2. Optimal logging schedule for (a$T_0'(s)<0$; (b) a nonincreasing $T_0(s)$ with $T_0'(s)$ **vanishing** over one or more subintervals of $(0, 1)$.

To gain further insight into the above result, we consider a *linear* $T_0(s)$ given by

$$T_0(s) = -t_s + (t_s - t_b)s \tag{4.5}$$

with $-t_b > -t_s$. For this $T_0(s)$, the condition (4.4) becomes

$$\frac{e^{-\delta t_0}\left[\dfrac{\{p(t_0 - T_0(0)) - c_0\} - \{p(t_0 - T_0(1)) - c_0\}}{T_0(0) - T_0(1)}\right]}{\int_0^1 e^{-\delta t_0}[p(t_0 - T_0(s)) - c_0]\,ds} = \delta. \tag{4.6}$$

The left-hand side of (4.6) gives the difference of the present value of net revenue from the first and last tree cut at t_0, averaged over their age difference, as a fraction of the present value of net revenue from the entire forest logged instantaneously at t_0. The optimal harvest time $T(s) \equiv t_0$ is that moment when this fraction equals the discount rate.

For cases with $T_0'(s) = 0$ over one or more segments of the cutting path, the ordered-site condition and the optimality of Fisher's age together require that $T(s) = t_0$ for the entire forest also, with t_0 determined again by (4.6). For example, if the initial age distribution is [Figure 2(b)]

$$T_0(s) = \begin{cases} -t_s & (0 \leqslant s \leqslant s_l), \\ -t_s + \dfrac{s - s_l}{s_r - s_l}(t_s - t_b) & (s_l \leqslant s \leqslant s_r), \\ -t_b & (s_r \leqslant s \leqslant 1) \end{cases} \tag{4.7}$$

with $0 < t_s < t_b$, then we must have $T(s) = t_0$ for $s_l \leqslant s \leqslant s_r$ with $\bar{A}(\delta) - t_b < t_0 < \bar{A}(\delta) - t_s$, since the inequality constraint $T'(s) \geqslant 0$ is binding there. With $T_0'(s) \equiv 0$ in the two remaining portions of the path, we should ideally choose

$$T(s) = \begin{cases} \bar{A}(\delta) - t_s > t_0 & (0 \leqslant s \leqslant s_l), \\ \bar{A}(\delta) - t_b < t_0 & (s_r \leqslant s \leqslant 1). \end{cases} \tag{4.8}$$

However, the ordered-site condition requires that $T(s) \leqslant t_0$ in $0 \leqslant s \leqslant s_l$ and $T(s) \geqslant t_0$ in $s_r \leqslant s \leqslant 1$. To minirnize the possible reduction in the present value of the forest, we take $T(s) = t_0$ for the whole forest [see Figure 2(b)]. In that case, $\lambda(s)$ is given by (3.11), and the condition $A(1) = 0$ for determining t_0 is again (4.6) for the class of p and c considered in this section.

4.4. $T_0'(s)$ *changes sign along the* logging path

For this case, the method for determining the optimal harvest schedule deduced from the maximum principle must be applied with some care. While we can simply describe the actual solution procedure, it will be more instructive to first show how we arrive at the results for some specific examples.

Consider the particular $T_0(s)$ as shown in Figure 3(a), with $T_0'(s) \geqslant 0$ for $0 \leqslant s \leqslant s_l$ and $s_r \leqslant s \leqslant 1$ and $T_0'(s) \leqslant 0$ for $s_l \leqslant s \leqslant s_r$. Clearly, we must have $T(s) = t_0$ for some constant t_0 in $s_l \leqslant s \leqslant s_r$, with $T_0(s_r) + \bar{A}(\delta) \leqslant t_0 \leqslant T_0(s_l) + \bar{A}(\delta)$. [If $s_l = 0$ and $s_r = 1$, then t_0 is determined by (4.4).] Since $s_r < 1$ in this case, and $t_0 \geqslant T_0(s_r) + \bar{A}(\delta)$, we must have $T(s) = t_0$ for $s_r \leqslant s \leqslant s_r^*$ as well [just as $T(s) = 0$ for $T_0'(s) > 0$ if $T_0(s) + \bar{A}(\delta) < 0$], with s_r^* determined by $T_0(s_r^*) = t_0 - \bar{A}(\delta)$, and of course $T(s) = T_0(s) + \bar{A}(\delta)$ for $s_r^* \leqslant s \leqslant 1$ for optimality. Similarly, with $t_0 \leqslant T_0(s_l) + \bar{A}(\delta)$, we must have $T(s) = t_0$ for $s_l^* \leqslant s \leqslant s_l$ also, with $T_0(s_l^*) = t_0 - \bar{A}(\delta)$, since $T(s)$ is a nondecreasing function of s. Therefore $T(s)$ is given by the heavy nonsmooth curve in Figure (3a). Note that $s_l^* = 0$ if $t_0 < T_0(s) + \bar{A}(\delta)$ in $0 \leqslant s \leqslant s_l$ and $s_r^* = 1$ if $t_0 > T_0(s) + \bar{A}(\delta)$ in $s_r \leqslant s \leqslant 1$.

As in the $T_0'(s) \geqslant 0$ case, we have $\lambda(s) \equiv 0$ in $0 \leqslant s \leqslant s_l^*$ and $s_r^* \leqslant s \leqslant 1$. In the range $s_l^* \leqslant s \leqslant s_r^*, \lambda(s)$ is determined by (3.5) and the continuity conditions $\lambda(s_l^*) = \lambda(s_r^*) = 0$ [which follow from the continuity of $T(s)$]. For the class of p and c considered in this section, we get from (3.5) and $\lambda(s_l^*) = 0$

$$\lambda(s) = -\int_{s_l^*(t_0)}^{s} e^{-\delta t_0}\{\dot{p}(t_0 - T_0(s)) - \delta[p(t_0 - T_0(s)) - c_0]\}\, ds \qquad (4.9)$$

with $\lambda(s_r^*) = 0$ giving a condition for determining t_0:

$$\frac{\int_{s_l^*(t_0)}^{s_r^*(t_0)} e^{-\delta t}\dot{p}(t_0 - T_0(s))\, ds}{\int_{s_l^*(t_0)}^{s_r^*(t_0)} e^{-\delta t_0}[p(t_0 - T_0(s)) - c_0]\, ds} = \delta. \qquad (4.10)$$

The economic interpretation of (4.10) is the same as that for (4.4), except that only the portion of the forest covered by the logging-path segment (s_l^*, s_r^*) is involved in (4.10).

It should be emphasized that both s_l^* and s_r^* depend on t_0, so that the final solution for the optimal harvest schedule is decidedly nontrivial except when $T_0(s)$ is a very simple function of s. A more detailed discussion of the computational problem can be found in the Appendix of this paper. A complete solution for the optimal harvest schedule will be worked out there for a piecewise linear $T_0(s)$. A systematic numerical method will also be outlined there for cases which do not have an exact elementary solution.

The simple $T_0(s)$ in Figure 3(a) does not give rise to some complications which may be encountered in more general cases. Consider the $T_0(s)$ in Figure 3(b), which has two path segments with $T_0'(s) < 0$, namely, $0 \leqslant s \leqslant s_{r1}$ and $s_{l2} \leqslant s \leqslant s_{r2}$. For each segment, we expect $T(s)$ to be a constant whenever T_0 is nonincreasing. Let the constant be t_1 in $0 \leqslant s \leqslant s_{r1}$ and t_2 in $s_{l2} \leqslant s \leqslant s_{r2}$. By arguments similar to those in the case shown in Figure 3(a), we have in fact $T(s) = t_1$ for $0 \leqslant s \leqslant s_{r1}^*(t_1)$ and $T(s) = t_2$ for $s_{l2}^*(t_2) \leqslant s \leqslant s_{r2}^*(t_2) (\leqslant 1)$ with $T_0(s_{l2}^*) = t_2 - \bar{A}(\delta)$ and $T_0(s_{rk}^*) = t_k - \bar{A}(\delta)$. The constants t_1 and t_2 are each determined by an integral condition

