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LECTURE NOTES ON PROBLEMS IN ELASTICITY:
II. LINEAR PLATE THEORY

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Cover photo courtesy of the U.B.C. Museum of Anthropology:

Haida totem pole; main figure, possibly bear, holding wolf between legs, frog in mouth, wolf between ears.

LECTURE NOTES ON PROBLEMS IN ELASTICITY:

II. LINEAR PLATE THEORY

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3. REISSNER'S PLATE THEORY

PROLOGUE

The boundary value problems (BVP) of linear elastostatics formulated in Part I of these lecture notes[1] (for the case of infinitesimal strains and displacements) generally do not admit an exact solution in terms of elementary or special functions. An accurate numerical solution for any fully three-dimensional problem is impractical by the capability of the current computing facilities; even a simpler two-dimensional problem, resulting from some kind of symmetry inherent in the geometry and loading of the problem, still requires computing at an unacceptable level. Historically, this situation gave considerable impetus to a search for an adequate approximate solution to many problems of technical interest. Among the many approaches to approximate solutions (not discussed herein), we mention only the popular Rayleigh-Ritz method (with only a few well chosen basis functions when high speed computation was not generally available). Of interest here, however, is another method which takes advantage of the special geometrical configuration of the elastic body to be analyzed. Intuition and experience gained from working with such bodies suggest simplifying approximations which not only reduce the dimensionality of the BVP (beyond the reduction due to symmetry), but also lead to a new and simple BVP, e.g., the well-known Dirichlet and Neumann problem for Laplace's equation. Examples of such approximate solutions include the one dimensional theory of beams and two dimensional theory of plates and shells.[2]

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(1) Reference [1] at the end of the Prologue.

(2) The approximations introduced for slender and thin bodies usually reduce the original BVP to a new BVP of a lower dimension called beam theory, plate theory, etc. The exact solution of this new BVP provides an approximate solution for the original problem. Since the new BVP is usually solvable, the approximating theory is often called an approximate solution of the original BVP.

In this part of the lecture notes, we introduce the reader to an approximate solution of the elastostatic problem for thin flat bodies, called flat plates. We will develop here what is known as plate theory from a modern viewpoint advanced by applied mathematicians since the early nineteen sixties [2,3].

The classical Germain-Kirchhoff theory for (thin) plates was originally developed by adapting a set of assumptions about the approximate behavior of elastic bodies which simplify the BVP of elastostatics. Unfortunately, these assumptions, known as the Euler-Bernoulli(-Germain)-Kirchhoff hypotheses, are inconsistent and often baffle students of the resulting very attractive theory as they cannot all be valid simultaneously. We begin our development of the Germain-Kirchhoff (thin) plate theory by solving the same elastostatic problem but for a transversely rigid body. Such a body is a limiting case of an orthotropic material with isotropy in planes normal to the thickness direction of the plate. The solution for this new problem turns out to be identical to the classical Germain-Kirchhoff theory of isotropic plates. Our approach here is to use the transversely rigid plate (which is readily understood and accepted) as an artificial device to introduce readers to the thin plate theory formulation without having to adapt a set of artificial and self-contradictory hypotheses.

As in the conventional development, we see that there cannot be an approximate solution of the elastostatic BVP by way of the thin plate theory alone unless the boundary conditions for the BVP are in a very special form. To applied mathematicians, the solution seems to be similar to having only the outer (asymptotic expansion) solution for a singular perturbation problem.[3] This similarity was eventually exploited in two ways. One of these involves the recognition that thin plate theory solution is the leading term of an

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(3) Strictly speaking, the dimensionless form of the elastostatic problem for plates is not in the typical form of a singular perturbation problem; highest order derivative terms are not multiplied by a small parameter.

outer asymptotic expansion of the solution for the original BVP in a dimensionless thickness parameter [2,3]. It then follows that higher order terms in this outer expansion may be obtained in a systematic way to get a more accurate approximate solution for thicker plates. The determination of the outer asymptotic expansion solution will be discussed in Chapter (2) of these notes.

However many terms we may obtain in the outer expansion, it is known from singular perturbation theory that they could not by themselves satisfy all the boundary conditions of the problem. A second direction for exploiting the similarity between plate theory and outer expansion solution is to seek the counterpart of the inner solution for the plate problem. In general, the latter may be required to satisfy all boundary conditions of the three-dimensional problem and match with the outer solution in an overlapping region of validity. For a linear problem, we may take the exact solution as the sum of an interior solution component and a boundary layer (edge zone) component. Away from the edge(s), the interior solution is identical with the outer solution and the boundary layer solution is (numerically) insignificant. Also, the boundary layer solution coincides with the difference between the inner and outer solution within a narrow region adjacent to the edge. To the extent that it is generally difficult to obtain the inner solution or the boundary layer solution for a given problem (almost as difficult as the exact solution for most problems), there has been considerable research effort in the last three decades to determine the correct interior solution without any reference to the boundary layer solution.

By the technique of matched asymptotic expansions for singular perturbation problems, it was found [2,3] that only the leading term of the outer solution may be determined without the boundary layer solution. That is, the leading term interior solution is made to satisfy an appropriate portion of the boundary conditions and is thereby completely determined even if we know nothing about the boundary layer solution. However, this is possible only for prescribed edge tractions (and not for prescribed displacements or mixed data) and only for the leading term solution. There are other approaches taken for the determination of the interior solution without simultaneously a complete solution of at least the leading term boundary layer solution. We report one such approach by E. Reissner in Chapter (3). The actual resolution of the difficult problem of determining the interior solution can be found in [4,5] and will be discussed in a later chapter.

The difficulty encountered in the process of finding the exact or accurate approximate solution for the three dimensional BVP of elastostatics has also stimulated an ad hoc two dimensional formulation of plate theory analogous to that for three dimensional elasticity theory. This two dimensional approach, which we will also describe in a later chapter, leads to a more general (but not necessarily more useful) theory of plates which reduces to the conventional theory and the Reissner's theory upon appropriate specializations. The more general theory does reveal a more complete structure of the conventional plate theory and facilitates in unexpected ways the solution of many plate problems.

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1. The Theory of Transversely Rigid Plates

1. Statement of the Problem

We consider here elastic bodies bounded by two parallel planes and a cylindrical surface normal to the planes. A cartesian coordinate system is chosen so that the two planes are given by $x_3 = \pm h/2$ and the cylindrical surface is given by $f(x_1,x_2) = 0$ and $|x_3| \le h/2$. Such a body is called a flat plate, and $x_3 = 0$ is its middle plane, if h is small compared to the characteristic dimension (which measures the slenderness) of the middle plane and to the minimum radius of curvature of the boundary curve Γ given by $f(x_1,x_2) = 0$. The bounding planes $x_3 = \pm h/2$ are called the top and bottom (or upper and lower) face of the plate, respectively, h is its uniform thickness, and the cylindrical boundary is the edge of the plate.

For simplicity, we consider here plates subject to no body force intensities, so that $f_k(x_1,x_2,x_3) \equiv 0$, $k = 1,2,3$, in its interior, and to only surface traction normal to its faces so that

$$\sigma_{31}(x_1,x_2,\pm h/2) \equiv \sigma_{32}(x_1,x_2,\pm h/2) \equiv 0 \qquad (1.1\text{-a,b})$$

$$\sigma_{33}(x_1,x_2,-h/2) = \sigma_b(x_1,x_2) \quad , \quad \sigma_{33}(x_1,x_2,h/2) = \sigma_t(x_1,x_2) \qquad (1.1\text{-c,d})$$

for all points (x_1,x_2) inside the boundary curve Γ. Along the edge of the plate, three appropriate boundary conditions are prescribed in terms of displacement and/or stress components. For prescribed edge displacements, we have

$$u_k(x_1,x_2,x_3) = u_k^* (x_1,x_2,x_3) \quad , \quad (k = 1,2,3) \qquad (1.2)$$

for $f(x_1,x_2) = 0$; for prescribed edge tractions, we have

$$\sigma_{\nu k}(x_1,x_2,x_3) = T_k(x_1,x_2,x_3) \quad , \quad (k = 1,2,3) \qquad (1.3)$$

for $f(x_1,x_2) = 0$ where $\vec{\nu}$ is the unit normal to the curve Γ in the x_1,x_2-plane.

Within the framework of the linear theory of elasticity, the stresses and displacements of elastic plates are determined by (1) strain-displacement relations,

$$e_{jj} = u_{j,j} \quad (j = 1,2,3, \text{ no sum}) \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad , \qquad (1.4)$$

(2) equilibrium equations

$$\sigma_{jk,j} = 0 \qquad (k = 1,2,3) \quad , \qquad (1.5)$$

where we have taken $f_k \equiv 0$, $k = 1,2,3$, for the analysis of this chapter, and (3) stress-strain relations. We are interested in plates which are homogeneous and transversely isotropic so that

$$e_{33} = \frac{1}{E_3}\sigma_{33} - \frac{\nu_3}{E}(\sigma_{11} + \sigma_{22}) \quad , \quad e_{j3} = \frac{1}{2G_3}\sigma_{j3} \qquad (j = 1,2) \qquad (1.6\text{-a,b})$$

$$e_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu_3\sigma_{33}) \quad , \quad e_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11} - \nu_3\sigma_{33}) \qquad (1.7\text{-a,b})$$

$$e_{12} = e_{21} = \frac{1}{2G}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12} \quad . \qquad (1.7\text{-c})$$

In this chapter, we consider only the special case of transversely rigid plates with $1/E_3 = \nu_3 = 1/G_3 = 0$. It will be shown that, for this special case, the solution of a specific problem is reduced to solving two boundary value problems for two dimensional biharmonic equations.

2. Integration of Stress-Displacement Relations

With $1/E_3 = \nu_3 = 0$, we have from (1.4) and (1.6-a) $e_{33} = u_{3,3} = 0$ or

$$u_3(x_1,x_2,x_3) = W(x_1,x_2) \tag{1.8}$$

where W is an unknown function of integration independent of x_3 . With $1/G_3 = 0$, we get from (1.4), (1.6-b) and (1.8) $e_{3j} = u_{j,3} + x_3 W_{,j} = 0$ for $j = 1,2$, or

$$u_j(x_1,x_2,x_3) = U_j(x_1,x_2) - x_3 W_{,j}(x_1,x_2) \tag{1.9}$$

where $U_1(x_1,x_2)$ and $U_2(x_1,x_2)$ are two other unknown functions of integration independent of x_3 . With (1.8) and (1.9), the dependence of the displacement components on x_3 is completely known. In particular, the transversely displacement u_3 is the same for every mid-plane parallel layer, x_3 = constant .

With $\nu_3 = 0$, the in-plane stress-strain relations and strain-displacement relations may be combined and inverted to give

$$\sigma_{ij} = n_{ij} + x_3 m_{ij} \quad , \quad (i,j = 1,2) \tag{1.10}$$

where

$$n_{11} = \frac{E}{1-\nu^2}[\varepsilon_{11} + \nu\varepsilon_{22}] \quad , \quad n_{22} = \frac{E}{1-\nu^2}[\varepsilon_{22} + \nu\varepsilon_{11}] \tag{1.11-a,b}$$

$$n_{12} = n_{21} = \frac{E}{2(1+\nu)}[\varepsilon_{12} + \varepsilon_{21}] \tag{1.11-c}$$

$$m_{11} = \frac{E}{1-\nu^2}[\kappa_{11} + \nu\kappa_{22}] \quad , \quad m_{22} = \frac{E}{1-\nu^2}[\kappa_{22} + \nu\kappa_{11}] \tag{1.12-a,b}$$

$$m_{12} = m_{21} = \frac{E}{2(1+\nu)}[\kappa_{12} + \kappa_{21}] \tag{1.12-c}$$

with

$$\varepsilon_{11} = U_{1,1} \quad , \quad \varepsilon_{22} = U_{2,2} \quad , \quad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}[U_{1,2} + U_{2,1}] \tag{1.13}$$

$$\kappa_{11} = -W_{,11} \quad , \quad \kappa_{22} = -W_{,22} \quad , \quad \kappa_{12} = \kappa_{21} = -W_{,12} \quad . \tag{1.14}$$

The new auxiliary quantities ε_{ij} and κ_{ij} are introduced here in anticipation of later developments in the theory of plates and shells. It is not difficult to see from the defining equations (1.13) and (1.14),

$$e_{ij} = \varepsilon_{ij} + x_3\kappa_{ij} \qquad (i,j = 1,2) \tag{1.15}$$

so that $\{\varepsilon_{ij}\}$ are the mid-plane strain components and $\{\kappa_{ij}\}$ have the dimension of curvature. Equations (1.10) and (1.15) show that the dependence of all in-plane stress and strain components on x_3 is also known to be linear in x_3 with the coefficients given in terms of the various first partial derivatives of U_1 , U_2 , and W .

3. Equilibrium Equations and Boundary Conditions on the Faces

Having exhausted the content of the strain-displacement relations (1.4) and the stress-strain relations (1.6) and (1.7), we now turn to the equilibrium equations (1.5). The first two equations corresponding to $j = 1$ and 2 may be written as

$$\sigma_{3j,3} = -\sigma_{1j,1} - \sigma_{2j,2} = -(n_{1j,1} + n_{2j,2}) - x_3(m_{1j,1} + m_{2j,2})$$

or

$$\sigma_{3j} = q_j(x_1,x_2) - x_3(n_{1j,1} + n_{2j,2}) - \frac{x_3^2}{2}(m_{1j,1} + m_{2j,2})$$

where $q_j(x_1,x_2)$ are two new unknown functions of x_1 and x_2. In view of the boundary conditions $\sigma_{3j}(x_1,x_2,\pm h/2) = 0$, $j = 1,2$, on the top and bottom face of the plate, we must have

$$n_{1j,1} + n_{2j,2} = 0 \quad , \quad \frac{h^2}{8}(m_{1j,1} + m_{2j,2}) - q_j = 0 \quad , \quad (j = 1,2) \quad . \qquad (1.16, 1.17)$$

These requirements reduce the expressions for σ_{31} and σ_{32} to

$$\sigma_{3k} = q_k(x_1,x_2)(1 - z^2) \quad , \quad z \equiv x_3/(h/2) \qquad (k = 1,2) \quad . \qquad (1.18)$$

The two equations (1.16) may be written with the help of (1.11) and (1.13) as two second order PDEs (partial differential equations) for U_1 and U_2. Together with appropriate boundary conditions along Γ, they completely determine U_j, ε_{ij} and n_{ij} $(i,j = 1,2)$.

Equations (1.17) enable us to express q_j in terms of mid-plane transverse displacement $W(x_1,x_2)$

$$q_k(x_1,x_2) = -\frac{Eh^2}{8(1-\nu^2)}(\nabla^2 W)_{,k} \qquad (k = 1,2) \quad . \tag{1.19}$$

To get an equation for the remaining unknown function $W(x_1,x_2)$, we integrate the remaining equilibrium equation with respect to x_3 to get

$$\sigma_{33} = \sigma_o(x_1,x_2) - \frac{h}{2}[q_{1,1} + q_{2,2}](z - \frac{1}{3}z^3) \quad , \quad z \equiv \frac{x_3}{(h/2)}$$

where we have made use of the results in (1.18) for σ_{31} and σ_{32}. The face conditions

$$\sigma_{33}(x_1,x_2,h/2) = \sigma_t(x_1,x_2) \quad , \quad \sigma_{33}(x_1,x_2,-h/2) = \sigma_b(x_1,x_2)$$

require

$$\sigma_o(x_1,x_2) = \frac{1}{2}[\sigma_t(x_1,x_2) + \sigma_b(x_1,x_2)] \tag{1.20}$$

and

$$q_{1,1} + q_{2,2} + \frac{3}{2h} p(x_1,x_2) = 0 \tag{1.21}$$

where

$$p(x_1,x_2) = \sigma_t - \sigma_b \quad . \tag{1.22}$$

With (1.20) and (1.21), $\sigma_{33}(x_1,x_2,x_3)$ is completely determined (without any knowledge of the three primary unknowns U_1, U_2 and W) to be:

$$\sigma_{33} = \frac{1}{2}[\sigma_t + \sigma_b] + \frac{3}{4}[\sigma_t - \sigma_b](z - \frac{1}{3}z^3) \quad , \quad z \equiv \frac{x_3}{(h/2)} \quad . \tag{1.23}$$

At the same time, the condition (1.21) gives a fourth order linear PDE for the remaining unknown $W(x_1,x_2)$:

$$-D\nabla^2\nabla^2 W + p = 0 \tag{1.24}$$

where $\nabla^2(\ ) \equiv (\ )_{,11} + (\ )_{,22}$ and $D = Eh^3/12(1-\nu^2)$. With two appropriate boundary conditions along Γ (or along every edge curve if there are several boundary curves), the two dimensional inhomogeneous biharmonic equations (1.24) completely determines $W(x_1,x_2)$.

4. Stress Resultants and Stress Couples

The form of (1.21) suggests that we set

$$q_k(x_1,x_2) = \frac{3}{2h} Q_k(x_1,x_2) \tag{1.25}$$

to transform it into

$$Q_{1,1} + Q_{2,2} + p = 0 \quad . \tag{1.26}$$

Evidently, we have

$$\int_{-h/2}^{h/2} \sigma_{3k}(x_1,x_2,x_3)dx_3 = \int_{-h/2}^{h/2} \frac{3}{2h} Q_k(x_1,x_2)(1-z^2)dx_3 = Q_k$$

so that Q_1 and Q_2 are the resultant transverse shear force per unit arclength along an x_1 = constant coordinate curve and an x_2 = constant coordinate curve, respectively. They are called the <u>transverse shear resultants</u> of the plate. With (1.25), the form of (1.17) in turn suggests that we set

$$\frac{h^2}{8} m_{ij} = \frac{3}{2h} M_{ij} \tag{1.27}$$

so that

$$\int_{-h/2}^{h/2} \sigma_{ij}\, dx_3 = h n_{ij} \quad , \quad \int_{-h/2}^{h/2} \sigma_{ij}\, x_3 dx_3 = \frac{h^3}{12} m_{ij} = M_{ij}$$

Evidently, the stress resultants N_{i1} and N_{i2}, defined by

$$N_{ij}(x_1,x_2) = hn_{ij} \qquad (i,j = 1,2) , \tag{1.28}$$

are the resultant in-plane normal and shear force per unit arc-length along an x_i = constant coordinate curve, respectively. The stress couples (or moment resultants) M_{i1} and M_{i2} are the resultant bending and twisting moment per unit arclength along an x_i = constant coordinate curve, respectively. In terms of N_{ij} and M_{ij}, the reduced equilibrium equations (1.16) and (1.17) become force and moment equilibrium equations for the stress resultants and couples:

$$N_{1j,1} + N_{2j,2} = 0 , \tag{1.29}$$

$$(j = 1,2)$$

$$M_{1j,1} + M_{2j,2} - Q_j = 0 \tag{1.30}$$

Recall that the stress resultants N_{ij} and midplane strain components, ε_{ij}, are determined by the lateral (in-plane) displacement components of the midplane of the plate, U_1 and U_2 :

$$N_{11} = \frac{Eh}{1 - \nu^2}(\varepsilon_{11} + \nu\,\varepsilon_{22}) = \frac{Eh}{1 - \nu^2}(U_{1,1} + \nu\,U_{2,2})$$

$$N_{22} = \frac{Eh}{1 - \nu^2}(\varepsilon_{22} + \nu\,\varepsilon_{11}) = \frac{Eh}{1 - \nu^2}(U_{2,2} + \nu U_{1,1}) \tag{1.31}$$

$$N_{12} = N_{21} = \frac{Eh}{1 + \nu}\varepsilon_{12} = \frac{Eh}{1 + \nu}\varepsilon_{21} = \frac{Eh}{2(1 + \nu)}(U_{2,1} + U_{1,2})$$

The two midplane parallel force equilibrium equations (1.29) can therefore be written as two second order partial differential equations (PDE) for U_1 and U_2 :

$$\frac{Eh}{1-\nu^2} [\nabla^2 U_1 - \frac{1}{2}(1+\nu)(U_{1,2} - U_{2,1})_{,2}] = 0$$

$$\frac{Eh}{1-\nu^2} [\nabla^2 U_2 + \frac{1}{2}(1+\nu)(U_{1,2} - U_{2,1})_{,1}] = 0 \qquad (1.32)$$

Provided that the appropriate boundary conditions for the PDEs (1.32) involve only quantities which can be expressed in terms U_1 and U_2 and/or their first derivatives(*), the determination of U_j, N_{ij} and ε_{ij} , which characterize the <u>extension and torsion</u> of the plate, is uncoupled from the determination of the remaining unknowns, W , κ_{ij} , M_{ij} , and Q_j , which characterize the <u>bending and twisting</u> of the plate.

For bending and twisting actions, we have from (1.12)

$$M_{11} = D(\kappa_{11} + \nu\kappa_{22}) = -D(W_{,11} + \nu W_{,22})$$

$$M_{22} = D(\kappa_{22} + \nu\kappa_{11}) = -D(W_{,22} + \nu W_{,11}) \qquad (1.33)$$

$$M_{12} = M_{21} = \frac{1}{2}D(1-\nu)(\kappa_{12} + \kappa_{21}) = -D(1-\nu)W_{,12}$$

and from (1.30)

$$Q_j = M_{1j,1} + M_{2j,2} = -D(\nabla^2 W)_{,j} \qquad (j = 1,2) \qquad (1.34)$$

(*) Boundary conditions along the edge of the plate will be discussed in the next section.

The transverse force equilibrium equation (1.26) then gives the single inhomogeneous biharmonic equation (1.24) for the transverse (midplane) displacement W :

$$- D \nabla^4 W + p = 0 \tag{1.24}$$

Equations (1.33), (1.34) and (1.24) do not involve quantities associated with plate extension and torsion. Therefore, plate bending and twisting are also uncoupled from plate extension and torsion, assuming, of course, that there is no coupling between the two types of plate actions in the boundary conditions.

5. Displacement Boundary Conditions Along an Edge of the Plate

While a variety of physically realizable boundary conditions are possible, we limit the discussion in this and the next section to two types that often occur in practice, namely, conditions on edge displacements and conditions on edge stresses. Together, they serve to illustrate the nature of the analysis and the difficulties involved. These same difficulties will recur later in isotropic plates.

(A) Homogeneous Displacement Conditions along $x_1 = \bar{x}_1$

We begin with conditions of no displacements at an edge $x_1 = \bar{x}_1$. From the theory of three dimensional elasticity, the appropriate boundary conditions in this case are

$$u_k(\bar{x}_1, x_2, x_3) = 0 \qquad (k = 1,2,3) \tag{1.35}$$

For a transversely rigid plate, these become

$$u_3(\bar{x}_1, x_2, x_3) = W(\bar{x}_1, x_2) = 0 \tag{1.36}$$

$$u_j(\bar{x}_1, x_2, x_3) = U_j(\bar{x}_1, x_2) - x_3 W_{,j}(\bar{x}_1, x_2) = 0$$

The conditions on the midplane parallel displacements must be satisfied identically in x_3. Therefore, they require

$$U_j(\bar{x}_1, x_2) = 0 \qquad (j = 1,2) \tag{1.37}$$

and $W_{,j}(\bar{x}_1, x_2) = 0$, $j = 1,2$. However, $W_{,2}(\bar{x}_1, x_2) = 0$ is satisfied automatically once the condition (1.36) is satisfied; there is then only one nontrivial condition on the derivative of W in the direction normal to the edge (in the midplane of the plate)

$$W_{,1}(\bar{x}_1, x_2) = 0 \tag{1.38}$$

(B) Homogeneous Displacement Conditions along a Curved Edge

More generally, instead of a straight edge coinciding with a cartesian coordinate line $x_k = \bar{x}_k$ $(k = 1 \text{ or } 2)$, we may have a curved edge given parametrically by $x_k = \bar{x}_k(s)$ with a counterclockwise unit tangent vector $\vec{t}(x_1, x_2)$ and an outward directed unit normal vector $\vec{\nu}(x_1, x_2)$ in the plane of the boundary curve Γ. The conditions (1.35) remain appropriate for the case of no edge displacements for such a curved edge; but they can be also taken in the form

$$x_j = \bar{x}_j(s): \quad \begin{cases} u_3(x_1, x_2, x_3) = 0 \\ u_\nu(x_1, x_2, x_3) \equiv \nu_1 u_1 + \nu_2 u_2 = 0 \\ u_t(x_1, x_2, x_3) \equiv -\nu_2 u_1 + \nu_1 u_2 = 0 \end{cases} \tag{1.35'}$$

where $\nu_1(x_1, x_2)$ and $\nu_2(x_1, x_2)$ are the directional cosines of the unit normal $\vec{\nu} = \nu_1 \vec{i}_1 + \nu_2 \vec{i}_2$ along Γ, $u_\nu \equiv \vec{u} \cdot \vec{\nu}$ and $u_t \equiv \vec{u} \cdot \vec{t}$. For a transversely rigid plate, the condition $u_3 \equiv 0$ along Γ again requires

$$W(x_1, x_2) = 0 \tag{1.36'}$$

along $x_k = \bar{x}_k(s)$, $k = 1,2,$ where s is taken henceforth to be an arc length variable measured from some reference point. The two conditions on the in-plane displacement components require

$$U_\nu(x_1,x_2) \equiv \nu_1 U_1 + \nu_2 U_2 = 0 \ ,$$
$$U_t(x_1,x_2) \equiv - \nu_2 U_1 + \nu_1 U_2 = 0 \ , \qquad (1.37')$$

(which imply $U_1 = U_2 = 0$) and

$$- W,_\nu \equiv - (\nu_1 W,_1 + \nu_2 W,_2) = 0 \ , \qquad (1.38')$$

$$-W,_t \equiv - (- \nu_2 W,_1 + \nu_1 W,_2) = 0 \ . \qquad (1.39)$$

From differential geometry of plane curves, we have

$$\vec{t} = \frac{dx_1}{ds} \vec{i}_1 + \frac{dx_2}{ds} \vec{i}_2 = -\nu_2 \vec{i}_1 + \nu_1 \vec{i}_2$$
$$\vec{\nu} = \frac{dx_2}{ds} \vec{i}_1 - \frac{dx_1}{ds} \vec{i}_2 = \nu_1 \vec{i}_1 + \nu_2 \vec{i}_2 \qquad (1.40)$$

so that

$$- W,_t \equiv - (W,_1 \frac{dx_1}{ds} + W,_2 \frac{dx_2}{ds}) = \frac{dW}{ds} \ .$$

Therefore, $- W,_t = 0$ along Γ is satisfied automatically once we have (1.36'), and the homogeneous displacement conditions (1.35') are satisfied if the four conditions (1.36'), (1.37') and (1.38') are satisfied. The two conditions (1.37') uniquely determine the solution U_1 and U_2 of the pair of governing PDE for plate extension and torsion (1.32). Similarly, the two conditions (1.36') and (1.38') uniquely determine the solution W of (1.24) for plate bending and twisting.

(C) Inhomogeneous Displacement Boundary Conditions

If along the edge curve Γ, we have the prescribed displacement conditions

$$u_3(x_1,x_2,x_3) = u_3^*(s,x_3)$$

$$u_\nu(x_1,x_2,x_3) = u_\nu^*(s,x_3)$$

$$u_t(x_1,x_2,x_3) = u_t^*(s,x_3)$$

where u_3^*, u_ν^* and u_t^* are prescribed functions, they require for a transversely rigid plate

$$W(x_1,x_2) = u_3^*(s,x_3)$$

$$U_\nu(x_1,x_2) - x_3 W,_\nu(x_1,x_2) = u_\nu^*(s,x_3)$$

$$U_t(x_1,x_2) - x_3 W,_t(x_1,x_2) = u_t^*(s,x_3)$$

along Γ, which may again be parametrically represented by $x_1 = \bar{x}_1(s)$ and $x_2 = \bar{x}_2(s)$. Several observations on the prescribed displacements are immediate:

(a) It is not possible for W, U_ν and U_t to satisfy the prescribed displacement conditions unless u_3^* is independent of x_3 and u_ν^* and u_t^* are linear in x_3.

(b) For a transversely rigid plate, the prescribed displacements must in fact be in the form consistent with the distributions of displacement components (1.8) and(1.9):

$$u_3^*(x_1,x_2,x_3) = W^*(s)$$

$$u_\nu^*(x_1,x_2,x_3) = U_\nu^*(s) + x_3\Phi_\nu^*(s)$$

$$u_t^*(x_1,x_2,x_3) = U_t^*(s) + x_3\Phi_t^*(s)$$

Other kinds of distributions across the plate thickness would mean that the plate may be stretched (or compressed) and sheared in the $\vec{i}_3$ direction along the edge (and therefore not transversely rigid, at least not at the edge).

(c) While W^*, U_ν^*, U_t^*, and Φ_ν^* may be prescribed arbitrarily, we must have

$$\Phi_t^* = -W,_s^*$$

which is also part of the constraint imposed by transverse rigidity. With $u_k^{(o)}$ satisfied all the restrictions stipulated above, we have along the boundary curve Γ.

$$W(x_1,x_2) = W^*(s), \qquad W,_\nu(x_1,x_2) = -\Phi_\nu^*(s) \tag{1.40}$$

for plate bending and twisting, and

$$U_\nu(x_1,x_2) = U_\nu^*(s), \qquad U_t(x_1,x_2) = U_t^*(s) \tag{1.41}$$

for plate extension and torsion.