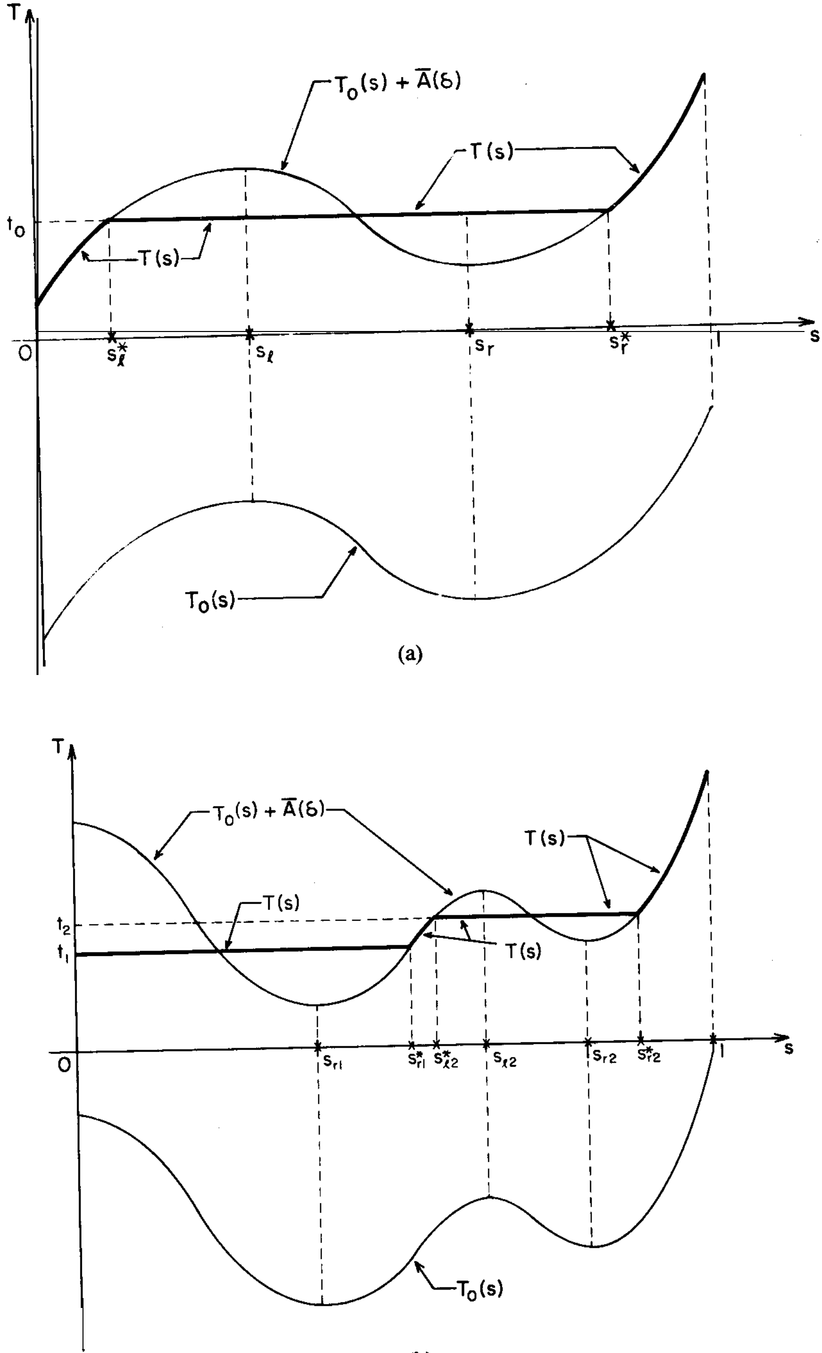


Figure 3. **Optimal logging** schedule for a $T_0(s)$ which changes sign (a) twice and (b) three times along the logging path.

similar to (4. 10), namely,

$$\frac{\int_0^{s_{r1}^*(t_1)} \dot{p}(t_1 - T_0(s))\, ds}{\int_0^{s_{r1}^*(t_1)} [p(t_1 - T_0(s)) - c_0]\, ds} = \delta \tag{4.11}$$

for t_1, and

$$\frac{\int_{s_{l2}^*(t_2)}^{s_{r2}^*(t_2)} \dot{p}(t_2 - T_0(s))\, ds}{\int_{s_{l2}^*(t_2)}^{s_{r2}^*(t_2)} [p(t_2 - T_0(s)) - c_0]\, ds} = \delta \tag{4.12}$$

for t_2. The optimal harvest schedule is then

$$T(s) = \begin{cases} t_1, & 0 \leqslant s \leqslant s_{r1}^*(t_1), \\ T_0(s) + \bar{A}(\delta), & s_{r1}^*(t_1) \leqslant s \leqslant s_{l2}^*(t_2), \\ t_2, & s_{l2}^*(t_2) \leqslant s \leqslant s_{r2}^*(t_2), \\ T_0(s) + \bar{A}(\delta), & s_{r2}^*(t_2) \leqslant s \leqslant 1. \end{cases} \tag{4.13}$$

However, if it should turn out that $s_{l2}^*(t_2) \leqslant s_{r1}^*(t_1)$, then the ordered-site property would require that $T(s) = \hat{t}_2$ for $0 \leqslant s \leqslant \hat{s}_{r2}(\hat{t}_2)\,(\leqslant 1)$, with $s_{r2} \leqslant \hat{s}_{r2}(\hat{t}_2) \leqslant s_{r2}^*$ determined by $T_0(\hat{s}_{r2}) = \hat{t}_2 - \bar{A}(\delta)$. The constant $\hat{t}_2$ is then determined by

$$\frac{\int_0^{\hat{s}_{r2}(\hat{t}_2)} \dot{p}(\hat{t}_2 - T_0(s))\, ds}{\int_0^{\hat{s}_{r2}(\hat{t}_2)} [p(\hat{t}_2 - T_0(s)) - c_0]\, ds} = \hat{\delta}, \tag{4.14}$$

and $T(s) = T_0(s) + \bar{A}(\delta)$ for $\hat{s}_{r2}(\hat{t}_2) \leqslant s \leqslant 1$.

We can now describe the general procedure for obtaining the optimal harvest schedule in general for the class of p and c considered in this section. Set $T(s) = t_k$ for each path segment $s_{lk} \leqslant s \leqslant s_{rk}$ where $T_0'(s) < 0$, with the segments ordered in increasing path length so that $s_{lk} > s_{r(k-1)}$. Find s_{lk}^* and s_{rk}^*, with $(s_{rk-1} \leqslant)\, s_{lk}^*(t_k) < s_{lk}$ and $s_{rk} < s_{rk}^*(t_k)\,(\leqslant s_{lk+1})$, by $T_0(s_{lk}^*) = T_0(s_{rk}^*) = t_k - \bar{A}(\delta)$. The optimal harvest schedule is then given by

$$T(s) = \begin{cases} t_k & (s_{lk}^* \leqslant s \leqslant s_{rk}^*), \\ \bar{T}_0(s) + \bar{A}(\delta) & (\text{otherwise}) \end{cases} \tag{4.15}$$

for $k = 1,2,\ldots,$ where t_k is determined by (4.12) with the lower and upper limits of integration replaced by $s_{lk}^*(t_k)$ and $s_{rk}^*(t_k)$, respectively.

If it should turn out that $s^*_{lk} < s^*_{r(k-1)}$, then we take $T(s) = \hat{t}_k$ for $\hat{s}_{l(k-1)} \leqslant s \leqslant \hat{s}_{rk}$ $(s^*_{r(k-2)} \leqslant \hat{s}_{l(k-1)} < s^*_{l(k-1)}$ and $s^*_{rk} < \hat{s}_{rk} \leqslant s^*_{l(k+1)})$, with $\hat{s}_{l(k-1)}$ and $\hat{s}_{rk}$ determined by $T_0(\hat{s}_{l(k-1)}) = T_0(\hat{s}_{rk}) = \hat{t}_k - \bar{A}(\delta)$. The constant $\hat{t}_k$ is determined by (4.12) with $\hat{s}_{l(k-1)}$ and $\hat{s}_{rk}$ as the lower and upper limit of integration, respectively. This' adjustment process is to be repeated whenever the ordered-site condition is violated, e.g., either $\hat{t}_k < T(s)$ for $s \leqslant \hat{s}_{l(k-1)}$ or $\hat{t}_k > T(s)$ for $s \geqslant \hat{s}_{rk}$.

5. Maximum feasible harvest rate and discontinuous age distributions

5. I. Maximum feasible harvest rate

In reality, a logging company is usually faced with a maximum feasible harvesting rate corresponding to a positive lower bound τ on $T'(s)$. Whether the optimal harvesting policy is given by the interior solution (3.8) now depends on whether the inequality constraint $T' \geqslant \tau > 0$ is binding. Again, we limit ourselves in this section to the case $p = p(A)$ and $c = c_0$, so that we have $\partial\mathcal{H}/\partial u \equiv 0$ and $\lambda \equiv 0$ for the interior solution. The corresponding harvesting schedule, $T(s) = \bar{A}(\delta) + T_0(s)$, is again a consequence of Fisher's rule (4.2). Consider first the case of a uniform initial age distribution so that $T_0'(s) \equiv 0$. For this case, we have $T'(s) = T_0'(s) \equiv 0$ for the interior solution, so that the inequality constraint is binding and the optimal harvest schedule is $T'(s) = \tau$ or

$$T = \tau s + t_0 \tag{5.1}$$

for $0 \leqslant s \leqslant 1$. For simplicity, we discuss only the case $t_0 \geqslant 0$, as the case $t_0 < 0$ can be analyzed in a way analogous to the same situation for $\tau = 0$ in Section 4. The adjoint variable is then given by (3.11), and the transversality conditions are satisfied by a value of t_0 determined by the integral condition (3.12). For the class of p and c considered here, (3.12) becomes

$$[p(\tau + t_0 - T_0) - c_0]e^{-\delta(\tau + t_0)} = [p(t_0 - T_0) - c_0]e^{-\delta t_0}. \tag{5.2}$$

Therefore, the optimal harvest schedule is to harvest at the maximum feasible rate $1/\tau$ (or minimum time per unit area, τ) throughout the entire forest [see (5.1)] starting at t_0 when the present value of the net revenue of the first tree cut equals that of the last tree cut.[3] This optimal harvesting policy is identical to that obtained in [7], as it should be. (Our method of derivation should be compared with that of [7]; the latter is effective mainly for a uniform $T_0(s)$.)