6. Stress Boundary Conditions Along an Edge of the Plate

(A) Homogeneous Stress Boundary Conditions along $x_j = \bar{x}_j$

For an $x_j = \bar{x}_j$ edge free of edge traction, we must have

$$\vec{\sigma}_j \Big|_{x_j = \bar{x}_j} = [\sigma_{jk}\vec{i}_k]_{x_j = \bar{x}_j} = \vec{0}$$

or

$$\sigma_{jk} \Big|_{x_j = \bar{x}_j} = 0 \qquad (k = 1,2,3) .$$

In view of the form of the stress distributions for a transversely rigid plate, the traction-free edge condition corresponds to five scalar conditions. Two of these are

$$N_{jk} \Big|_{x_j = \bar{x}_j} = 0 , \qquad (k = 1,2), \tag{1.42}$$

for the extension and torsion actions of the plate. The remaining three are

$$M_{jk} \Big|_{x_j = \bar{x}_j} = Q_j \Big|_{x_j = \bar{x}_j} = 0 , \ (k = 1,2) \tag{1.43}$$

for problems associated with plate bending and twisting.

The two codnitions (1.42) may be expressed in terms of the in-plane displacement components by (1.31) to get

$$N_{jj}\Big|_{x_j = \bar{x}_j} = \frac{Eh}{1 - \nu^2} (U_{j,j} + \nu U_{k,k})\Big|_{x_j = \bar{x}_j} = 0$$

$$(k \neq j, \text{ no sum}) \qquad (1.44)$$

$$N_{jk}\Big|_{x_j = \bar{x}_j} = \frac{Eh}{2(1 + \nu)} (U_{j,k} + U_{k,j})\Big|_{x_j = \bar{x}_j} = 0 \; .$$

They are appropriate boundary conditions for the fourth order system of PDEs (1.32).

The three conditions (1.43) may be expressed in terms of the transverse displacement W by way of (1.33) and (1.34). We see from the resulting conditions that they are in fact three independent conditions on W at the plate edge. (For the special case of vanishing Poisson's ratio $(\nu = 0)$, they effectively require the first, second and third derivative of W with respect to x_j to vanish along $x_j = \bar{x}_j$. The situtation for $\nu \neq 0$ is more complicated but qualitatively similar). Only two of these can be satisfied by W along an edge of the plate as it is the solution of a fourth order PDE. This paradox (which also appears in theory of isotropic plates as we shall see later) must be removed if the results obtained for plate bending and twisting are to be usable.

(B) Contracted Stress Boundary Conditions for Bending and Twisting of Plates

The difficulty of too many boundary conditions for the theory of plate bending (and twisting) is particularly challenging and cannot be side-stepped because the three conditions (1.43) on the bending moment resultant, twisting moment resultant and transverse shear resultant are expected on physical ground. It remained unresolved for thirty-five years dating from 1815, the time when Sophie Germain was awarded the prize set up by the French Academy of Sciences for her theory on plate bending. In 1850, G. Kirchhoff showed that the two conditions on the twisting moment and transverse shear resultant in (1.43) should be combined into a single contracted condition for an effective (or equivalent) transverse shear resultant

$$Q_j^e = Q_j + \frac{\partial M_{jk}}{\partial x_k} \qquad (k \neq j \;,\; \text{no sum}) \tag{1.45}$$

As we shall see in a later section, the contraction is required by calculus of variations to be consistent with the "idealization" in treating the elastic material as transversely rigid.

(C) Stress Free Conditions along a Curved Edge

For a curved edge Γ , it is not difficult to see that the two conditions corresponding to (1.42) are

$$N_{\nu\nu}(x_1(s),x_2(s)) = 0 \;, \quad N_{\nu t}(x_1(s),x_2(s)) = 0 \tag{1.42'}$$

with the edge curve Γ parametrized by $x_k = \bar{x}_k(s)$, $k = 1,2,$ and with $\vec{N}_\nu = \nu_1\vec{N}_1 + \nu_2\vec{N}_2$ (where $\vec{N}_j = N_{j1}\vec{i}_1 + N_{j2}\vec{i}_2 + Q_j\vec{i}_3$) so that

$$
\begin{aligned}
N_{\nu\nu} &= \vec{\nu}\cdot \vec{N}_\nu = \nu_1^2 N_{11} + \nu_2^2 N_{22} + 2\nu_1\nu_2 N_{12} \\
N_{\nu t} &= \vec{t}\cdot\vec{N}_\nu = (\nu_1^2 - \nu_2^2)N_{12} + 2\nu_1\nu_2 (N_{22} - N_{11})
\end{aligned}
\qquad (1.46)
$$

The two conditions corresponding to (1.44) for a curved edge can be obtained from (1.42') and (1.31).

For plates with a single curved boundary Γ , the two PDEs (1.32) for the interior of Γ and the two stress boundary conditions (1.42') along Γ (together with boundedness conditions on stresses and displacements) define a linear BVP (boundary value problem) in PDE. The BVP is <u>homogeneous</u> and therefore has only a rigid body displacement solution. A proof of this claim is analogous to the proof of the uniqueness theorem for the second fundamental problem in three dimensional linear elasticity theory.

We will show in a later section that the two appropriate contracted stress-free conditions for a <u>curved</u> edge are

$$M_{\nu\nu}(x_1(s), x_2(s)) = 0 , \quad Q^e_\nu(x_1(s), x_2(s)) = 0 \qquad (1.47)$$

where

$$Q^e_\nu = Q_\nu + \frac{\partial M_{\nu t}}{\partial s} . \qquad (1.48)$$

For plate bounded by a single edge curve Γ , the two stress-free conditions (1.47) and the inhomogeneous biharmonic equation for W define a well-posed linear BVP in PDEs provided that the applied surface load intensity $p(x_1,x_2)$ is self-equilibrating.

(D) Inhomogeneous Stress Conditions

Instead of the edge being traction free, it may be subject to externally applied edge stresses so that

$$\begin{aligned} \vec{\sigma} &\equiv \sigma_{\nu\nu}\vec{\nu} + \sigma_{\nu t}\vec{t} + \sigma_{\nu 3}\vec{i}_3 \\ &= T_\nu(s,x_3)\,\vec{\nu} + T_t(s,x_3)\vec{t} + T_3(s,x_3)\vec{i}_3 \equiv \vec{T}(s,x_3) \end{aligned} \tag{1.49}$$

along Γ . But from (1.10), $\sigma_{\nu\nu}$ and $\sigma_{\nu t}$ are linear in x_3 and from (1.18), $\sigma_{\nu 3}$ must be quadratic in x_3 . Therefore, the prescribed inhomogeneous stress conditions along Γ cannot be satisfied unless T_ν and T_t are linear in x_3 and T_3 is proportional to $(1 - z^2)$ with $z = x_3/(h/2)$. Unlike the situation with prescribed edge displacement, it seems possible to apply other kinds of edge load distributions along Γ whether or not the plate is transversely rigid. While the behaviour of elastic material with directional rigidity under arbitrary surface traction is not well understood, it is possible to appeal to the so-called Saint Venant Principle which allows the edge load conditions be satisfied in resultant force and moment per unit arc length, i.e.

$$N_{\nu\nu}(x_1(s),\ x_2(s)) = N^*_{\nu\nu}\ (s)\quad ,\ N_{\nu t}(x_1(s),\ x_2(s)) = N^*_{\nu t}\ (s) \tag{1.50}$$

and

$$\begin{aligned} Q^e_\nu(x_1(s),\ x_2(s)) &= Q^{(o)}_\nu + \frac{\partial M^{(o)}_{\nu t}}{\partial s}\ , \\ M_{\nu\nu}(x_1(s),\ x_2(s)) &= M^*_{\nu\nu}\ (s) \end{aligned} \tag{1.51}$$

where

$$N^*_{\nu\nu} = \int_{-h/2}^{h/2} T_\nu\ dx_3\ ,\quad N^*_{\nu t} = \int_{-h/2}^{h/2} T_t(s)dx_3 \tag{1.52}$$

$$\begin{aligned} M^*_{\nu\nu} &= \int_{-h/2}^{h/2} T_\nu x_3 dx_3,\quad M^*_{\nu t} = \int_{-h/2}^{h/2} T_t x_3 dx_3 \\ Q^*_\nu &= \int_{-h/2}^{h/2} T_3(1 - z^2)dx_3 \end{aligned} \tag{1.53}$$

The principle assures that at distance away from the edge greater than h, the error in the stress and displacement distributions due the replacement of (1.49) by (1.50) and (1.51) is small of order h/L where L is the characteristic length scale in the midplane. We will discuss this principle in more detail later.

7. Stress Function Formulation for the In-plane Problem

With U_1 and U_2 as the primary unknowns, the governing differential equations for the extension and torsion of transversely rigid plates are the two coupled second order PDEs (1.32). The solution of these equations depends on the value of Poisson's ratio, i.e., it contains ν as a parameter. It turns out that the (in-plane) stress distributions for the stress BVP of extension and torsion do not depend on the elastic moduli E and ν. This is true for the present problem (for plates with no distributed load in the interior of Γ) and for many problems with distributed interior load intensities. Therefore, it is important for both theoretical considerations as well as applications to reformulate the problem of extension and torsion so that the stress BVP does not involve the elastic moduli. This is accomplished with the help of the Airy stress function $F(x_1,x_2)$.

It is not difficult to verify by direct substitution that the two equilibrium equations (1.29) for extension and torsion are satisfied identically if we set

$$N_{11} = F_{,22} \quad , \quad N_{22} = F_{,11} \quad , \quad N_{12} = N_{21} = -F_{,12} \tag{1.54}$$

for an arbitrary (four times continuously differentiable) function F. To determine F, we observe that the three mid-plane strain measures $\varepsilon_{ij} = \varepsilon_{ji}$ $(i,j = 1,2)$, defined in terms of two displacement components U_1 and U_2 by (1.13), satisfy the following compatibility equation

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0 \quad . \tag{1.55}$$

With the stress strain relations (1.31) inverted to give

$$\begin{aligned} &\varepsilon_{11} = A(N_{11} - \nu N_{22}) \quad , \quad \varepsilon_{22} = A(N_{22} - \nu N_{11}) \quad , \\ &\varepsilon_{12} = \varepsilon_{21} = \frac{N_{12}}{2G} = A(1 + \nu)N_{12} \quad , \quad A = \frac{1}{Eh} \quad , \end{aligned} \tag{1.56}$$

the compatibility equation (1.55) gives a single PDE for the stress function F . After dividing both sides of that PDE by A , we get

$$\nabla^2\nabla^2 F = 0 \quad , \tag{1.57}$$

in the region R bounded by the boundary curve Γ . The two relevant stress boundary conditions given by (1.50) may also be expressed in terms of the stress function F :

$$\begin{aligned} &\nu_1^2 F_{,22} + \nu_2^2 F_{,11} - 2\nu_1\nu_2 F_{,12} = N^*_{\nu\nu}(s) \\ &-(\nu_1^2 - \nu_2^2)F_{,12} + 2\nu_1\nu_2(F_{,11} - F_{,12}) = N^*_{\nu t}(s) \quad . \end{aligned} \tag{1.58}$$

The PDE (1.57) and the stress boundary conditions (1.58) do not involve the elastic moduli E and ν. Therefore, the solution F of the stress BVP is independent of ν and E. The same conclusion also holds when the plate is subject to certain classes of in-plane distributed loads in the interior of Γ. The verification of this claim and the restrictions on the distributed loads are left as an exercise.

8. Polar Coordinates

For many problems of interests (especially those involving circular or annular plates), the solution is easier to obtain if polar coordinates (r,θ) are used insteand of cartesian coordinates (x_1,x_2) . With the coordinate transformation formulas

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad ,$$

$$\vec{i}_r = \cos\theta\vec{i}_1 + \sin\theta\vec{i}_2 \quad , \quad \vec{i}_\theta = -\sin\theta\vec{i}_1 + \cos\theta\vec{i}_2 \quad ,$$

$$\vec{\nabla}(\;) = \vec{i}_1(\;)_{,1} + \vec{i}_2(\;)_{,2} = \vec{i}_r(\;)_{,r} + \vec{i}_\theta[\frac{1}{r}(\;)]_{,\theta} \quad , \tag{1.59}$$

$$\nabla^2(\;) = \vec{\nabla}\cdot\vec{\nabla}(\;) = (\;)_{,rr} + \frac{1}{r}(\;)_{,r} + \frac{1}{r^2}(\;)_{,\theta\theta} \quad ,$$

and the component representations

$$\vec{U} = U_1\vec{i}_1 + U_2\vec{i}_2 + W\vec{i}_3 = U_r\vec{i}_r + U_\theta\vec{i}_\theta + W\vec{i}_3 \quad ,$$

$$\vec{N}_k = N_{k1}\vec{i}_1 + N_{k2}\vec{i}_2 + Q_k\vec{i}_3 \quad , \quad (k = 1,2) \quad ,$$

$$\vec{N}_r = N_{rr}\vec{i}_r + N_{r\theta}\vec{i}_\theta + Q_r\vec{i}_3 = (\vec{i}_r\cdot\vec{i}_1)\vec{N}_1 + (\vec{i}_r\cdot\vec{i}_2)\vec{N}_2 \quad ,$$

$$\vec{N}_\theta = N_{\theta r}\vec{i}_r + N_{\theta\theta}\vec{i}_\theta + Q_\theta\vec{i}_3 = (\vec{i}_\theta\cdot\vec{i}_1)\vec{N}_1 + (\vec{i}_\theta\cdot\vec{i}_2)\vec{N}_2 \quad , \tag{1.60}$$

$$\vec{M}_k = \vec{i}_3 \times (M_{k1}\vec{i}_1 + M_{k2}\vec{i}_2) \quad , \quad (k = 1,2) \quad ,$$

$$\vec{M}_r = \vec{i}_3 \times (M_{rr}\vec{i}_r + M_{r\theta}\vec{i}_\theta) = (\vec{i}_r\cdot\vec{i}_1)\vec{M}_1 + (\vec{i}_r\cdot\vec{i}_2)\vec{M}_2 \quad ,$$

$$\vec{M}_\theta = \vec{i}_3 \times (M_{\theta r}\vec{i}_r + M_{\theta\theta}\vec{i}_\theta) = (\vec{i}_\theta\cdot\vec{i}_1)\vec{M}_1 + (\vec{i}_\theta\cdot\vec{i}_2)\vec{M}_2 \quad ,$$

(where Cauchy's formula has been used for the corresponding three dimensional stress measures), it is not difficult to transform the results of preceding sections into corresponding results in polar coordinates. We summarize these results below, with the derivations left as an exercise.