When the initial age distribution of the forest is not uniform, the situation is more complicated but similar to the unconstrained-harvest-rate case discussed in Sections 4.2-4.4.

(i) $T_0'(s) \geqslant \tau$. The optimal harvest schedule in this case is again given by the interior solution $T(s) = T_0(s) + \bar{A}(\delta)$, with the same forest economic interpreta-

[3] It is not difficult to show $t_0 < \bar{A}(\delta)$.

tion. The optimal time to harvest a particular tree is when it reaches the Fisher age $\bar{A}(\delta)$ determined by (4.2).

(ii) $T_0'(s) \leqslant \tau$. For the case $T_0'(s) < \tau$ (except possibly at isolated points), the inequality constraint is binding throughout the solution domain and we have $T(s) = \tau s + t_0$ with t_0 chosen so that the integral condition (3.12) is satisfied. As such, the optimal harvest schedule for this case is qualitatively identical to a forest with a uniform initial age distribution. As in that case, the (transversality) condition (3.12) may be written as

$$\frac{\int_0^1 \dot{p}(\tau s + t_0 - T_0(s))\, e^{-\delta(\tau s + t_0)}\, ds}{\int_0^1 [p(\tau s + t_0 - T_0(s)) - c_0]\, e^{-\delta(\tau s + t_0)}\, ds} = \delta. \tag{5.3}$$

But unlike the uniform-&(s) case, the integrals cannot be evaluated explicitly except for simple forms of $T_0(s)$. Even as it stands, the condition (5.3) has a simple economic interpretation. Logging should begin at the instant t_0 when the relative discounted net marginal yield (i.e., the discounted marginal yield as a fraction of the discounted net future revenue) of the entire forest equals the discount rate. Once started, the logging should proceed at the maximum feasible rate $1/\tau$.

For the case where $T_0'(s) = \tau$ for one or more segments of the logging path, the argument leading to the final solution is similar to that of Section 4.3 for the case $T_0'(s) = 0$ for one or more segments of the logging path. The optimal harvest schedule is again $T(s) = \tau s + t_0$ with t_0 determined by (5.3).

(iii) $T_0'(s) - \tau$ *changes sign along the logging path.* It is now clear that the solution for the $\tau > 0$ case is exactly the same as that for the $\tau = 0$ case of Section 4.4 with $T_0(s)$ replaced by $T_0(s) - \tau s$ in the results there. Thus, we have effectively described the optimal harvest schedule for all possible continuous distributions of $T_0(s)$ for the class of *p* and c considered here. Discontinuous initial age distributions will be considered briefly in Section 5.2.

5.2. *Discontinuous initial age distributions*

For a discontinuous initial age distribution, we obtain the optimal harvest policy for a one-parameter family of continuous age distributions which has as its limiting behavior the given discontinuous distribution. For an age distribution with only simple jump discontinuities (which is the only kind we expect to encounter in practice), the limiting behavior of the optimal policy for the related continuous distributions should give the correct solution for the discontinuous distribution. A specific example- suffices to illustrate the procedure.

For the discontinuous initial distribution

$$T_0(s) = \begin{cases} -t_n & (0 \leqslant s < s_i), \\ -t_f & (s_i < s \leqslant 1), \end{cases} \tag{5.4}$$

we consider the one-parameter (ϵ) family of continuous initial distributions

$$\hat{T}_0(s;\epsilon) = \begin{cases} -t_n & (0 \leqslant s \leqslant s_i - \epsilon), \\ -t_n - \dfrac{s - s_i + \epsilon}{2\epsilon}(t_f - t_n) & (s_i - \epsilon \leqslant s \leqslant s_i + \epsilon), \\ -t_f & (s_i + \epsilon \leqslant s \leqslant 1), \end{cases} \tag{5.5}$$

where $0 < \epsilon \ll 1$. Evidently, $\hat{T}_0(s;\epsilon)$ is a good approximation of $T_0(s)$ for $\epsilon \ll 1$ and tends to $T_0(s)$ as $\epsilon \to 0$.

If $t_n > t_f$ (old trees first), we have $\hat{T}_0'(s;\epsilon) = 0$ except in the small interval $s_i - \epsilon \leqslant s \leqslant s_i + \epsilon$ where $\hat{T}_0' = -(t_f - t_n)/2\epsilon > 0$. If there is no upper limit to the harvest rate, then we have $T(s;\epsilon) = \hat{T}_0(s;\epsilon) + \bar{A}(\delta)$ for the class of p and c considered in this section. The limiting behavior of $T(s;\epsilon)$ as $\epsilon \to 0$ is to harvest all trees when they reach the Fisher age, which we expect to be the correct optimal policy.

The more interesting case is $t_n < t_f$ (young trees first), again with $\tau = 0$. For the continuous distribution $\hat{T}_0(s;\epsilon)$, the results of Section 4.2 give immediately $T(s;\epsilon) = t_0$ with t_0 determined by the condition [see Figure 2(b)]

$$\frac{e^{-\delta t_0}\left\{\dot{p}(t_0+t_n)(s_i-\epsilon)+\dot{p}(t_0+t_f)(1-s_i-\epsilon)+2\epsilon\dfrac{p(t_0+t_f)-p(t_0+t_n)}{t_f-t_n}\right\}}{e^{-\delta t_0}\left\{[p(t_0+t_n)-c_0](s_i-\epsilon)+[p(t_0+t_f)-c_0](1-s_i-\epsilon)+\int_{s_i-\epsilon}^{s_i+\epsilon}[p(t_0-T_0(s))-c_0]\,ds\right\}} = \delta, \tag{5.6}$$

which follows from (4.4) and (5.5). As $\epsilon \to 0$, the condition (5.6) tends to

$$\frac{\{s_i\dot{p}(t_0+t_n)+(1-s_i)\dot{p}(t_0+t_f)\}e^{-\delta t_0}}{\{s_i[p(t_0+t_n)-c_0]+(1-s_i)[p(t_0+t_f)-c_0]\}e^{-\delta t_0}} = \delta \tag{5 7}$$

Given that there are only young trees of the same age along the near end of the path $(s < s_i)$ and only old trees of the same age along the far end $(s_i < s)$, the condition (5.7) continues to have the same economic content as (4.4), which applies only to continuous initial age distributions. To deduce the same result for the discontinuous $T_0(s)$ without using $\hat{T}_0(s;\epsilon)$ as an intermediate step would require a more elaborate form of the (conventional) maximum principle.

The solution procedure described for the discontinuous distribution (5.4) can be used for other initial age distributions with simple jump discontinuities and also when there is a maximum feasible harvest rate. We will not pursue a discussion along these lines, as we do not anticipate any unusual complications.

6. Unit harvest cost varying with harvest rate

6.1. U-shaped *unit harvest cost*

Instead of c being a constant, we consider in this section unit harvest cost functions which depend only on T' with the conventional U-shaped graph, i.e., $c(T') > 0$ is convex in T' with a minimum at $T' = \tau_{\min} > 0$ (so that $\min[c(T')] = c(\tau_{\min})$). This class of unit cost functions includes both the effect of a fixed cost component and an overload cost component. We limit ourselves here to cost functions which become unbounded as $u \equiv T'$ tends to infinity.

With $p = p(A) \equiv p(T - T_0)$ as before, we have from (3.4) and $V = p - c$ that $\partial\mathcal{H}/\partial u = -e^{-\delta T}\dot{c}(u) + \lambda$, where $u = T'$ and a dot on top of a function indicates differentiation with respect to its argument. An interior solution of the optimal-control problem requires

$$\lambda(s) = e^{-\delta T}\dot{c}(u) = e^{-\delta T(s)}\dot{c}(T'(s)), \tag{6.1}$$

with the transversality conditions $\lambda(0) = \lambda(1) = 0$ satisfied by taking

$$T'(0) = T'(1) = \tau_{\min}. \tag{6.2}$$

[No other choice is possible, as c has a unique stationary (minimum) point and $e^{-\delta T}$ never vanishes.] Thus, we have reproduced and extended the principal result of [7] for the same class of problems, namely, *harvesting should start and end with a harvest rate giving a minimum unit harvest cost.* The extension of the result of Heaps and Neher consists of removing the restriction of unlimited harvest capacity and instituting the requirement of ordered site access. If there is an upper bound to the feasible harvest rate, the interior solution (6.1) with $T'(0) = T'(1) = \tau_{\min}$ may not be appropriate and the inequality constraint on the control $u \equiv T'$ may be binding. We shall return to a discussion of the optimal solution for the case of limited harvest capacity later.

From (6.1) and the qualitative behavior of the class of $c(T')$ of interest here, we see that T' remains positive along the entire logging path for the interior solution, independent of the initial age distribution. Therefore, the optimal harvest schedule is determined by inserting (6.1) into the differential equation (3.5) for the adjoint variable λ and solving the resulting second-order ODE for T with the boundary conditions $T'(0) = T'(1) = \tau_{\min}$. As an alternate solution process, we may solve (6.1) for T' to get the unique solution

$$T' = f(\lambda e^{\delta T}) \tag{6.3}$$

(since $\dot{c}$ is a monotone increasing function of its argument), and then write (3.5) as

$$\lambda' = -\{\dot{p}(T - T_0) - \delta p(T - T_0) + \delta c(f(\lambda e^{\delta T}))\}e^{-\delta T}. \tag{6.4}$$

The second-order system of two first-order equations, (6.3) and (6.4), and the two transversality conditions, $\lambda(0)=\lambda(1)=0$, define a two-point boundary-value problem for $\lambda(s)$ and $T(s)$ and can be solved by available methods.

Before moving on to a discussion of the effect of limited harvesting capacity, we note that (6.4) and $\lambda(0)=0$ yield

$$\lambda(s) = -\int_0^s\{\dot{p}(T-T_0)-\delta[p(T-T_0)-c(T')]\}e^{-\delta T}ds. \tag{6.5}$$

The condition $\lambda(1)=0$ then gives

$$\frac{\int_0^1\dot{p}(T-T_0)e^{-\delta T}ds}{\int_0^1[p(T-T_0)-c(T')]e^{-\delta T}ds} = \delta. \tag{6.6}$$

Whether we have a uniform initial age distribution or an upper bound on the harvest rate, *the optimalpolicy logs the forest on a schedule which makes the relative discounted marginal net yield of the forest equal the discount rate with the first and last tree sites logged at the rate* $1/\tau_{\min}$ *(for a minimum unit harvest cost there).* For simplicity, we have tacitly assumed $T(s)\geqslant 0$; otherwise, the entire path segment $0\leqslant s\leqslant\bar{s}$ where T(s) < 0 should be clear-cut immediately (for the case $T'\geqslant\tau=0$).

When there is an upper limit to the harvesting capacity (a lower bound $\tau>0$ on T'), the optimal harvest schedule depends on the sign of $T'-\tau$. The optimal harvest schedule continues to be the interior solution defined by the two-point boundary-value problem (6.3), (6.4), and (3.6) if $T'\geqslant\tau$ for the entire forest. The situation is more complicated if $T'<\tau$ for some portion of the path. For example, if $\tau_{\min}<\tau$, then the inequality constraint $T'\geqslant\tau$ is binding for an initial segment of the logging path $0\leqslant s\leqslant\bar{s}$, so that we have

$$T(s) = \tau s + t_0 \tag{6.7}$$

and

$$\lambda(s) = -\int_0^s\{\dot{p}(\tau s+t_0-T_0)-\delta p(\tau s+t_0-T_0)+\delta c(\tau)\}e^{-\delta(\tau s+t_0)}ds \tag{6.8}$$

there. However, for $\bar{s}\leqslant s\leqslant 1$, the condition (6.1) is admissible and the optimal harvest policy satisfies (6.3) and (6.4) with $\lambda(1)=0$. The two unknown parameters t_0 and $\bar{s}$ are determined by the continuity[4] of λ and T at the junction $s=\bar{s}$. A similar procedure for determining the optimal harvest schedule applies when $T'-\tau$ becomes negative in one or more segments of the logging path which may or may not include an end point.

[4] With $c(u)\to$ oo as u $\to\infty$, T must be continuous for all s in (0,l); the continuity of λ follows.

6.2. Linear unit cost function and singular solution

Among the unit cost functions $c(T')$ with a qualitatively different dependence on the harvest rate than the conventional U-shaped ones considered in Section 6.1, one class deserves some discussion. When overloading is not an issue, the unit harvest cost may be a linear function of T', i.e., $c = c_0 + c_f T'$. With unit price again a monotone increasing concave function of tree age ($A \equiv T - T_0$) only, and with $p - c \equiv [p(A) - c_0] - c_f T'$, the Hamiltonian may be written as

$$\mathcal{H} \equiv p_0(A)e^{-\delta T} + \left(A - c_f e^{-\delta T}\right)u, \tag{6.9}$$

where

$$T' \equiv u, \qquad p_0(A) \equiv p(A) - c_0. \tag{6.10}$$

The differential equation for the adjoint variable A becomes

$$\lambda^{I} = -e^{-\delta T}\left[\dot{p}_0 - \delta p_0 + \delta c_f u\right]. \tag{6.11}$$

Note that $\partial\mathcal{H}/\partial u \neq 0$ for any u unless $\lambda(s)$ and $T(s)$ satisfy the relation

$$\lambda(s) - c_f e^{-\delta T} \equiv 0. \tag{6.12}$$

We call the solution which satisfies (6.12) for all s in $[0,1]$ the *singular solution* and denote it by $\lambda_s(s)$ and $T_s(s)$. Evidently, we have $\partial\mathcal{H}/\partial u \equiv 0$ for the singular solution.

Upon differentiating both sides of the singular solution (6.12), we get

$$\lambda_s' + c_f \delta T_s' e^{-\delta T_s} \equiv 0$$

or, in view of (6.11) for A',

$$\frac{\dot{p}_0(A)}{p_0(A)} \equiv \frac{\dot{p}(A)}{p(A) - c_0} = \delta. \tag{6.13}$$

The singular solution therefore requires the harvest schedule to satisfy a *modified Fisher rule* in which the fixed cost component $c_f T'$ is excluded from the unit harvest cost in the calculation of the relative marginal yield. The condition (6.13) determines a *modified Fisher age* $A_s(\delta)$ for each tree site, and the *singular solution* is simply $T_s(s) = A_s(\delta) + T_0(s)$ with $T_s'(s) = T_0'(s)$. The corresponding harvest schedule is to log a tree when it reaches the modified Fisher age.

The singular solution by itself cannot be the optimal harvest schedule since the corresponding adjoint variable $\lambda_s(s)$ [see (6.12)] does not satisfy the two transversality conditions A(O) = A (1) = 0. Therefore, at least in a segment adjacent to each of the end points of the logging path, the current shadow price $\lambda e^{\delta T}$ is not a constant. Since $T' \equiv u \geqslant \tau\ (\geqslant 0)$ and $\lambda - c_f e^{-\delta T} < 0$ near the end points [because of the transversality conditions $\lambda(0) = \lambda(1) = 0$], the Hamiltonian $\mathcal{H}$ is maximized if u is minimized. Therefore, $T(s) = \tau s + t_0$ whenever it is not the singular solution $T_s(s)$.

For a uniform initial age distribution and a limited harvesting capacity so that $T' \geqslant \tau > 0$, we have

$$T(s) = \tau s + t_0 \qquad (0 \leqslant s \leqslant 1), \tag{6.14}$$

as the singular solution $T_s(s) = A_s(\delta) + T_0$ violates the inequality constraint for the entire path. With (6.14), we get from (6.11) and $\lambda(0) = 0$

$$\lambda(s) = \left[e^{-\delta(\tau s + t_0)} \left\{ c_f - \frac{p_0(\tau s + t_0 - T_0)}{\tau} \right\} \right]_0^s. \tag{6.15}$$

The condition $\lambda(1) = 0$ yields

$$e^{-\delta(\tau + t_0)} \{ p(\tau + t_0 - T_0) - c(\tau) \} = e^{-\delta t_0} \{ p(t_0 - T_0) - c(\tau) \}, \tag{6.16}$$

which determines t_0. The condition (6.16) again requires logging to start at the instant t_0 (with the same maximum feasible harvest rate $1/\tau$ thereafter) when the discounted future net revenue of the first tree cut equals the same present value of the last tree cut.

When there is no limit on the harvest rate, so that $\tau = 0$, we have $T' \equiv 0$ whether the inequality constraint on the harvest rate $(T' \geqslant 0)$ is or is not binding [given $T_s' = (A_s(\delta) + T_0)' \equiv 0$]. Therefore, we get the same optimal harvesting schedule for the entire path without insisting on the singular solution. (This is not true for a nonuniform initial age distribution.) With $T' \equiv 0$ and $T = t_0$ so that the entire forest is clear-cut instantaneously at some t_0, the equation for the (discounted) shadow price (6.11) and the transversality condition $\lambda(0) = 0$ imply

$$\lambda(s) = -e^{-\delta t_0} [\dot{p}_0(t_0 - T_0) - \delta p_0(t_0 - T_0)] s. \tag{6.17}$$

The remaining transversality condition $\lambda(1) = 0$ gives a condition on t_0 for the singular solution with the same economic content as (6.13). Again, logging should begin when the trees reach the modified Fisher age $A_s(\delta)$, and the entire forest is to be clear-cut instantaneously. [With (6.13), we have from (6.17) $\lambda(s) \equiv 0$, which also follows from (6.15) if we set $\tau = 0$.]

We now turn to the case of a nonuniform $T_0(s)$ and examine separately the three cases: (i) $T_0' \leqslant \tau$, (ii) $T_0' \geqslant \tau$, and (iii) T_0' changes sign along the logging path.