(A) Extension and Torsion

$$A\nabla^2\nabla^2 F = 0 \tag{1.57'}$$

$$N_{rr} = \frac{1}{r}F_{,r} + \frac{1}{r^2}F_{,\theta\theta} \quad , \quad N_{\theta\theta} = F_{,rr} \quad , \quad N_{r\theta} = N_{\theta r} = -\left(\frac{1}{r}F_{,\theta}\right)_{,r} \tag{1.61}$$

$$\varepsilon_{rr} = U_{r,r} = A(N_{rr} - \nu N_{\theta\theta}) \quad , \quad \varepsilon_{\theta\theta} = \frac{1}{r}(U_{\theta,\theta} + U_r) = A(N_{\theta\theta} - \nu N_{rr})$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2r}[U_{r,\theta} + rU_{\theta,r} - U_\theta] = A(1+\nu)N_{r\theta} \quad . \tag{1.62}$$

Also, the stress resultants satisfy the following differential equations of force equilibrium:

$$(rN_{rr})_{,r} + N_{\theta r,\theta} - N_{\theta\theta} = 0 \quad , \quad (rN_{r\theta})_{,r} + N_{\theta\theta,\theta} + N_{\theta r} = 0 \quad , \tag{1.63}$$

while the mid-plane strain components satisfy the following compatibility condition

$$(r\varepsilon_{\theta\theta})_{,rr} + \frac{1}{r}\varepsilon_{rr,\theta\theta} - \varepsilon_{rr,r} - \frac{2}{r}(r\varepsilon_{r\theta})_{,\theta r} = 0 \quad . \tag{1.64}$$

(B) Bending and Twisting

$$-D\nabla^2\nabla^2 W + p = 0 \tag{1.24}$$

$$\kappa_{\theta\theta} = -\frac{1}{r}W_{,r} - \frac{1}{r^2}W_{,\theta\theta} \quad , \quad \kappa_{rr} = -W_{,rr} \quad , \quad \kappa_{\theta r} = \kappa_{r\theta} = -(\frac{1}{r}W_{,\theta})_{,r} \qquad (1.65)$$

$$M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu\kappa_{rr}) \quad , \quad M_{rr} = D(\kappa_{rr} + \nu\kappa_{\theta\theta}) \quad , \quad M_{r\theta} = M_{\theta r} = D(1-\nu)\kappa_{\theta r} \qquad (.166)$$

$$Q_r = \frac{1}{r}[(rM_{rr})_{,r} + M_{\theta r,\theta} - M_\theta] = -D(\nabla^2 W)_{,r}$$

(1.67)

$$Q_\theta = \frac{1}{r}[(rM_{r\theta})_{,r} + M_{\theta\theta,\theta} + M_{\theta r}] = -\frac{1}{r}D(\nabla^2 W)_{,\theta} \quad .$$

Note that the expressions (1.65) for κ_{rr} , $\kappa_{\theta\theta}$ and $\kappa_{r\theta} = \kappa_{\theta r}$ in terms W satisfy the following two compatibility equations:

$$(r\kappa_{\theta\theta})_{,r} - \kappa_{r\theta,\theta} - \kappa_{rr} = 0 \quad , \quad (r\kappa_{\theta r})_{,r} - \kappa_{rr,\theta} + \kappa_{r\theta} = 0 \tag{1.68}$$

while the governing PDE (1.24) for W is a consequence of transverse force equilibrium

$$(rQ_r)_{,r} + Q_{\theta,\theta} + rp = 0 \quad . \tag{1.69}$$

9. A Circular Plate Under Uniform Net Face Pressure Distribution

Consider a (transversely rigid) circular plate with its edge given by $r = \sqrt{x_1^2 + x_2^2} = a$ and $|x_3| \le h/2$. The plate is subject to a normal traction at the two faces with a net pressure distribution uniform for the entire mid-plane, i.e., $p(x_1,x_2) = p_o$ (a constant). Suppose the plate is constrained at its only edge so that $\vec{u} = \vec{0}$ there. The condition of no displacement at $r = a$ and $|x_3| \le h/2$ is most simply expressed in a polar coordinate formulation:

$$U_r(a,\theta) = U_\theta(a,\theta) = 0 \qquad (1.70a)$$

$$(0 \le \theta \le 2\pi)$$

$$W(a,\theta) = W_{,r}(a,\theta) = 0 \quad . \qquad (1.70b)$$

A polar coordinate formulation also allows us to take advantage of the expected polar symmetry in stresses and displacements resulting from the uniform pressure distribution by simply taking $\partial(\ )/\partial\theta \equiv 0$. To calculate the displacement and stress distributions of the plate, we observe that the portion of the plate problem associated with in-plane extension and torsion involves no external load inside the cylindrical boundary surface and no displacement along the edge. The homogeneous BVP for U_r and U_θ has only a trivial solution $U_r \equiv U_\theta \equiv 0$ which induces no stresses in the plate. Therefore, we only have to focus our attention to the bending and twisting portion of the problem.

With $\partial(\ )/\partial\theta \equiv 0$, the biharmonic equation (1.24) for $W(r,\theta) \equiv W(r)$ is simplified to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right) = \frac{p_o}{D} \quad , \tag{1.71}$$

and may be solved by setting

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} = V \tag{1.72}$$

so that (1.71) becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = \frac{p_o}{D} \quad . \tag{1.73}$$

The general solution of the equidimensional ODE (1.73) for constant p_o/D is

$$V = c_1 + c_2 \ln r + \frac{p_o r^2}{4D} \quad . \tag{1.74}$$

With $V = [d(rdW/dr)/dr]/r$, we have from (1.72) and (1.74)

$$W = c_4 + c_3 \ln r + \frac{1}{4}c_2 r^2 + \frac{1}{4}c_1 r^2(\ln r - 1) + \frac{p_o}{64D}r^4 \quad . \tag{1.75}$$

For the transverse displacement W to be bounded for $0 \le r \le a$, we must have $c_3 = 0$. With

$$Q_r = -D[\nabla^2 W]_{,r} = -DV_{,r} = -Dc_1\frac{1}{r} - \frac{p_o}{2}r \quad ,$$

the transverse shear stress σ_{r3} is bounded for $0 \le r \le a$ only if $c_1 = 0$. After setting $c_1 = c_3 = 0$, the expression (1.75) for W becomes

$$W = c_4 + \frac{1}{4}c_2 r^2 + \frac{p_o}{64D}r^4 \quad . \tag{1.76}$$

To determine c_2 and c_4, we apply the two boundary conditions (1.70b) to get

$$c_4 + \frac{1}{4}c_2 a^2 + \frac{p_o a^4}{64D} = 0$$

$$\frac{1}{2}c_2 a + \frac{p_o}{16D}a^3 = 0 \quad .$$

The second condition determines $c_2 = -p_o a^2/8D$ and the first in turn gives $c_4 = p_o a^4/64D$. Upon substituting these expressions into (1.76), we get

$$W(r) = \frac{p_o a^4}{64D}[1 - (\frac{r}{a})^2]^2 \quad . \tag{1.77}$$

The stress couples and transverse shear resultants can now be calculated from (1.65)-(1.67) to be

$$M_{rr} = -\frac{p_o a^2}{16}\left[(1+\nu) - (3+\nu)\frac{r^2}{a^2}\right] \quad , \quad M_{\theta\theta} = -\frac{p_o a^2}{16}\left[(1+\nu) - (1+3\nu)\frac{r^2}{a^2}\right]$$

$$Q_r = \frac{1}{2}p_o r \quad , \quad Q_\theta \equiv 0 \quad , \quad M_{r\theta} \equiv M_{\theta r} \equiv 0 \quad . \tag{1.78}$$

2. PARAMETRIC EXPANSIONS AND THE LINEAR THEORY OF ISOTROPIC PLATES

1. Dimensionless Equations of Linear Elastostatics

We saw in chapter (1) that the solution process for boundary value problems of linear elastostatics is simplified substantially when the plate is transversely rigid. The solution for this limiting case also turns out to be a good approximation of the solution for plates which are not transversely rigid (at least away from the edge(s) of the plate) as long as the plates are not "transversely soft", i.e. E_3 and G_3 are not much smaller than the corresponding elastic moduli in the in-plane directions. In the context of matched asymptotic expansions, the results for a transversely rigid plate is just the leading term of the outer expansion of the exact solution for the corresponding isotropic plate. We will obtain in this chapter this leading term outer solution as well as higher order correction terms. We will also discuss briefly the nature of the complementary inner solution of the problem.

To keep our discussion relatively simple, we consider only isotropic plates with a uniform thickness h, with no body force intensities (so that $f_i(x_1, x_2, x_3) \equiv 0$) and with prescribed surface traction at the top and bottom faces of the plate. The analysis in this chapter can be extended, in a straightforward way, to other plates with more general loadings.

Let L be a typical in-plane scale length. It may be the wavelength of the surface traction or a representative lineal dimension of the plate in its midplane. An elastic body is said to be a plate if

$$\varepsilon \equiv \frac{h}{2L} << 1 \quad . \tag{2.1}$$

We now introduce dimensionless spatial variables

$$x = x_1/L \quad , \quad y = x_2/L \quad , \quad z = x_3/(h/2) \tag{2.2}$$

and scale the stress and displacement quantities by

$$\sigma_{ij} = \sigma_o s_{ij}(x,y,z) \quad , \quad \sigma_{33} = \sigma_2 s_{33}(x,y,z) \quad ,$$

$$\sigma_{3j} = \sigma_{j3} = \sigma_1 s_{3j}(x,y,z) = \sigma_1 s_{j3}(x,y,z) \quad , \tag{2.3}$$

$$u_3 = u_o W(x,y,z) \quad , \quad u_j = v_o V_j(x,y,z) \quad ,$$

where $i,j = 1,2$ and the dimensionless quantities s_{ij} , s_{3j} , W and V_j are $O(1)$ quantities.

In terms of the dimensionless quantities, the equilibrium equations take the form

$$\varepsilon\sigma_o[s_{11,x} + s_{21,y}] + \sigma_1[s_{31,z}] = 0 \quad , \quad \varepsilon\sigma_o[s_{12,x} + s_{22,y}] + \sigma_1[s_{32,z}] = 0 \quad ,$$

$$\varepsilon\sigma_1[s_{13,x} + s_{23,y}] + \sigma_2[s_{33,z}] = 0 \quad . \tag{2.4}$$

In general, σ_{3m} , $m = 1,2,3$, vary considerably across the plate thickness as the prescribed tractions on the top and bottom face may differ substantially. Their contributions to the equilibrium equations cannot dominate or remain insignificant as either case may lead to physically unacceptable consequences. Therefore, we take

$$\sigma_1 = \varepsilon\sigma_o \quad , \qquad \sigma_2 = \varepsilon^2\sigma_o \tag{2.5}$$

and therewith

$$s_{11,x} + s_{21,y} + s_{31,z} = 0 \quad , \quad s_{12,x} + s_{22,y} + s_{32,z} = 0 \quad ,$$

$$s_{13,x} + s_{23,y} + s_{33,z} = 0 \quad . \tag{2.4'}$$

Correspondingly, the strain-displacement relations and stress-strain relations for an isotropic medium may be combined to give

$$v_oV_{1,x} = \frac{L\sigma_o}{E}[s_{11} - \nu(s_{22} + \varepsilon^2 s_{33})] \ , \ v_oV_{2,y} = \frac{L\sigma_o}{E}[s_{22} - \nu(s_{11} + \varepsilon^2 s_{33})] \tag{2.6a}$$

$$v_o[V_{1,y} + V_{2,x}] = \frac{L\sigma_o}{E}[2(1+\nu)s_{12}] \tag{2.6b}$$

$$v_oV_{1,z} + \varepsilon u_oW_{,x} = \frac{L\sigma_o}{E}\varepsilon^2[2(1+\nu)s_{13}] \ , \ v_oV_{2,z} + \varepsilon u_oW_{,y} = \frac{L\sigma_o}{E}\varepsilon^2[2(1+\nu)s_{33}] \tag{2.6c}$$

$$u_oW_{,z} = \frac{\sigma_oL}{E}\varepsilon[\varepsilon^2 s_{33} - \nu(s_{11} + s_{22})] \quad . \tag{2.6d}$$

Evidently, u_m , $m = 1,2,3$, generally vary considerably over the span of the plate. To allow for this variation, we take, according to (2.6a,b),

$$v_o = \frac{L\sigma_o}{E} \qquad (2.7)$$

so that the relations in (2.6c) become

$$V_{1,2} + \varepsilon \frac{u_o}{v_o} W_{,x} = \varepsilon^2[2(1+\nu)s_{13}] \ , \quad V_{2,z} + \varepsilon \frac{u_o}{v_o} W_{,y} = \varepsilon^2[2(1+\nu)s_{23}] \ .$$

The terms on the right of these relations are small, of higher order in ε. It follows $\varepsilon u_o/v_o$ is $O(1)$ at most; otherwise W is approximatel uniform over the span of the plate. To allow for plate bending, we take

$$\varepsilon u_o = v_o = \frac{\sigma_o L}{E} \quad \text{or} \quad u_o = \frac{2\sigma_o L^2}{Eh} \qquad (2.8)$$

as V_j , $j = 1,2$, cannot be uniform across the plate thickness in general.

With (2.7) and (2.8), the stress-displacement relations (2.6a-d) become

$$V_{1,x} = s_{11} - \nu(s_{22} + \varepsilon^2 s_{33}) \ , \quad V_{2,y} = s_{22} - \nu(s_{11} + \varepsilon^2 s_{33}) \qquad (2.6a')$$

$$V_{1,y} + V_{2,x} = 2(1+\nu)s_{12} \qquad (2.6b')$$

$$V_{1,z} + W_{,x} = \varepsilon^2[2(1+\nu)s_{13}] \ , \quad V_{2,z} + W_{,y} = \varepsilon^2[2(1+\nu)s_{23}] \qquad (2.6c')$$

$$W_{,z} = \varepsilon^2[\varepsilon^2 s_{33} - \nu(s_{11} + s_{22})] \ . \qquad (2.6d')$$

We note from (2.5), (2.7) and (2.8) that of the five scale factors, σ_1, σ_2, u_o and v_o are now determined in terms of σ_o. The scale factor σ_o is to be determined by the external load magnitude. In the present case, the external loads consist of the face tractions at $z = \pm 1$ and the edge traction along the cylindrical face(s), or the edge(s), of the plate defined (parametrically) by $x_j = \bar{x}_j(s)$, $j = 1,2$, and $|x_3| \le h/2$ for some given functions $\bar{x}(s)$ and $\bar{x}_2(s)$ of the arclength variable s. For this chapter, we restrict ourselves to cases where σ_o is determined by the face tractions. The possibility that edge loads may be dominant will be analyzed in chapter (4).

For simplicity, we consider, as in chapter (1), only cases where the applied face tractions are in the direction normal to the faces so that

$$z = \pm 1 : \quad \sigma_o \varepsilon s_{31} = \sigma_o \varepsilon s_{32} = 0$$

$$\sigma_o \varepsilon^2 s_{33} = \begin{Bmatrix} \sigma_t(x_1,x_2) \\ \sigma_b(x_1,x_2) \end{Bmatrix} .$$

Suppose $\bar{p}_o$ is the maximum absolute value of σ_t and σ_b. Let

$$\sigma_t(x_1,x_2) \equiv \bar{p}_o p_+(x,y) , \qquad \sigma_b(x_1,x_2) \equiv \bar{p}_o p_-(x,y)$$

with $|p_\pm| \le 1$. From the boundary conditions at the two faces, we have

$$\sigma_o \varepsilon^2 = \bar{p}_o \qquad \text{or} \qquad \sigma_o = \frac{\bar{p}_o}{\varepsilon^2} = \frac{\bar{p}_o L^2}{(h/2)^2} \tag{2.9}$$

and therewith

$$\sigma_1 = \sigma_o \varepsilon = \frac{\bar{p}_o}{\varepsilon} = \frac{\bar{p}_o L}{(h/2)} \quad , \quad \sigma_2 = \sigma_o \varepsilon^2 = \bar{p}_o$$

$$\frac{u_o}{(h/2)} = \frac{\bar{p}_o}{E\varepsilon^4} = \frac{\bar{p}_o L^4}{E(h/2)^4} \quad , \quad \frac{v_o}{(h/2)} = \frac{\bar{p}_o}{E\varepsilon^3} = \frac{\bar{p}_o L^3}{E(h/2)^3} \qquad (2.10)$$

The face boundary conditions themselves become

$$z = \pm 1 : \qquad s_{31} = s_{32} = 0 \quad , \quad s_{33} = p_{\pm}(x,y). \qquad (2.11)$$

The boundary conditions along the plate edge(s) $\{x_j = \bar{x}_j(s) \, , \, |x_3| \le h/2\}$ will be discussed in a later section.

2. Leading Term of a Perturbation Series Solution in ε

The dimensionless form of the equations of linear elastostatics of a flat isotropic plate of uniform thickness suggests that we seek a parametric series solution[1] of the boundary value problem in powers of ε :

$$\{s_{mk}(x,y,z;\varepsilon),\ V_j(x,y,z;\varepsilon),\ W(x,y,z;\varepsilon)\}$$
$$= \sum_{n=0}^{\infty} \{s_{mk}^{(n)}(x,y,z),\ V_j^{(n)}(x,y,z),\ W^{(n)}(x,y,z)\}\varepsilon^n \quad . \tag{2.12}$$

Under the assumption that differentiations with respect to x, y and z do not change order of magnitude, substitution of (2.12) into (2.4') and (2.6') shows that the leading terms of the series are determined by

$$W^{(0)}_{,z} = 0 \quad , \qquad V^{(0)}_{1,z} + W^{(0)}_{,x} = 0 \quad , \qquad V^{(0)}_{2,z} + W^{(0)}_{,y} = 0 \quad ,$$

$$s^{(0)}_{11} = \frac{V^{(0)}_{1,x} + \nu V^{(0)}_{2,y}}{1-\nu^2} \, , \qquad s^{(0)}_{22} = \frac{V^{(0)}_{2,y} + \nu V^{(0)}_{1,x}}{1-\nu^2} \, , \qquad s^{(0)}_{12} = \frac{V^{(0)}_{1,y} + V^{(0)}_{2,x}}{2(1+\nu)} \tag{2.13}$$

$$s^{(0)}_{3k,z} = -\,(s^{(0)}_{1k,x} + s^{(0)}_{2k,y}) \qquad (k = 1,2,3) \quad .$$

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(1) Consistent with the form of (2.6'), we may take the parametric series in powers of ε^2 . However, the complementary inner solution of section (6) requires that terms involving odd powers of ε be included as well.

Except for a change in notation and the order of appearance, these are exactly the same as the equations for a transversely rigid plate. Since the boundary conditions on the two faces are also identical, the leading term perturbation solution for the isotropic plate is identical to the exact solution for the transversely rigid plate obtained in chapter (1). We summarize this solution below using a different notation to indicate that it is only the leading term of a parametric series solution:

$$W^{(0)}(x,y,z) = w^{(0)}(x,y) \tag{2.14a}$$

$$\begin{aligned} V_1^{(0)}(x,y,z) &= v_1^{(0)}(x,y) - zw_{,x}^{(0)}(x,y) \quad , \\ V_2^{(0)}(x,y,z) &= v_2^{(0)}(x,y) - zw_{,y}^{(0)}(x,y) \end{aligned} \tag{2.14b}$$

$$s_{ij}^{(0)}(x,y,z) = n_{ij}^{(0)}(x,y) + zm_{ij}^{(0)}(x,y) \qquad (i,j = 1,2) \tag{2.14c}$$

$$\begin{aligned} n_{11}^{(0)} &= \frac{v_{1,x}^{(0)} + \nu v_{2,y}^{(0)}}{1 - \nu^2} \quad , \quad n_{22}^{(0)} = \frac{v_{2,y}^{(0)} + \nu v_{1,x}^{(0)}}{1 - \nu^2} \quad , \\ n_{12}^{(0)} &= n_{21}^{(0)} = \frac{v_{1,y}^{(0)} + v_{2,x}^{(0)}}{2(1 + \nu)} \end{aligned} \tag{2.14d}$$

$$\begin{aligned} m_{11}^{(0)} &= -\frac{w_{,xx}^{(0)} + \nu w_{,yy}^{(0)}}{1 - \nu^2} \quad , \quad m_{22}^{(0)} = -\frac{w_{,yy}^{(0)} + \nu w_{,xx}^{(0)}}{1 - \nu^2} \quad , \\ m_{12}^{(0)} &= m_{21}^{(0)} = \frac{w_{,xy}^{(0)}}{1+\nu} \end{aligned} \tag{2.14e}$$

$$s_{3j}^{(0)} = \frac{1}{2} q_j^{(0)}(x,y)(1 - z^2) \qquad (j = 1,2) \tag{2.14f}$$

$$q_1^{(0)} = m_{11,x}^{(0)} + m_{21,y}^{(0)} = - \frac{(\nabla^2 w^{(0)})_{,x}}{1 - \nu^2}$$

(2.14g)

$$q_2^{(0)} = m_{12,x}^{(0)} + m_{22,y}^{(0)} = - \frac{(\nabla^2 w^{(0)})_{,y}}{1 - \nu^2}$$

$$s_{33}^{(0)} = p_e(x,y) + \frac{3}{2} p_o (z - \frac{1}{3} z^3) \tag{2.14h}$$

$$p_e = \frac{1}{2}(p_+ + p_-) \quad , \quad p_o = \frac{1}{2}(p_+ - p_-) \quad . \tag{2.14i}$$

The three leading term midplane displacement components $v_1^{(0)}$, $v_2^{(0)}$ and $w^{(0)}$ are determined by

$$n_{11,x}^{(0)} + n_{21,y}^{(0)} = \frac{1}{1-\nu^2} [\nabla^2 v_1^{(0)} - \frac{1}{2}(1+\nu)(v_{1,y}^{(0)} - v_{2,x}^{(0)})_{,y}] = 0 \quad ,$$

(2.15)

$$n_{12,x}^{(0)} + n_{22,y}^{(0)} = \frac{1}{1-\nu^2} [\nabla^2 v_2^{(0)} + \frac{1}{2}(1+\nu)(v_{1,y}^{(0)} - v_{2,x}^{(0)})_{,x}] = 0 \quad ,$$

and

$$q_{1,x}^{(0)} + q_{2,y}^{(0)} + 3p_o = - \frac{\nabla^4 w^{(0)}}{1-\nu^2} + 3p_o = 0 \quad , \tag{2.16}$$

respectively, supplemented by boundary conditions along the (cylindrical) edge(s) of the plate.

In the above solution, we have implicitly assumed that the dimensionless applied face pressures $p_{\pm}(x,y)$ are independent of ε. If they are functions of ε, then they should be expanded in powers of ε as well with the leading term $p_{\pm}^{(0)}(x,y)$ entering (2.14i) instead of p_+ and p_- themselves. The higher order terms $p_{\pm}^{(k)}(x,y)$ will enter into the solution process for the higher order terms in the series (2.12).

It is customary in two dimensional elasticity to use a stress function formulation for the stretching parts of the problem. Following this practice, we set

$$n_{11}^{(0)} = F_{,yy}^{(0)} \quad , \quad n_{22}^{(0)} = F_{,xx}^{(0)} \quad , \quad n_{12}^{(0)} = n_{21}^{(0)} = -F_{,xy}^{(0)} \qquad (2.17)$$

so that in-plane equilibrium (2.15) is satisfied identically. To determine the unknown stress function $F^{(0)}(x,y)$, we observe that the three relations (2.14d) imply a compatibility condition among the three $n_{ij}^{(0)}$:

$$(n_{11} - \nu n_{22})_{,yy} + (n_{22} - \nu n_{11})_{,xx} - 2(1 + \nu)n_{12,xy} = 0 \ . \qquad (2.18)$$

When expressed in terms of $F^{(0)}$, (2.18) becomes

$$\nabla^4 F^{(0)} = 0 \ . \qquad (2.18')$$

The compatibility condition (2.18) also ensures the integrability of (2.14d) for $V_1^{(0)}$ and $v_2^{(0)}$ given $n_{ij}^{(0)}$. An inhomogeneous term would be added to the two in-plane equilibrium equations (2.15), and hence to the stress function representation (2.17), if the faces of the plate are not free of shear stresses.

Having completed the reduction of the equations for the leading terms of the series in (2.12) to two biharmonic equations, one homogeneous equation for $F^{(0)}(x,y)$ and one inhomogeneous equation for $w^{(0)}(x,y)$, we note that the equations for the $0(\varepsilon)$ terms in the series in (2.12) are determined by a set of equations which is identical to (2.13) in form with the superscript (0) replaced by (1). The reason is, of course, that only ε^2 and ε^4 appear in the dimensionless form of the original equations of linear elastostatics (2.4') and (2.6'). It is not difficult to see then that the $0(\varepsilon)$ terms in the series in (2.12) are also given in terms of two quantities, $w^{(1)}(x,y)$ and $F^{(1)}(x,y)$, which are solutions of biharmonic equations. In the case $p_{\pm}(x,y)$ are independent of ε, we have

$$\nabla^4 w^{(1)} = 0 \quad , \quad \nabla^4 F^{(1)} = 0 \quad . \tag{2.19a,b}$$

The $0(\varepsilon)$ terms of the series, $W^{(1)}$, $V_j^{(1)}$, $s_{ij}^{(1)}$ and $s_{3j}^{(1)}$ are given in terms of $w^{(1)}$ and $F^{(1)}$ just as $W^{(0)}$, $V_j^{(0)}$, $s_{ij}^{(0)}$ and $s_{3j}^{(0)}$ are given in terms of $F^{(0)}$ and $w^{(0)}$ in (2.14 a-g). Because p_+ and p_- are independent of ε, we have instead of (2.14h) and (2.14i)

$$s_{33}^{(1)} \equiv 0 \quad . \tag{2.19c}$$

3. The $O(\varepsilon^2)$ Term

The $O(\varepsilon^2)$ terms of the series in (2.12) are determined by the system of equations,

$$W^{(2)}_{,z} = -\nu(s^{(0)}_{11} + s^{(0)}_{22}) \quad , \tag{2.20a}$$

$$V^{(2)}_{1,z} + W^{(2)}_{,x} = 2(1+\nu)s^{(0)}_{13} \; , \quad V^{(2)}_{2,z} + W^{(2)}_{,y} = 2(1+\nu)s^{(0)}_{23} \tag{2.20b}$$

$$s^{(2)}_{11} = \frac{V^{(2)}_{1,x} + \nu V^{(2)}_{2,y} + \nu(1+\nu)s^{(0)}_{33}}{1-\nu^2} \; , \quad s^{(2)}_{22} = \frac{V^{(2)}_{2,y} + \nu V^{(2)}_{1,x} + \nu(1+\nu)s^{(0)}_{33}}{1-\nu^2} \tag{2.20c}$$

$$s^{(2)}_{12} = s^{(2)}_{21} = \frac{V^{(2)}_{1,y} + V^{(2)}_{2,x}}{2(1+\nu)} \tag{2.20d}$$

$$s^{(2)}_{3k,z} = -(s^{(2)}_{1k,x} + s^{(2)}_{2k,y}) \quad , \quad (k = 1,2,3) \quad , \tag{2.20e}$$

which follows from (2.12), (2.4') and (2.6'). The system of differential equations is supplemented by the boundary conditions on the two faces,

$$z = \pm 1 : \quad s^{(2)}_{3k} \equiv 0 \quad , \quad (k = 1,2,3) \tag{2.20f}$$

and the boundary conditions at the edge(s) of the plate. Except for some additional inhomogeneous terms which are known from the last section, these equations are identical to (2.13) for the leading terms solution.