Case (i) is easily disposed of, since the inequality constraint $T' \geqslant \tau$ is binding. The optimal harvest schedule is $T(s) = \tau s + t_0$ with t_0 determined by (6.11) and

$\lambda(0) = \lambda(1) = 0$. In particular, logging should begin at the instant t_0 when we have

$$\frac{\int_0^1 e^{-\delta T}\dot{p}(T-T_0)\,ds}{\int_0^1 e^{-\delta T}\left[p(T-T_0)-c_0-c_f\tau\right]ds} = \delta \tag{6.18}$$

and should be at the maximum feasible harvest rate.

The singular solution $T_s = A_s(\delta)+T_0(s)$ is admissible for *case (ii)* (since $T_s' = T_0' \geqslant \tau$) except near the two ends of the path. Therefore, we have

$$T(s) = \begin{cases} \tau s + t_1 & (0 \leqslant s < s_1), \\ T_s(s) \equiv A_s(\delta)+T_0(s) & (s_1 < s < s_2), \\ \tau s + t_2 & (s_2 < s \leqslant 1). \end{cases} \tag{6.19}$$

From (6.11) [or (6.12)], we get the corresponding $\lambda(s)$ as

$$\lambda(s) = \begin{cases} \left[\frac{1}{\tau}e^{-\delta(\tau s+t_1)}\{c_f\tau - p_0(\tau s + t_1 - T_0(s))\}\right]_0^s & (0 \leqslant s \leqslant s_1), \\ \lambda_s(s) \equiv c_f e^{-\delta T_s} & (s_1 < s < s_2), \\ \left[\frac{1}{\tau}e^{-\delta(\tau s+t_2)}\{c_f\tau - p_0(\tau s + t_2 - T_0(s))\}\right]_s^1 & (s_2 \leqslant s \leqslant 1). \end{cases} \tag{6.20}$$

By the ordered-site property, we must have $\tau s_1 + t_1 \leqslant A_s(\delta)+T_0(s_1)$ and $A_s(\delta)+T_0(s_2) \leqslant \tau s_2 + t_2$. For $T(s)$ to be as close to the interior (singular) solution as possible, we take $\tau s_1 + t_1 = A_s(\delta)+T_0(s_1)$ and $\tau s_2 + t_2 = A_s(\delta)+T_0(s_2)$, or

$$t_1 = A_s(\delta) + T_0(s_1) - \tau s_1, \qquad t_2 = A_s(\delta) + T_0(s_2) - \tau s_2. \tag{6.21}$$

The continuity of T at s_1 and s_2 requires that λ be continuous there also, so that

$$\left[\frac{1}{7}e^{-\delta[\tau s + t_1(s_1)]}\{c(\tau) - p(\tau s + t_1(s_1) - T_0(s))\}\right]_0^{s_1} = c_f e^{-\delta T_s(s_1)},$$
$$\left[\frac{1}{7}e^{-\delta[\tau s + t_2(s_2)]}\{c(\tau) - p(\tau s + t_2(s_2) - T_0(s))\}\right]_{s_2}^{1} = c_f e^{-\delta T_s(s_2)}. \tag{6.22}$$

The two conditions in (6.22) determine s_1 and s_2 and thereby the optimal harvest schedule for this case.

The situation for *case (iii)* is much more complicated, as the optimal harvest schedule consists of the singular solution T_s over segments of the logging path, and corner solutions $\tau s + t_k$, $k = 1,2,\ldots$, over the rest of the path. The actual location of those path segments is determined by continuity conditions on $T(s)$ and the corresponding $\lambda(s)$ at the junctions of contiguous path segments. To the extent that the actual solution process is analogous to those described in Sections 4.4 and 5.1 for the case $c_f = 0$, it is not necessary to discuss this case in more detail.

6.3. *An exact solution for a class of unit cost functions*

It is of some interest to note that an exact solution of the optimal harvest schedule is possible for the following class of U-shaped unit harvest cost functions:

$$c(u) = c_f u + c_0 + \frac{c_l}{u}, \tag{6.23}$$

where c_f, c_0, and c_l are known non-negative constants corresponding to the fixed cost component, the constant unit harvest cost component, and the overload cost component, respectively. The U-shaped $c(u)$ has a unique minimum at $u = \sqrt{c_l/c_f} \equiv \tau_{\min}$, and

$$\lambda = \dot{c}(u)e^{-\delta T} = \left(c_f - \frac{c_l}{u^2}\right)e^{-\delta T} \tag{6.24}$$

vanishes at $u = T' = \sqrt{c_l/c_f}$. Upon differentiating (6.24) with respect to s and using the resulting expression to eliminate λ' from (6.4), we get (with $u = T'$) a second order differential equation for $T(s)$ which may be written as

$$\frac{T''}{(T')^2} + \delta = \frac{T'}{2c_l}\{\delta[p(T-T_0)-c_0] - \dot{p}(T-T_0)\}. \tag{6.25}$$

If the initial age distribution is *uniform*, so that $T_0(s)$ is a constant, (6.25) may be integrated once immediately to give

$$2c_l\delta s + 2c_l\left(\frac{1}{\tau_{\min}} - \frac{1}{T'}\right)$$

$$= p(T(0)-T_0) - p(T-T_0) + \delta\int_{T(0)}^{T}[p(\xi-T_0)-c_0]\,d\xi, \tag{6.26}$$

where the transversality condition $T'(0) = \tau_{\min} \equiv \sqrt{c_l/c_f}$ has been used to express the constant of integration in terms of the yet unknown **starting** time of the harvest schedule, $T(0)$ (which is not to be confused with the initial age distribution T_0). The other transversality condition $T'(1) = \tau_{\min}$ becomes

$$\frac{\left[p(T - T_0)\right]_{T(0)}^{T(1)}}{\int_{T(0)}^{T(1)}\left[p(\xi - T_0) - c_0\right] d\xi - 2c_l} = \delta; \tag{6.27}$$

it determines $T(1)$ in terms of the unknown starting time $T(0)$. To get the actual optimal harvest schedule $T(s)$, we consider for simplicity only the case $\tau_{\min} \geq \tau$. In that case we get $T(s)$ by solving the first-order ODE (6.26) for $T(s)$. While an exact solution for $T(s)$ does not seem possible, an exact solution of $s(T)$ is straightforward once we make use of the relation $1/T' = ds/dT$ to transform (6.26) into a first-order *linear* ODE for s as a function of T:

$$\frac{ds}{dT} - \delta s = \frac{1}{\tau_{\min}} + \frac{1}{2c_l}\left\{ p(T - T_0) - p(T(0) - T_0) - \delta\int_{T(0)}^{T}\left[p(\xi - T_0) - c_0\right] d\xi\right\}. \tag{6.28}$$

The exact solution of (6.28) is

$$2c_l\delta s = \left[\frac{2c_l}{\tau_{\min}} - p(T(0) - T_0) + c_0\right]\left[e^{\delta[T - T(0)]} - 1\right] + \delta\int_{T(0)}^{T}\left[p(\xi - T_0) - c_0\right] d\xi, \tag{6.29}$$

where we have made use of $s(T(0)) = 0$ to eliminate the constant of integration. Finally the condition $s(T(1)) = 1$ gives the relation

$$2c_l\delta = \left[\frac{2c_l}{\tau_{\min}} - p(T(0) - T_0) + c_0\right]\left[e^{\delta[T(1) - T(0)]} - 1\right] + \delta\int_{T(0)}^{T(1)}\left[p(\xi - T_0) - c_0\right] d\xi. \tag{6.30}$$

With $T(1)$ already determined in terms of $T(0)$ by (6.27), the condition (6.30) is an equation for the single unknown constant $T(0)$. Alternately, we may write (6.27) as

$$\frac{p(T(1) - T_0) - p(T(0) - T_0)}{P_0(T(1) - T_0) - P_0(T(0) - T_0) - 2c_l} = \delta, \tag{6.31}$$

where

$$P_0(x) \equiv \int_0^x\left[p(t) - c_0\right] dt,$$

and use (6.27) itself to simplify (6.30) to get

$$\left\{\frac{2c_l}{\tau_{\min}}-\left[p\left(T(0)-T_0\right)-c_0\right]\right\}\left\{e^{-\delta[T(0)-T_0]}-e^{-\delta[T(1)-T_0]}\right\}$$

$$=e^{-\delta[T(1)-T_0]}\left[p\left(T(0)-T_0\right)-p\left(T(1)-T_0\right)\right]. \tag{6.32}$$

The two equations (6.31) and (6.32) simultaneously determine T(O)- T_0 and $T(1)-T_0$ once $p(\cdot)$ is specified. When we have $T(0)$, Equation (6.29) then gives s as a function of T. This function must be monotone increasing by the ordered-site requirement. Therefore, we get a well-defined inverse function $T(s)$, the optimal harvest schedule sought.

The above solution process for the special class of problems treated in this section involves considerably less machine computation than that required by a direct numerical solution of the BVP for λ and T. The solution process has also enabled us to deduce the simple dependence of $T(s)$ on the uniform initial age of the trees, T_0. From (6.29), we see that a change of T_0 simply shifts the entire $T(s)$ curve upward or downward. A direct numerical solution is generally necessary for problems with a nonuniform $T_0(s)$ and/or a more general cost function $c(u)$.

6.4. *Numerical results*

To gain some insight into the effect of the economic and biological parameters such as the discount rate δ, the size of the forest F_0 (the number of trees), etc., numerical results have been generated for a unit price function of the form

$$p(A)=p_0F_0\left(1-\sigma e^{-A/A_0}\right) \tag{6.33}$$

and the unit harvest cost function (6.23). For our purpose, it is more convenient to take this cost function in the form

$$c(u)=C_fu+C_0F_0+\nu C_0F_0^2u^{-1} \tag{6.34}$$

with $c_f\equiv C_f$, $c_0=C_0F_0$, and $c_l=\nu C_0F_0^2$ so that

$$\tau_{\min}=F_0\sqrt{\frac{\nu C_0}{C_f}},\qquad c(\tau_{\min})=F_0\left[C_0+2\sqrt{\nu C_0C_f}\right]. \tag{6.35}$$

For the numerical results presented here, we have fixed

$$p_0 = \$1200/(\text{ tree})\ (\text{unit path length}),$$

$$C_0 = \$200/(\text{ tree})\ (\text{unit path length}),$$

$$C_f = \$50/\text{hr} = \$100{,}000/\text{yr},$$

$$\sigma = \tfrac{5}{4},$$

$$A_0 = 60 \text{ yr},$$

$$\nu = 5 \times 10^{-5}\ \text{yr/(tree)(unit path length)}.$$

For the choice of parameter values used to generate the numerical results, we have $\tau_{\min} = F_0 \times 10^{-4}$ yr/(unit path length). With the logging path normalized to unit length, $\tau_{\min}$ is also the time required to harvest the entire forest at a rate which gives a minimum unit harvesting cost. For our set of parameter values, this rate would log 100,000 trees in 10 years time, which should not strain the available harvesting capacity of a big logging company. Therefore, we will assume $\tau_{\min} > \tau$, and that the interior solution is appropriate. Harvesting at the most efficient rate, the parameter values used here give a net profit of nearly \$750 for a 120 year-old tree and no net profit for trees which are younger than 29 years. The corresponding Fisher age is 37.67. . ., 59.07. . ., and 87.26. . . for $\delta = 0.1$, 0.025, and 0.0 1, respectively.

In arriving at (6.33) and (6.34) for our continuous model of a forest with discrete tree stands, we have let the land area assigned to each tree of the forest be A; the net discounted revenue from the stumpage of a logging path increment $(\sigma, \sigma + d\sigma)$ within an assigned area for a single tree before path-length *normalization* is taken to be $(p_1 - c_1)e^{-\delta T} d\sigma/\Delta$, where p_1 and c_1 are the gross price function and harvest cost function, respectively, for a single tree stand. To normalize the entire path to unit length, we set $\sigma = F_0 \Delta s$ so that the present value of net revenue from the entire forest is

$$P = \int_0^{F_0\Delta} \frac{1}{\Delta}(p_1 - c_1)e^{-\delta T} d\sigma = \int_0^1 (p - c)e^{-\delta T} ds$$

with $p - c \equiv F_0(p_1 - c_1)$.

(a) Uniform initial age distribution. We have taken $T_0 \equiv 0$ in all cases presented here. As previously observed, a different (uniform) initial tree age simply shifts the graph of $T(s)$ upward or downward.

In Figure 4, we show the graph of T' and T for $\delta = 10\%$ for three different forests: $F_0 = 10^3, 10^4$, and 10^5. We see from these graphs that small forests are logged at a nearly uniform rate close to $\tau_{\min}$ along the entire logging path. Here, $\tau_{\min}$ is sufficiently small so that trees are logged at nearly their Fisher age. Large forests are logged at a rate substantially faster than $\tau_{\min}$ away from the two ends

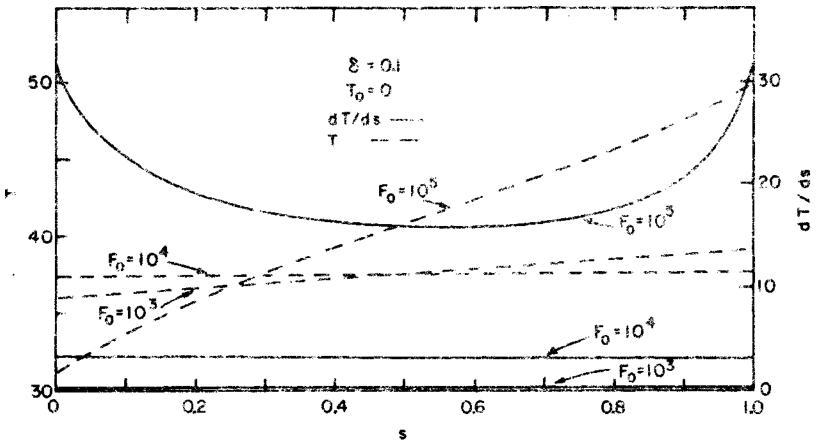


Figure 4. Optimal harvest schedules for different-size forests with a uniform initial age distribution.

of the logging path. But to avoid excessively high harvest cost, the optimal harvest rate results in a 19-year gap between the first and the last tree harvested. Thus, the first tree is logged well before its Fisher age.

Figure 5 shows the effect of discount rate δ on T' and T. As δ increases from 2.5% to 10%, the harvest rate away from the end points increases, reflecting the fact that we should harvest the trees sooner when the discount rate is high.

The above numerical results were generated by the exact solution of Section 6.3 and then confirmed by solving the BVP for T and λ [defined by (6.3) and (6.4) for $0 \leqslant s \leqslant 1$ with boundary conditions $\lambda(0) = \lambda(1) = 0$] numerically by the spline-collocation BVP solver COLSYS (see [13]). The close agreement between the two sets of results serves as a check for our calling program for COLSYS, which is needed for problems with nonuniform initial age distributions below.

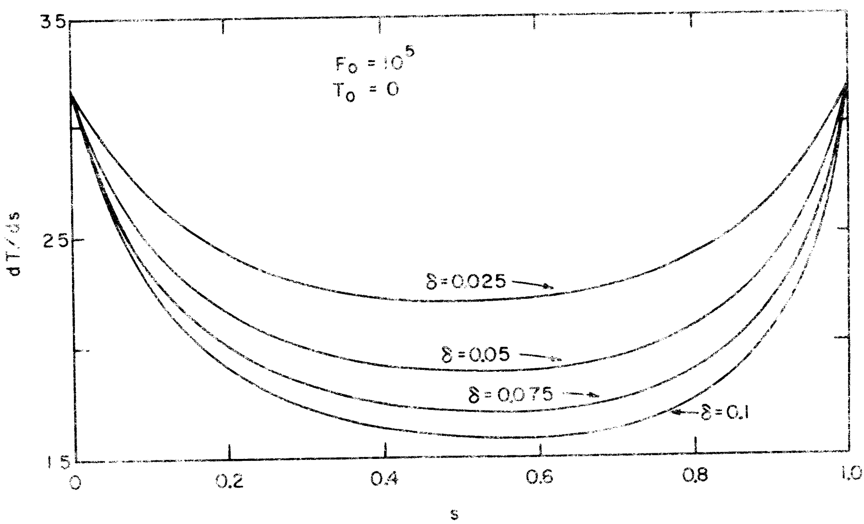
(b) Nonuniform initial age distribution. The results for three different types of initial age distributions $T_0(s)$ will be described below, and the qualitative features of the corresponding harvest schedules will be noted. The numerical results were generated by the BVP solver COLSYS with four-significant-figure accuracy.

(i) $T_0(s) = \alpha s$: In Figure 6(a) and (b), graphs of T' are shown for different values of the parameter α, $-30 \leqslant \alpha \leqslant 30$, and for $\delta = 0.1$ and $\delta = 0.025$, respectively. For $\delta = 0.10$ [Figure 6(a)], the harvest rate stays close to τ_{min} throughout the logging path when $\alpha = 30$, logging the trees near their Fisher age (see Table 1). As α decreases T' deviates more and more from τ_{min} away from the end points. For $\alpha < 0$ the ordered-site-access constraint prevents the older trees located near the end of the logging path from being cut first, and a high overload cost constrains the logging company from harvesting the entire forest instantaneously. In order to get to the older trees not too long after their (individual) optimal age, the younger trees at the start of the logging path would have to be harvested well before their Fisher age.

Table 1
Harvest Schedule for the End Trees When $T_0(s) = \alpha s$

	$\delta = 0.10$		$\delta = 0.025$	
α	$T(0) = A(0)$	$A(1) = T(1) - \alpha$	$T(0) = A(0)$	$A(1) = T(1) - \alpha$
−30	22.40	66.57	37.65	88.27
−20	25.42	60.57	41.22	82.88
−10	28.33	55.22	44.79	77.68
0	31.06	49.99	48.34	72.71
10	33.49	45.23	51.83	67.99
20	35.56	41.15	55.20	63.57
30	37.25	37.93	58.40	59.51
$\bar{A}(\delta)$[a]	37.67		59.07	

[a] The Fisher age is calculated with $c = c(\tau_{min})$.