With $s_{11}^{(0)}$ and $s_{22}^{(0)}$ given by (2.14c) and (2.17), equation (2.20a) may be integrated to give

$$W^{(2)}(x,y,z) = w^{(2)}(x,y) - \nu z \nabla^2 F^{(0)} + \frac{\nu z^2}{2(1-\nu)} \nabla^2 w^{(0)} \qquad (2.21a)$$

where $w^{(2)}$ is an arbitrary function to be determined in the subsequent development. With (2.21a) and $s_{k3}^{(0)}$ given by (2.14f), we get from (2.20b)

$$V_1^{(2)}(x,y,z) = v_1^{(2)}(x,y) - z w_{,x}^{(2)} + \frac{1}{2}\nu z^2 \nabla^2 F_{,x}^{(0)} - \frac{6z-(2-\nu)z^3}{6(1-\nu)} \nabla^2 w_{,x}^{(0)}$$

$$\qquad (2.21b)$$

$$V_2^{(2)}(x,y,z) = v_2^{(2)}(x,y) - z w_{,y}^{(2)} + \frac{1}{2}\nu z^2 \nabla^2 F_{,y}^{(0)} - \frac{6z-(2-\nu)z^3}{6(1-\nu)} \nabla^2 w_{,y}^{(0)} .$$

It follows from (2.21b), (2.20c) and (2.20d)

$$s_{11}^{(2)} = n_{11}^{(2)} + z m_{11}^{(2)} + \frac{\nu z^2}{2(1-\nu^2)} \nabla^2 (F_{,xx}^{(0)} + \nu F_{,yy}^{(0)}) + t_o(z) \nabla^2 m_{11}^{(0)} + \frac{\nu}{1-\nu} s_{33}^{(0)}$$

$$s_{22}^{(2)} = n_{22}^{(2)} + z m_{11}^{(2)} + \frac{\nu z^2}{2(1-\nu^2)} \nabla^2 (F_{,yy}^{(0)} + \nu F_{,xx}^{(0)}) + t_o(z) \nabla^2 m_{22}^{(0)} + \frac{\nu}{1-\nu} s_{33}^{(0)} \qquad (2.21c)$$

$$s_{12}^{(0)} = s_{21}^{(0)} = n_{12}^{(2)} + z m_{12}^{(2)} + \frac{\nu z^2}{2(1+\nu)} \nabla^2 F_{,xy}^{(0)} + t_o(z) \nabla^2 m_{12}^{(0)}$$

where $s_{33}^{(0)}(x,y,z)$ is a known function given by (2.14h) and (2.14i), $n_{ij}^{(2)}(x,y)$ and $m_{ij}^{(2)}(x,y)$ are given in terms of $v_j^{(2)}$ and $w^{(2)}$ by the same relations, (2.14d) and (2.14e), which give $n_{ij}^{(0)}$ and

$m_{ij}^{(0)}$ in terms of $v_j^{(0)}$ and $w^{(0)}$, and

$$t_o(z) = \frac{6z - (2 - \nu)z^3}{6(1 - \nu)} \quad . \tag{2.21d}$$

From (2.20e) with $k = 1$, we have

$$-s_{31,z}^{(2)} = (n_{11,x}^{(2)} + n_{21,y}^{(2)}) + z(m_{11,x}^{(2)} + m_{21,y}^{(2)}) + \frac{\nu z^2}{2(1-\nu^2)} [\nabla^4 F^{(0)}]_{,x}$$

$$- \frac{t_o(z)}{1-\nu^2} [\nabla^4 w^{(0)}]_{,x} + \frac{\nu}{1 - \nu} s_{33,x}^{(0)} ,$$

or

$$s_{31}^{(2)} = \frac{1}{2} q_1^{(2)}(x,y) - z(n_{11,x}^{(2)} + n_{21,y}^{(2)}) - \frac{1}{2} z^2 (m_{11,x}^{(2)} + m_{21,y}^{(2)})$$

$$+ t_1(z)[3p_{o,x}] - \frac{\nu}{1 - \nu} [zp_{e,x} + \frac{3}{4} p_{o,x}(z^2 - \frac{1}{6} z^4)] ,$$

where $t_1(z) = [12z^2 - (2 - \nu)z^4]/24(1 - \nu)$. The boundary conditions $s_{31}^{(2)} = 0$ at $z = \pm 1$ require

$$n_{11,x}^{(2)} + n_{21,y}^{(2)} + \frac{\nu}{1 - \nu} p_{e,x} = 0 \tag{2.21e}$$

$$q_1^{(2)}(x,y) = m_{11,x}^{(2)} + m_{21,y}^{(2)} - \frac{20-3\nu}{8(1-\nu)} p_{o,x} = - \frac{\nabla^2 w_{,x}^{(2)}}{1-\nu^2} - \frac{20-3\nu}{8(1-\nu)} p_{o,x} \quad . \tag{2.21f}$$

These requirements simplify the expression for $s_{31}^{(2)}$ to read

$$s_{31}^{(2)}(x,y,z) = \frac{1}{2} q_1^{(2)}(x,y)(1-z^2) + t_2(z) p_{o,x} \tag{2.21g}$$

with $t_2(z) = [(44-15\nu)z^2 - 4(1-\nu)z^4]/16(1-\nu)$. Similarly, we have also from (2.20e) with $k = 2$

$$s_{32}^{(2)}(x,y,z) = \frac{1}{2}q_2^{(2)}(x,y)(1 - z^2) + t_2(z)p_{o,y} \qquad (2.21h)$$

with

$$q_2^{(2)}(x,y) = m_{12,x}^{(2)} + m_{22,y}^{(2)} - \frac{20-3\nu}{8(1-\nu)}p_{o,y} = -\frac{\nabla^2 w_{,y}^{(2)}}{1-\nu^2} - \frac{20-3\nu}{8(1-\nu)}p_{o,y} \qquad (2.21i)$$

$$n_{12,x}^{(2)} + n_{22,y}^{(2)} + \frac{\nu}{1-\nu}p_{e,y} = 0 \quad . \qquad (2.21j)$$

Finally, we have from (2.20e) with $k = 3$

$$s_{33,z}^{(2)} = -\frac{1}{2}[q_{1,x}^{(2)} + q_{2,y}^{(2)}](1 - z^2) - t_2(z)\nabla^2 p_o \quad .$$

Integration and the boundary condition $s_{33}^{(2)} \equiv 0$ on $z = \pm 1$ give

$$s_{33}^{(2)}(x,y,z) = [\frac{3}{2}(z - \frac{1}{3}z^3)t_3(1) - t_3(z)]\nabla^2 p_o \qquad (2.21k)$$

where $t_3(z) = [5(44 - 15\nu)z^3 - 12(1 - \nu)z^5]/240(1 - \nu)$ with

$$q_{1,x}^{(2)} + q_{2,y}^{(2)} + 3t_3(1)\nabla^2 p_o = 0 \qquad (2.21\ell)$$

or

$$-\frac{1}{1-\nu^2}\nabla^4 w^{(2)} + \frac{8 - 33\nu}{80(1-\nu)}\nabla^2 p_o = 0 \quad . \qquad (2.21\ell')$$

While the reduction of the equations for the $O(\varepsilon^2)$ terms of the perturbation series in (2.12) is essentially completed, we note the possible simplifications of the results when p_o and/or p_e are uniform in the midplane of the plate. When $p_{o,x} \equiv p_{o,y} \equiv 0$, the expressions (2.21f)-(2.21h), (2.21k) and (2.21ℓ) are simplified to

$$s_{3j}^{(2)}(x,y,z) = \frac{1}{2} q_j^{(2)}(x,y)(1-z^2) \ , \quad (j=1,2) \ , \quad s_{33}^{(2)}(x,y,z) \equiv 0 \qquad (2.22a,b)$$

with

$$q_1^{(2)}(x,y) = m_{11,x}^{(2)} + m_{21,y}^{(2)} = -\frac{\nabla^2 w_{,x}^{(2)}}{1-\nu^2}$$

$$q_2^{(2)}(x,y) = m_{12,x}^{(2)} + m_{22,y}^{(2)} = -\frac{\nabla^2 w_{,y}^{(2)}}{1-\nu^2} \qquad (2.22c)$$

$$q_{1,x}^{(2)} + q_{2,y}^{(2)} = -\frac{1}{1-\nu^2} \nabla^4 w^{(2)} = 0 \quad . \qquad (2.22d)$$

When $p_{e,x} \equiv p_{e,y} \equiv 0$, we introduce the stress function representation

$$n_{11}^{(2)} = F_{,yy}^{(2)} \ , \quad n_{22}^{(2)} = F_{,xx}^{(2)} \ , \quad n_{12}^{(2)} = n_{21}^{(2)} = -F_{,xy}^{(2)} \qquad (2.23a)$$

and the compatibility condition among the expressions for $n_{ij}^{(2)}$ in terms of $V_j^{(2)}$ again gives

$$\nabla^4 F^{(2)} = 0 \quad . \qquad (2.23b)$$

The stress function representation (2.23a) would have to include some load terms if $p_{e,x}$ or $p_{e,y}$ is not zero and the differential equation for $F^{(2)}$ would be inhomogeneous.

We note also that the equations for the determination of $O(\varepsilon^3)$ terms of the perturbation series in (2.12) are identical to those for the $O(\varepsilon^2)$ terms with all superscripts increased by one. Given that the solutions for the $O(\varepsilon)$ terms are identical in form to those for the $O(1)$ terms (except for $s_{33}^{(1)} \equiv 0$), the solutions for the $O(\varepsilon^3)$ terms are therefore identical in form to the solution for the $O(\varepsilon^2)$ terms obtained in this section, with all superscripts in the $O(\varepsilon^2)$ term solutions increased by one and with $s_{33}^{(1)}$ and other inhomogeneous terms involving the face tractions vanishing identically.

4. Higher Order Correction Terms

From (2.12), (2.4') and (2.6'), we obtain for $n \geq 4$ the following equations for the determination of the $O(\varepsilon^n)$ terms of the perturbation series in (2.12):

$$W^{(n)}_{,z} = -\nu[s^{(n-2)}_{11} + s^{(n-2)}_{22}] + s^{(n-4)}_{33} \tag{2.24a}$$

$$V^{(n)}_{1,z} + W^{(n)}_{,x} = 2(1+\nu)s^{(n-2)}_{13} \quad , \qquad V^{(n)}_{2,z} + W^{(n)}_{,y} = 2(1+\nu)s^{(n-2)}_{23} \tag{2.24b}$$

$$s^{(n)}_{11} = \frac{V^{(n)}_{1,x} + V^{(n)}_{2,y} + \nu(1+\nu)s^{(n-2)}_{33}}{1-\nu^2} \quad , \qquad s^{(n)}_{22} = \ldots . \tag{2.24c}$$

$$s^{(n)}_{12} = s^{(n)}_{21} = \frac{V^{(n)}_{1,y} + V^{(n)}_{2,x}}{2(1+\nu)} \tag{2.24d}$$

$$s^{(n)}_{3k,z} = -[s^{(n)}_{1k,x} + s^{(n)}_{2k,y}] \qquad (k = 1,2,3) \quad . \tag{2.24e}$$

This system of differential equations is supplemented by the boundary conditions

$$z = \pm 1 : \qquad s^{(n)}_{3k} = 0 \qquad (k = 1,2,3) \quad , \tag{2.24f}$$

when p_+ and p_- are independent of ε , and boundary conditions along the edge(s) of the plate. The above system, which holds for $n \geq 4$, differs from (2.13) of section (2) and (2.20) of section (3) again only in the inhomogeneous terms which are known from a previous section. Integration of the system can proceed as before, leading to its solution which differs from the solution for $n = 2$ case obtained in section (3) only in the particular solution involving the face pressure distributions p_+ and p_-. These particular solutions are polynomials in z of degree $n + 3$ when n is even; they are absent when n is odd. We summarize below the results for the special case where p_e or p_o is uniformly distributed while the other vanishes identically.

When n is even and greater than 2, the solution of the system (2.24) is

$$W^{(n)}(x,y,z) = \{w^{(n)}(x,y) + \frac{\nu z^2}{2(1-\nu)}\nabla^2 w^{(n-2)}\} - \nu z\{\nabla^2 F^{(n-2)}\} \tag{2.25a}$$

$$V_1^{(n)}(x,y,z) = \{v_1^{(n)}(x,y) + \frac{1}{2}\nu z^2\nabla^2 F^{(n-2)}_{,x}\} - \{zw^{(n)}_{,x} + t_o(z)\nabla^2 w^{(n-2)}_{,x}\}$$

$$V_2^{(n)}(x,y,z) = \{v_2^{(n)}(x,y) + \frac{1}{2}\nu z^2\nabla^2 F^{(n-2)}_{,y}\} - \{zw^{(n)}_{,y} + t_o(z)\nabla^2 w^{(n-2)}_{,y}\} \tag{2.25b}$$

$$s_{11}^{(n)}(x,y,z) = \{n_{11}^{(n)} + \frac{\nu z^2}{2(1-\nu^2)}\nabla^2[F^{(n-2)}_{,xx} + \nu F^{(n-2)}_{,yy}]\} + \{zm_{11}^{(n)} + t_o(z)\nabla^2 m_{11}^{(n-2)}\}$$

$$s_{22}^{(n)}(x,y,z) = \{n_{22}^{(n)} + \frac{\nu z^2}{2(1-\nu^2)}\nabla^2[F^{(n-2)}_{,xx} + \nu F^{(n-2)}_{,yy}]\} + \{zm_{22}^{(n)} + t_o(z)\nabla^2 m_{22}^{(n-2)}\} \tag{2.25c}$$

$$s_{12}^{(n)}(x,y,z) = s_{21}^{(n)}(x,y,z) = \{n_{12}^{(n)} + \frac{\nu z^2}{2(1+\nu)}\nabla^2 F^{(n-2)}_{,xy}\} + \{zm_{12}^{(n)} + t_o(z)\nabla^2 m_{12}^{(n-2)}\} \tag{2.25d}$$

$$s_{k3}^{(n)}(x,y,z) = s_{3k}^{(n)}(x,y,z) = \frac{1}{2} q_k^{(n)}(x,y)(1 - z^2) \qquad (k = 1,2) \tag{2.25e}$$

$$s_{33}^{(n)}(x,y,z) \equiv 0 \tag{2.25f}$$

where $t_o(z)$ is given by (2.21d) and where

$$m_{11}^{(n)}(x,y) = -\frac{w_{,xx}^{(n)} + \nu w_{,yy}^{(n)}}{1-\nu^2} \quad , \quad m_{22}^{(n)}(x,y) = -\frac{w_{,yy}^{(n)} + \nu w_{,xx}^{(n)}}{1-\nu^2}$$

$$m_{12}^{(n)}(x,y) = m_{21}^{(n)}(x,y) = -\frac{w_{,xy}^{(n)}}{1+\nu} \tag{2.25g}$$

$$q_1^{(n)}(x,y) = m_{11,x}^{(n)} + m_{21,y}^{(n)} = -\frac{1}{1-\nu^2}\nabla^2 w_{,x}^{(n)} \quad ,$$

$$q_2^{(n)}(x,y) = m_{12,x}^{(n)} + m_{22,y}^{(n)} = -\frac{1}{1-\nu^2}\nabla^2 w_{,y}^{(n)} \tag{2.25h}$$

$$q_{1,x}^{(n)} + q_{2,y}^{(n)} = -\frac{1}{1-\nu^2}\nabla^4 w^{(n)} = 0 \tag{2.25i}$$

$$\frac{v_{1,x}^{(n)} + \nu v_{2,y}^{(n)}}{1-\nu^2} = n_{11}^{(n)}(x,y) = F_{,yy}^{(n)} \quad ,$$

$$\frac{v_{2,y}^{(n)} + \nu v_{1,x}^{(n)}}{1-\nu^2} = n_{22}^{(n)}(x,y) = F_{,xx}^{(n)} \quad , \tag{2.25j}$$

$$\frac{v_{1,y}^{(n)} + v_{2,x}^{(n)}}{2(1+\nu)} = n_{12}^{(n)}(x,y) = n_{21}^{(n)}(x,y) = -F_{,xy}^{(n)}$$

$$[n_{11}^{(n)} - \nu n_{22}^{(n)}]_{,yy} + [n_{22}^{(n)} - \nu n_{11}^{(n)}]_{,xx} - [2(1+\nu)n_{12}^{(n)}]_{,xy} = \nabla^4 F^{(n)} = 0 \tag{2.25k}$$

with the stress function representation (2.25j) automatically satisfying in-plane equilibrium

$$n_{11,x}^{(n)} + n_{21,y}^{(n)} = 0 \quad , \quad n_{12,x}^{(n)} + n_{22,y}^{(n)} = 0 \quad , \quad n_{12}^{(n)} = n_{21}^{(n)} \quad . \tag{2.25ℓ}$$

5. Boundary Conditions Along an Edge of the Plate

In the last three sections, we have obtained a class of solutions for the equations of linear elastostatics which satisfies the prescribed traction conditions on the two faces of a thin plate. It remains to specify a solution which also satisfies the prescribed boundary conditions along the edge(s) of the plate.