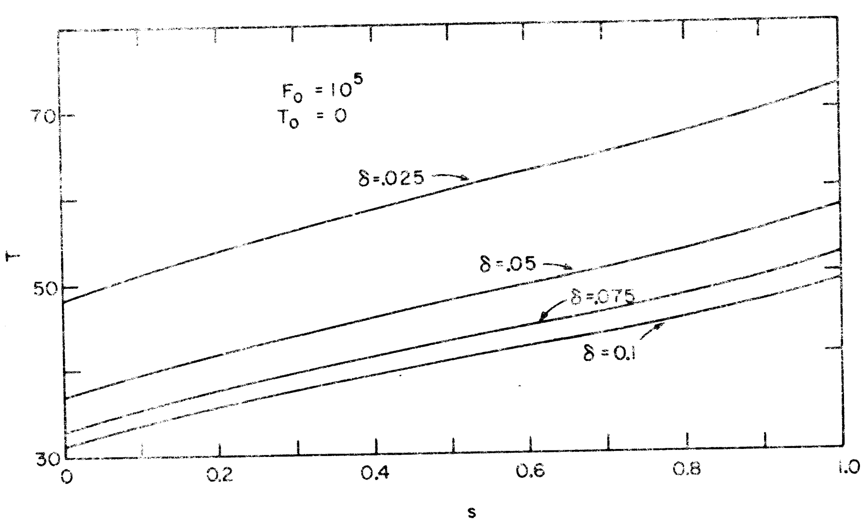


Figure 5. Optimal harvest schedules for different discount rates and a uniform initial age distribution.

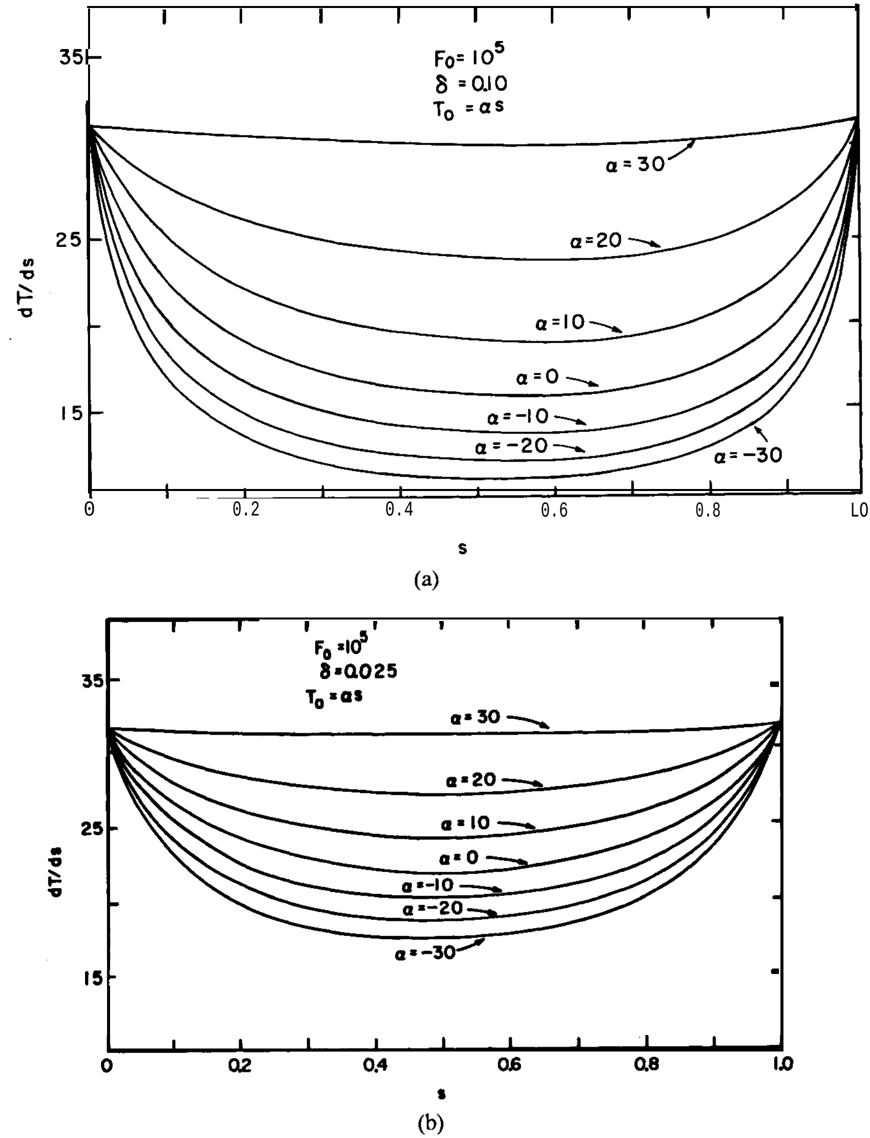


Figure 6. Harvest time per unit site for linear initial age distributions: (a) $\delta = 0.10$ and (b) $\delta = 0.025$.

For a lower discount rate, say $\delta = 0.025$ [see Figure 6(b) and Table 1] the same qualitative features persist but the harvest rate away from the two end points is, for the same α, considerably slower than the harvest rate for $\delta = 0.10$. The difference simply reflects the fact that trees should be cut sooner when the discount rate is higher.

(ii) $T_0(s) = \beta s(s - 1)$. For a negative β, the tree age declines as we move through the first half of the logging path; it then increases with path length thereafter. Thus, we expect the optimal schedule to start logging somewhat later

Table 2
Harvest Schedule for $T_0(s) = \beta s(s-1)$

	$\delta = 0.10$			$\delta = 0.025$		
β	$A(0) = T(0)$	$A(\frac{1}{2}) = T(\frac{1}{2}) + \frac{1}{4}\beta$	$T(1) = A(1)$	$A(0) = T(0)$	$A(\frac{1}{2}) = T(\frac{1}{2}) + \frac{1}{4}\beta$	$A(1) = T(1)$
− 100	42.69	33.5 1	66.62	63.69	53.04	89.37
0	31.06	40.82	49.99	48.34	60.56	72.7 1
100	17.11	49.54	35.48	33.22	68.97	57.28
$\bar{A}(\delta)$[a]		37.67			59.07	

[a]The Fisher age is calculated with $c = c(\tau_{min})$.

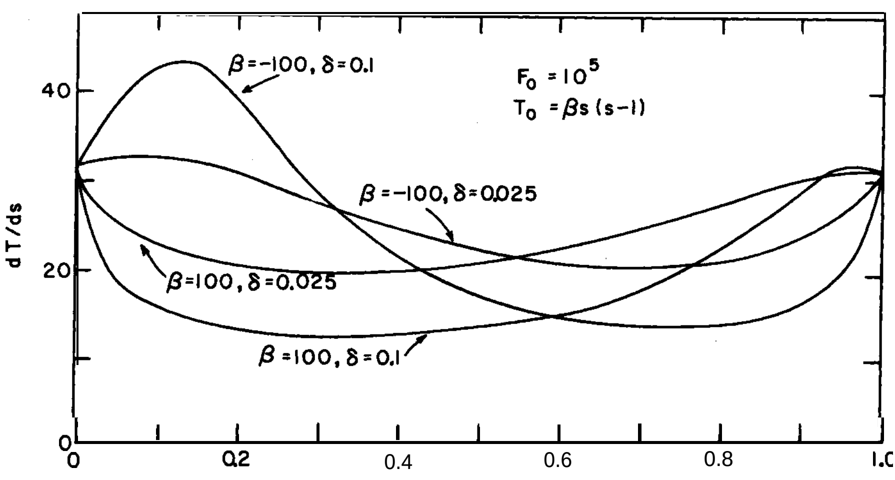


Figure 7. Optimal harvest schedules for quadratic initial age distributions.

than the Fisher age of the trees near the start of the logging path, to allow the trees near the middle to grow. For the trees near the end to be cut not too long after their Fisher age, the trees in the middle would have to be cut a little sooner than their Fisher age (see Table 2). For the one case with a negative β shown in Figure 7, the tree age declines so sharply near the start of the logging path that the harvest rate has to be slower than τ_{min} for a while, allowing time for the young trees to grow.

For positive values of β, tree age increases with path length until the midpoint and then decreases. In order for the trees in the middle not to get too old, the trees near the start of the logging path would have to be harvested well before their Fisher age (see Table 2). Near the end point, a slower harvesting rate (than $\tau_{\min}$) is again necessary to allow the young trees to grow almost to their Fisher age (see Figure 7).

(iii) $T_0(s)=\gamma\tanh([s-\frac{1}{2}]/\epsilon)$. For $\gamma>0$, we have $-T_0'<0$, so that the tree age declines along the logging path. When the initial age profile has no abrupt changes ($\epsilon=1.0$), the harvest schedule (see Table 3 and Figure 8(a)) calls for all trees to be cut around their Fisher age. For the profile chosen, it is necessary to cut the first few trees sooner, and the last few later, than their respective Fisher age. When $-T_0(s)$ has a sharp drop as in the case $\epsilon=0.1$, the optimal harvest schedule allows for this by harvesting much more slowly in the area of abrupt change.