Consider first the leading term (approximate) solution of section (2). It is identical to the solution for a transversely rigid plate which is traction free on the faces obtained in chapter (1). Therefore, the discussion of boundary conditions of sections (5) and (6) of that chapter applies also to the present leading term perturbation solution for an isotropic plate. We recall in particular the following situations for displacement boundary conditions:

(i) Homogeneous displacement conditions along an edge, so that we have $u_j \equiv 0$, $j = 1,2,3$, there, are satisfied by requiring $v_j^{(0)}$, $w^{(0)}$ and the derivative normal to the edge (but in the midplane of the plate) to vanish along the edge. These four conditions along with the two biharmonic equations for $w^{(0)}$ and $F^{(0)}$ can be shown to define a well-posed boundary value problem.

(ii) Inhomogeneous displacement conditions along an edge cannot be satisfied unless the prescribed displacement data is such that u_3 is uniform across the plate thickness and u_1 and u_2 are linear in z . In addition to the consistency requirement on the approximate solution (2.14a) and (2.14b) with regard to its z dependence, the multiplication factor of z in the displacement tangent to the edge is also required to be the edgewise derivative of u_3 along the edge of the midplane. When all these highly restrictive requirements are satisfied, the resulting boundary conditions on $w^{(0)}$ and $v_j^{(0)}$ lead to a well defined boundary value problem for $F^{(0)}$ and $w^{(0)}$.

(iii) Stress free boundary conditions require $n_{\nu\nu}^{(0)}$, $n_{\nu t}^{(0)}$, $m_{\nu\nu}^{(0)}$, $m_{\nu t}^{(0)}$ and $q_{\nu}^{(0)}$ to vanish along the edge where ν and t indicate directions (in the middle plane) normal and tangent to the edge. The biharmonic equation (2.18') together with the two boundary conditions on $n_{\nu\nu}^{(0)}$ and $n_{\nu t}^{(0)}$ define a well-posed boundary value problem for plate extension and torsion. However, the remaining three conditions are one too many for the biharmonic equation for $w^{(0)}$ for plate bending. By variational considerations (later in chapter (3) and (5)), it is found that the three conditions should be contracted to two boundary conditions on $m_{\nu\nu}^{(0)}$ and $q_{\nu}^{(0)} + \partial m_{\nu t}^{(0)}/\partial s$.

(iv) Inhomogeneous stress conditions along an edge of the plate in general cannot be satisfied by our leading term solution. If the data is consistent with $\sigma_o s_{ij}^{(0)}$ and $\sigma_1 s_{3j}^{(0)}$ (j=1,2) in the dependence on z, then four (contracted) inhomogeneous stress conditions may be specified to give a well defined boundary value problem for $w^{(0)}$ and $F^{(0)}$.

The fact that the leading term solution of the series in (2.12) cannot satisfy arbitrary stress or displacement boundary conditions (which are admissible in the context of elastostatics of solids) suggests that the regular perturbation solution (2.12) is incomplete and that we have a singular perturbation problem (although it is not obviously so from the structure of the differential equations). To obtain the missing portion of the solution, we will have to abandon the assumption that differentiations with respect to the dimensionless spatial variables x, y and z do not change order of magnitude. In other words, the missing portion of the solution is expected to be boundary layers adjacent to the edge(s) of the plate.

Before we proceed to discuss the supplementary boundary layer solution, we will show that higher order correction terms in the perturbation series (2.12) do not change the situations described above on the basis of the leading term approximation. To see the effect of the higher order correction terms, we begin with the perturbation series for u_3 :

$$u_3(x_1,x_2,x_3) = u_o W(x,y,z;\varepsilon) = u_o \sum_{n=0}^{\infty} W^{(n)}(x,y,z)\varepsilon^n$$

$$= u_o \{ \sum_{n=0}^{\infty} w^{(n)}(x,y)\varepsilon^n + \sum_{n=2}^{\infty} [\frac{\nu z^2}{2(1-\nu)} \nabla^2 w^{(n-2)} - \nu z \nabla^2 F^{(n-2)}]\varepsilon^n \}$$

$$= u_o\{w(x,y;\varepsilon) - \nu\varepsilon^2 z \nabla^2 F + \frac{\nu\varepsilon^2 z^2}{2(1-\nu)} \nabla^2 w\} \qquad (2.26a)$$

where

$$w(x,y;\varepsilon) = \sum_{n=0}^{\infty} w^{(n)}(x,y)\varepsilon^n \quad , \quad -\frac{1}{1-\nu^2}\nabla^4 w + 3p_o = 0 \tag{2.26b}$$

$$F(x,y;\varepsilon) = \sum_{n=0}^{\infty} F^{(n)}(x,y)\varepsilon^n \quad , \quad \nabla^4 F = 0 \ . \tag{2.26c}$$

It is now clear that even with the entire series, the perturbation solution[2] cannot satisfy arbitrarily prescribed (admissible) displacement data. If the prescribed distribution of u_3 is quadratic in z, the coefficients of the z and z^2 term must be related to the midplane displacement $w(x,y;\varepsilon)$ and stress function $F(x,y;\varepsilon)$ in a specific way and the contribution of these z dependent terms is $O(\varepsilon^2)$ compared to $w(x,y;\varepsilon)$. Similarly, we have

$$\begin{aligned} u_1(x_1,x_2,x_3) &= v_o V_1(x,y,z;\varepsilon) \\ &= v_o\{v_1(x,y;\varepsilon) - zw_{,x} + \varepsilon^2[\tfrac{1}{2}\nu z^2\nabla^2 F - t_o(z)\nabla^2 w]_{,x}\} \end{aligned} \tag{2.26d}$$

$$\begin{aligned} u_2(x_1,x_2,x_3) &= v_o V_2(x,y;\varepsilon) \\ &= v_o\{v_2(x,y;\varepsilon) - zw_{,y} + \varepsilon^2[\tfrac{1}{2}\nu z^2\nabla^2 F - t_o(z)\nabla^2 w]_{,y}\} \end{aligned} \tag{2.26e}$$

$$\begin{aligned} \sigma_{11}(x_1,x_2,x_3) &= \sigma_0 s_{11}(x,y,z;\varepsilon) \\ &= \sigma_0\{n_{11}(x,y;\varepsilon) + zm_{11}(x,y;\varepsilon) + \varepsilon^2[\tfrac{1}{2}\nu z^2\nabla^2\mu_{11} + t_o(z)\nabla^2 m_{11} + \tfrac{\nu}{1-\nu}s_{33}^{(0)}]\} \end{aligned} \tag{2.26f}$$

$$\begin{aligned} \sigma_{22}(x_1,x_2,x_3) &= \sigma_0 s_{22}(x,y,z;\varepsilon) \\ &= \sigma_0\{n_{22}(x,y;\varepsilon) + zm_{22}(x,y;\varepsilon) + \varepsilon^2[\tfrac{1}{2}\nu z^2\nabla^2\mu_{22} + t_o(z)\nabla^2 m_{22} + \tfrac{\nu}{1-\nu}s_{33}^{(0)}]\} \end{aligned} \tag{2.27g}$$

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(2) Though the results here are for the case where p_e and p_o are both uniformly distributed, the main conclusions are not affected by arbitrarily distributed p_e and p_o.

$$\sigma_{12}(x_1,x_2,x_3) = \sigma_0 s_{12}(x,y,z;\varepsilon)$$

(2.26h)

$$= \sigma_0\{n_{12}(x,y;\varepsilon) + zm_{12}(x,y;\varepsilon) + \varepsilon^2[\frac{1}{2}\nu z^2\nabla^2\mu_{12} + t_o(z)\nabla^2 m_{12}]\}$$

$$\sigma_{j3}(x_1,x_2,x_3) = \sigma_1 s_{j3}(x,y,z;\varepsilon) = \sigma_1\{\frac{1}{2}q_j(x,y;\varepsilon)(1-z^2)\} \quad (j=1,2) \qquad (2.26i)$$

$$\sigma_{33}(x_1,x_2,x_3) = \sigma_2(x,y;\varepsilon) = \sigma_2\{p_e + \frac{3}{2}p_o(z - \frac{1}{3}z^3)\} \equiv \sigma_2 s_{33}^{(0)}(x,y,z) \qquad (2.26j)$$

where

$$\sigma_0\mu_{11} = \frac{1}{1-\nu^2}(F_{,xx} + \nu F_{,yy}) \qquad \sigma_0\mu_{22} = \frac{1}{1-\nu^2}(F_{,yy} + \nu F_{,xx}) \quad ,$$

$$\sigma_0\mu_{12} = \sigma_0\mu_{21} = \frac{1}{1-\nu}F_{,xy} \ . \qquad (2.26k)$$

It should be evident from the expressions (2.26a)-(2.26h) that the perturbation solution (2.12) cannot satisfy prescribed (stress, displacement or mixed) edge conditions if the prescribed quantities are not distributed in z consistent with these expressions.

6. Boundary Layer Solution

As long as the regular perturbation solution (2.12) do not (and cannot) satisfy arbitrary admissible displacement or stress boundary conditions along an edge of the plate consisting of the cylindrical surface $\{x_j = \bar{x}_j(s)\ ,\ j=1,2\ ,\ |x_3| \le h/2\}$, we would have to find the complementary inner expansion of the exact solution which is valid adjacent to the edge. The linearity of our problem allows us to seek the boundary layer component of the exact solution to supplement to so-called interior solution obtained in previous section. For simplicity, we seek such a boundary layer solution only in a region adjacent to an edge corresponding to $x_1 = 0$.[3]

For a boundary layer, differentiation with respect to x_1 (direction normal to the edge) will change order of magnitude. Therefore, we set

$$\bar{x} = \frac{x_1}{\lambda(\varepsilon)L} \quad , \tag{2.27a}$$

with $\lambda(\varepsilon) << 1$ and $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$, while y and z are defined as before. We now scale the stress components by

$$\sigma_{ij} = \bar{\sigma}_0\, \bar{s}_{ij}(\bar{x},y,z)\ ,\ \sigma_{j3} = \sigma_{3j} = \bar{\sigma}_1\, \bar{s}_{3j}(\bar{x},y,z)\ ,\ \sigma_{33} = \bar{\sigma}_2\, \bar{s}_{33}(\bar{x},y,z) \tag{2.27b}$$

so that $\bar{s}_{ij} = 0(1)$ at most, and write the equilibrium equations of linear elasticity as

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[3] For a more general discussion, see the article by K.O. Friedrich & R.F. Dressler, in Comm. on Pure & Appl. Math. 14, 1961, 1-36.

$$\frac{\bar{\sigma}_0}{\lambda(\varepsilon)}[\bar{s}_{11,\bar{x}} + \lambda(\varepsilon)\bar{s}_{21,y}] + \frac{\bar{\sigma}_1}{\varepsilon}\bar{s}_{31,z} = 0$$

$$\frac{\bar{\sigma}_0}{\lambda(\varepsilon)}[\bar{s}_{12,\bar{x}} + \lambda(\varepsilon)\bar{s}_{22,y}] + \frac{\bar{\sigma}_1}{\varepsilon}\bar{s}_{32,z} = 0$$

$$\frac{\bar{\sigma}_1}{\lambda(\varepsilon)}[\bar{s}_{13,\bar{x}} + \lambda(\varepsilon)\bar{s}_{23,y}] + \frac{\bar{\sigma}_2}{\varepsilon}\bar{s}_{33,z} = 0 \quad .$$

A possible choice of the parameters (or, in the language of matched asymptotic expansions, a distinguished limit of the equations), which would lead to a boundary layer phenomenon, is

$$\lambda(\varepsilon) = \varepsilon \quad , \quad \bar{\sigma}_0 = \bar{\sigma}_1 = \bar{\sigma}_2 \quad . \tag{2.27c}$$

The corresponding dimensionless equilibrium equations are

$$\bar{s}_{1j,\bar{x}} + \varepsilon\bar{s}_{2j,y} + \bar{s}_{3j,z} = 0 \qquad (j = 1,2,3) \quad . \tag{2.28}$$

It is implicit in the above development that differentiations with respect to $\bar{x}$, y and z do not change order of magnitude.

For the dimensionless form of the stress-displacement relations, we limit ourselves to the case where the two midplane parallel displacement components are expected to be of the same order of magnitude. In that case, we set

$$u_j(x_1,x_2,x_3) = \bar{v}_0\bar{V}_j(\bar{x},y,z;\varepsilon) \quad , \quad u_3(x_1,x_2,x_3) = \bar{u}_0\bar{W}(\bar{x},y,z;\varepsilon) \tag{2.27d}$$

so that the strain-displacement relations take the form

$$\frac{\bar{u}_0}{\varepsilon}\bar{W}_{,z} = \frac{\bar{\sigma}_0 L}{E}(\bar{s}_{33} - \nu\bar{s}_{11} - \nu\bar{s}_{22})$$

$$\frac{\bar{v}_0}{\varepsilon}\bar{V}_{1,x} = \frac{\bar{\sigma}_0 L}{E}(\bar{s}_{11} - \nu\bar{s}_{22} - \nu\bar{s}_{33})$$

$$\bar{v}_0\bar{V}_{2,y} = \frac{\bar{\sigma}_0 L}{E}(\bar{s}_{22} - \nu\bar{s}_{11} - \nu_3\bar{s}_{33})$$

$$\frac{\bar{v}_0}{\varepsilon}[\varepsilon\bar{V}_{1,y} + \bar{V}_{2,\bar{x}}] = \frac{\bar{\sigma}_0 L}{E}2(1+\nu)\bar{s}_{12}$$

$$\frac{\bar{u}_0}{\varepsilon}\bar{W}_{,\bar{x}} + \frac{\bar{v}_0}{\varepsilon}\bar{V}_{1,z} = \frac{\bar{\sigma}_0 L}{E}2(1+\nu)\bar{s}_{13}$$

$$\bar{u}_0\bar{W}_{,y} + \frac{\bar{v}_0}{\varepsilon}\bar{V}_{2,z} = \frac{\bar{\sigma}_0 L}{E}2(1+\nu)\bar{s}_{23} \quad .$$

In order for $\bar{W}$ and $\bar{V}_1$ to vary with z and $\bar{x}$ to leading order, we take

$$\bar{u}_0 = \bar{v}_0 = \frac{\bar{\sigma}_0 h}{2E} \quad . \tag{2.27e}$$

With (2.27e), the stress-displacement relations become

$$\bar{W}_{,z} = (\bar{s}_{33} - \nu\bar{s}_{11} - \nu\bar{s}_{22}) \quad , \quad \bar{V}_{1,\bar{x}} = (\bar{s}_{11} - \nu\bar{s}_{22} - \nu\bar{s}_{33}) \quad . \tag{2.29a,b}$$

$$\varepsilon\bar{V}_{2,y} = (\bar{s}_{22} - \nu\bar{s}_{11} - \nu\bar{s}_{33}) \tag{2.29c}$$

$$\varepsilon\bar{V}_{1,y} + \bar{V}_{2,\bar{x}} = 2(1+\nu)\bar{s}_{12} \quad , \quad \varepsilon\bar{W}_{,y} + \bar{V}_{2,z} = 2(1+\nu)\bar{s}_{23} \tag{2.29d,e}$$

$$\bar{W}_{,\bar{x}} + \bar{V}_{1,z} = 2(1+\nu)\bar{s}_{13} \quad . \tag{2.29f}$$

The remaining undetermined scale factor $\bar{\sigma}_0$ will be fixed by the magnitude of the edge stresses or edge displacements.