For $\gamma<0$, we have $-T_0'>0$, so that the tree age increases along the logging path. Given the ordered-site-access requirement, the best we can do is to harvest the entire forest instantaneously at an appropriate time. But the prohibitive (overload) cost for doing so forces us to log at a finite but varying rate. In the examples with $\delta=0.1$ shown in Figure 8(b) (see also Table 3), the discount rate is sufficiently large to make it worthwhile to cut down the young trees at the front end of the logging path at a financial loss in order to get to the older trees at the back sooner. Note that the harvest rate varies gradually whether the initial age profile has abrupt changes or not.

7. Concluding remarks

Optimal harvest schedules for unit price and cost functions more general than those considered in Sections 4, 5, and 6 can be analyzed in a straightforward manner within the framework of the formulation of Section 3. No conceptual difficulties are expected, but numerical methods will often be necessary for the actual optimal solution. However, for typical once-and-for-all forest logging, the time scale involved is sufficiently short to render the fluctuation of p and c with

Table 3
Harvest Schedule for $T_0(s)=\gamma\tanh([s-\frac{1}{2}]/\epsilon)$
[with $T(s)=A(s)+T_0(s)$]

		$\delta=0.10$			$\delta=0.025$		
γ	ϵ	$A(0)$	$A(\frac{1}{2})$	$A(1)$	$A(0)$	$A(\frac{1}{2})$	$A(1)$
-25	1.0	24.62	43.80	62.60	40.21	62.34	84.56
	0.1	20.21	51.30	82.00	34.22	68.18	102.22
0		31.06	40.82	49.99	48.34	60.56	72.71
25	1.0	35.94	38.22	40.17	56.10	59.25	62.34
	0.1	34.00	34.15	41.30	57.40	56.94	59.53
$\bar{A}(\delta)=$			37.67			59.07	

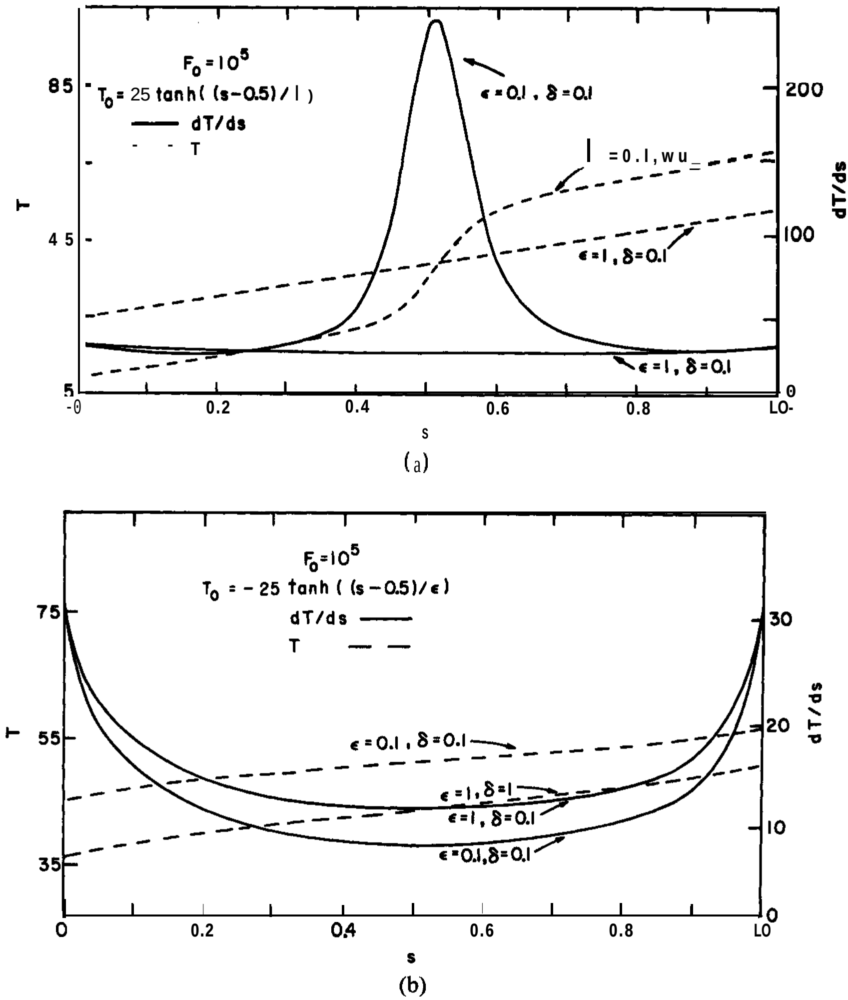


Figure 8. Optimal harvest schedules for $T_0(s) = \gamma \tanh([s-\frac{1}{2}]/\epsilon)$ with (a) $\gamma > 0$, (b) $\gamma < 0$.

time insignificant. Variations of p and c with tree site, as well as the dependence of c on tree age, if significant, are merely fine tuning in our formulation and are not expected to lead to substantive qualitative changes in the optimal policy. Substantive changes are expected if p depends on T', so that the logging company is not just a price taker but has some influence on the market price through the amount of timber it supplies. For this case, optimal harvest rate depends on a proper balance between the unit harvest price and unit harvest cost at that harvest rate. This balance is attained with the current shadow price $\lambda e^{\delta T}$ equal to the marginal yield of net revenue for an extra unit harvest time per tree site.

Our model does have some serious limitations for the case of a price taker. For example, the model requires the entire forest to be harvested eventually. Under where

$$x = \frac{1}{A_0}[\bar{A}(\delta) - t_0], \tag{A.8}$$

$$\Lambda(s_l) = \frac{1}{1-s_l}\left[e^{-30(1-s_l)/A_0} - s_l e^{-30(1-s_l)/(s_l A_0)}\right]. \tag{A.9}$$

Since (A.7) has at most one solution, the only feasible harvesting schedule is

$$T(s) = \begin{cases} T_0(s) + \bar{A}(\delta) & (0 \leqslant s \leqslant s_l^*), \\ t_0 = \bar{A}(\delta) - A_0 x & (s_l^* \leqslant s \leqslant 1), \end{cases} \tag{A.10}$$

where

$$s_l^* = 1 + \frac{t_0 - \bar{A}(\delta)}{30}. \tag{A.11}$$

For $A_0 = 60$, $\sigma = \frac{30}{19}$, $v_0 = 950$, $\delta = 0.1$, and $s_l = \frac{1}{2}$, we have

$$\begin{aligned} \bar{A}(\delta) &= 60\ln\left[\frac{30}{19}\left(\frac{1+60\delta}{60\delta}\right)\right] = 60\ln\left(\tfrac{35}{19}\right) \\ t_0 &= 15.6545\ldots, \qquad s_l^* = 0.2998485\ldots, \end{aligned} \tag{A.12}$$

and therewith

$$T(s) = \begin{cases} 60\ln\left(\tfrac{35}{19}\right) - 30(1-s) & (0 \leqslant s \leqslant 0.2998\ldots), \\ 15.6545\ldots & (0.2998\ldots \leqslant s \leqslant 1). \end{cases} \tag{A.13}$$

2. *A general procedure for a numerical solution*

For an initial age distribution $T_0(s)$ with

$$T_0'(s) \begin{cases} > 0, & 0 \leqslant s < s_l, \\ = 0, & s = s_l, \\ < 0, & s_l < s < s_r, \\ = 0, & s = s_r, \\ > 0, & s_r < s \leqslant 1, \end{cases} \tag{A.14}$$

we have

$$T(s) = \begin{cases} T_0(s)+\bar{A}(\delta), & 0 \leqslant s \leqslant s_l^* \ (<s_l), \\ t_0, & s_l^* \leqslant s \leqslant s_r^*, \\ T_0(s)+\bar{A}(\delta), & (s_r<)\ s_r^* \leqslant s \leqslant 1, \end{cases} \tag{A.15}$$

with $\lambda(s) \equiv 0$ in $0 \leqslant s \leqslant s_l^*$ and $s_r^* \leqslant s \leqslant 1$. In the range $s_l^* \leqslant s \leqslant s_r^*$, the constraint on the control variable is binding, so that $T(s)=t_0$ and

$$\begin{aligned} \lambda' &= -\frac{\partial H}{\partial T} = -\left[\dot{V}(T-T_0)-\delta V(T-T_0)\right]e^{-\delta T} \\ &= -\left[\dot{V}(t_0-T_0(s))-\delta V(t_0-T_0(s))\right]e^{-\delta t_0}, \end{aligned} \tag{A.16}$$

with

$$\lambda(s_l^*) = 0, \qquad \lambda(s_r^*) = 0. \tag{A.17}$$

By the continuity of T(s), s_l^* and s_r^* are the smallest and the largest of three roots of

$$T_0(s) + \bar{A}(\delta) = t_0. \tag{A.18}$$

For a numerical solution, we pick an initial approximation for $t_0, T_0(s_r)+\bar{A}(\delta)<t_0<T_0(s_l)+\bar{A}(\delta)$. Solve (A.18) for the three roots to get s_l^* and s_r^* [with $T_0'(s_l^*)>0$, $T_0'(s_r^*)>0$, and $s_l^*<s_r^*$]. Solve. the IVP defined by(A.16) and $\lambda(s_l^*)=0$. Iterate on t_0 (by Newton's method or the secant method) until $\lambda(s_r^*)=0$.

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