The form of (2.28) and (2.29) suggests that we seek parametric series solution of these equations in powers of ε :

$$\{\bar{s}_{ij}(\bar{x},y,z;\varepsilon)\ ,\ \bar{V}_j(\bar{x},y,z;\varepsilon),\ \bar{W}(\bar{x},y,z;\varepsilon)\} \qquad (2.30)$$
$$= \sum_{n=0}^{\infty} \{\bar{s}_{ij}^{(n)}(\bar{x},y,z)\ ,\ \bar{V}_j^{(n)}(\bar{x},y,z)\ ,\ \bar{W}^{(n)}(\bar{x},y,z)\}\varepsilon^n\ .$$

The equations governing the leading terms of these expansions separate themselves into two uncoupled groups. One group consists of the stress components $\{\bar{s}_{11}^{(0)},\ \bar{s}_{22}^{(0)},\ \bar{s}_{33}^{(0)}\ ,\ \bar{s}_{13}^{(0)} = \bar{s}_{31}^{(0)}\}$ and the displacement components $\bar{W}^{(0)}$ and $\bar{V}_1^{(0)}$ and the equations for them are in the form of plane strain. The other group consists of the stress components $\{\bar{s}_{12}^{(0)},\ \bar{s}_{32}^{(0)}\}$ and the displacement component $\bar{V}_2^{(0)}$ and the equations for them are in the form of the Saint Venant torions problem.

For the <u>torsion group</u>, we have

$$\bar{s}_{12,\bar{x}}^{(0)} + s_{32,z}^{(0)} = 0\ ,\quad \bar{V}_{2,\bar{x}}^{(0)} = 2(1+\nu)\bar{s}_{12}^{(0)}\ ,\quad \bar{V}_{2,z}^{(0)} = 2(1+\nu)\bar{s}_{32}^{(0)} \qquad (2.31)$$

with the variable y as a parameter. The only equilibrium equation can be satisfied by way of a stress function $\psi^{(0)}$ with

$$\bar{s}_{12}^{(0)} = \psi_{,z}^{(0)}\quad ,\quad \bar{s}_{32}^{(0)} = -\psi_{,\bar{x}}^{(0)}\ . \qquad (2.32)$$

The stress function $\psi^{(0)}$ is then determined by the compatibility condition $\bar{V}^{(0)}_{2,\bar{x}z} = \bar{V}^{(0)}_{2,z\bar{x}}$ which, in view of the last two equations of (2.31), becomes

$$\nabla^2\psi^{(0)} \equiv \psi^{(0)}_{,\bar{x}\bar{x}} + \psi^{(0)}_{,zz} = 0 \quad . \tag{2.33}$$

This is the conventional approach to the torsion type problem. Alternatively, we can use the last two equations of (2.31) to transform the first into a simple equation for $\bar{V}_2$:

$$\frac{1}{2(1+\nu)} \nabla^2 \bar{V}^{(0)}_2 = \frac{1}{2(1+\nu)} [\bar{V}^{(0)}_{2,\bar{x}\bar{x}} + \bar{V}^{(0)}_{2,zz}] = 0 \ . \tag{2.34}$$

The solution domain for this problem is the semi-infinite strip $|z| \le 1$ and $\bar{x} \ge 0$, (keeping in mind that y is a parameter in the solution of the problem). The boundary conditions for this problem are the shear free condition on the two faces

$$z = \pm 1 : \qquad \bar{s}^{(0)}_{32} = 0 \quad , \tag{2.35a}$$

and (typically) either prescribed stress condition

$$\bar{x} = 0 : \qquad \bar{s}^{(0)}_{12} = \bar{T}^{(0)}_2(y,z) \quad , \tag{2.35b}$$

or the prescribed displacement condition

$$\bar{x} = 0 : \qquad \bar{V}^{(0)}_2 = \bar{D}^{(0)}_2(y,z) \quad , \tag{2.35c}$$

along the edge $x_1 = 0$. In (2.35b) and (2.35c), we have made use of appropriate expansions for the edge data:

$$\bar{\sigma}_0 \bar{s}_{12}\Big|_{\bar{x}=0} = T_2(x_2,x_3) = t_2 \bar{T}_2(y,z;\varepsilon) = t_2 \sum_{n=0}^{\infty} \bar{T}_2^{(n)}(y,z)\varepsilon^n \qquad (2.35d)$$

$$\bar{v}_0 \bar{V}_2\Big|_{\bar{x}=0} = D_2(x_2,x_3) = d_2 \bar{D}_2(y,z;\varepsilon) = d_2 \sum_{n=0}^{\infty} \bar{D}_2^{(0)}(y,z)\varepsilon^n \qquad (2.35e)$$

where t_2 and d_2 are the magnitude factor chosen so that $|\bar{T}_2| \le 1$ and $|\bar{D}_2| \le 1$. In addition to (2.35a) and (2.35b) (or (2.35c)), we stipulate also that the solution tends to zero as $\bar{x} \to \infty$ (in the case of a supplementary boundary layer solution) or matches with the (outer) solution of sections (2)-(4) in an intermediate overlapping region of validity. Well-known completeness of the eigenfunctions of the relevant eigenvalue problems allows the solution to fit essentially arbitrarily prescribed edge data on σ_{12}, u_2 or any other admissible boundary condition.

For the <u>plane strain group</u>, we have

$$\bar{s}^{(0)}_{11,\bar{x}} + \bar{s}^{(0)}_{31,z} = 0 \quad , \quad \bar{s}^{(0)}_{13,\bar{x}} + \bar{s}^{(0)}_{33,z} = 0 \quad , \qquad (2.36a,b)$$

$$\bar{W}^{(0)}_{,z} = \bar{s}^{(0)}_{33} - \nu(\bar{s}^{(0)}_{11} + \bar{s}^{(0)}_{22}) \quad , \quad \bar{V}^{(0)}_{1,\bar{x}} = \bar{s}^{(0)}_{11} - \nu(\bar{s}^{(0)}_{22} + \bar{s}^{(0)}_{33}) \quad , \qquad (2.36c,d)$$

$$\bar{s}^{(0)}_{22} - \nu(\bar{s}^{(0)}_{11} + \bar{s}^{(0)}_{33}) = 0 \quad , \quad \bar{W}^{(0)}_{,\bar{x}} + \bar{V}^{(0)}_{1,z} = 2(1+\nu)\bar{s}^{(0)}_{13} \quad , \qquad (2.36e,f)$$

again with y as a parameter. The conventional approach to this plane strain problem is to satisfy the two equilibrium equation by a stress function $\chi^{(0)}$ with

$$\bar{s}_{11}^{(0)} = \chi^{(0)}_{,zz} \;, \quad \bar{s}_{33}^{(0)} = \chi^{(0)}_{,\bar{x}\bar{x}} \;, \quad \bar{s}_{13}^{(0)} = \bar{s}_{31}^{(0)} = -\chi^{(0)}_{,\bar{x}z} \quad . \tag{2.37a}$$

From (2.36e), we have also

$$\bar{s}_{22}^{(0)} = \nu[\bar{s}_{11}^{(0)} + \bar{s}_{33}^{(0)}] = \nu\nabla^2\chi^{(0)} \equiv \nu[\chi^{(0)}_{,\bar{x}\bar{x}} + \chi^{(0)}_{,zz}] \quad . \tag{2.37b}$$

The single compatibility condition $(\bar{W}^{(0)}_{,z})_{,\bar{x}\bar{x}} + (\bar{V}^{(0)}_{1,\bar{x}})_{,zz} = (\bar{W}^{(0)}_{,\bar{x}} + \bar{V}^{(0)}_{1,z})_{,\bar{x}z}$, expressed in terms of stress components by (2.36c)-(2.36f), gives a biharmonic equation for $\chi^{(0)}$:

$$(1-\nu^2)\nabla^4\chi^{(0)} = (1-\nu^2)[\chi^{(0)}_{,\bar{x}\bar{x}\bar{x}\bar{x}} + 2\chi^{(0)}_{,\bar{x}\bar{x}zz} + \chi^{(0)}_{,zzzz}] = 0 \quad . \tag{2.38}$$

The solution domain is again the semi-infinite strip $|z| \le 1$ and $\bar{x} \ge 0$ (keeping in mind that y is a parameter in the solution of the problem). For the boundary conditions for this problem we now choose to consider the boundary layer solution supplementing the interior solution of sections (2)-(5). In that case, we have

$$z = \pm 1 : \quad \bar{s}_{31}^{(0)} = \bar{s}_{33}^{(0)} = 0 \tag{2.39a}$$

and, for prescribed tractions at the edge, say,

$$\bar{x} = 0 : \quad \bar{s}_{11}^{(0)} = \bar{\chi}^{(0)}_{,zz} = \bar{T}_1^{(0)}(y,z) \quad , \quad \bar{s}_{13}^{(0)} = \bar{\chi}^{(0)}_{,\bar{x}z} = \bar{T}_3^{(0)}(y,z) \quad , \tag{2.39b}$$

where we have similar to (2.35d)

$$\begin{aligned} \bar{\sigma}_0 \bar{s}_{ij}\Big|_{\bar{x}=0} &= T_j(x_2,x_3) = t_j \bar{T}_j(y,z;\varepsilon) \\ &= t_j \sum_{n=0}^{\infty} \bar{T}_j^{(n)}(y,z)\varepsilon^n \qquad (j = 1,3) \end{aligned} \tag{2.39c}$$

where the amplitude factors t_1 and t_3 are chosen so that $|T_j| \leq 1$. As a supplementary boundary layer solution, we also want all stresses and displacement to decay to zero as $\bar{x} \to \infty$. Completeness and expansion theorems for the above boundary value problem for $\chi^{(0)}$ are also available so that the solution can effectively fit arbitrary admissible edge data.[4]

Having the lead term solution, we may proceed, if we wish, to obtain the coefficient of higher powers of ε in the expansions of (2.30). We will not give further details of this rather straightforward solution process.

[illegible] be found in R.D. Gregory, *J. of Elasticity*

7. Composite Solution and Edge Conditions

The linearity of our boundary value problem of elastostatics allows us to superpose the interior solution of sections (2)-(4) and the boundary layer solution of section (6). In particular, we have

$$u_3 = u_0 \sum_{n=0}^{\infty} W^{(n)}(x,y,z)\varepsilon^n + \bar{u}_0 \sum_{n=0}^{\infty} \bar{W}^{(n)}(\bar{x},y,z)\varepsilon^n$$

$$u_j = v_0 \sum_{n=0}^{\infty} V_j^{(n)}(x,y,z)\varepsilon^n + \bar{u}_0 \sum_{n=0}^{\infty} \bar{V}_j^{(n)}(\bar{x},y,z)\varepsilon^n$$

$$\sigma_{ij} = \sigma_0 \sum_{n=0}^{\infty} s_{ij}^{(n)}(x,y,z)\varepsilon^n + \bar{\sigma}_0 \sum_{n=0}^{\infty} \bar{s}_{ij}^{(n)}(\bar{x},y,z)\varepsilon^n \qquad (2.40)$$

$$\sigma_{3j} = \sigma_0 \varepsilon \sum_{n=0}^{\infty} s_{3j}^{(n)}(x,y,z)\varepsilon^n + \bar{\sigma}_0 \sum_{n=0}^{\infty} \bar{s}_{3j}^{(n)}(\bar{x},y,z)\varepsilon^n$$

$$\sigma_{33} = \sigma_0 \varepsilon^2 \sum_{n=0}^{\infty} s_{33}^{(n)}(x,y,z)\varepsilon^n + \bar{\sigma}_0 \sum_{n=0}^{\infty} \bar{s}_{33}^{(n)}(\bar{x},y,z)\varepsilon^n$$

where $i,j = 1,2$, $\bar{u}_0 = \bar{\sigma}_0 h/2E$, $u_0 = \sigma_0 L^2/E(h/2)$ and $v_0 = \varepsilon u_0 = \sigma_0 L/E$. The composite solution satisfies the precribed traction conditions on the two faces. It remains for it to satisfy the boundary condition along the edge $x_1 = 0$ of the plate (as the boundary layer solution was obtained for that special case).

To be specific, we consider he the stress boundary value problem with

$$x_1 = 0: \quad \sigma_{1j} = T_j(x_2, x_3) = t_j \bar{T}_j(y,z;\varepsilon) = t_j \sum_{n=0}^{\infty} \bar{T}_j^{(n)}(y,z)\varepsilon^n \quad (j = 1,2,3). \quad (2.41)$$

We limit our discussion here to the case of edgewise uniform edge loads, $T_j = \bar{T}_j(z)$, which dominate the face tractions so that $p_0 = 0(t_j\varepsilon^2)$ at most, $j = 1,2,3$; otherwise, we have $\sigma_0 = \bar{\sigma}_0 = p_0/\varepsilon^2$. If only the leading term in each expansion is retained, the fitting of the edge data (2.41) requires

$$\sigma_0 s_{11}^{(0)}(0,z) + \bar{\sigma}_0 \bar{s}_{11}^{(0)}(0,z) = t_1 \bar{T}_1^{(0)}(z)$$

$$\sigma_0 s_{12}^{(0)}(0,z) + \bar{\sigma}_0 \bar{s}_{12}^{(0)}(0,z) = t_2 \bar{T}_2^{(0)}(z) \qquad (2.42)$$

$$\sigma_0 \varepsilon s_{13}^{(0)}(0,z) + \bar{\sigma}_0 \bar{s}_{13}^{(0)}(0,z) = t_3 \bar{T}_3^{(0)}(z) \quad .$$

When the edge loads dominate, the magnitude factors t_j specify σ_0 and $\bar{\sigma}_0$ and the fitting of edge data $\bar{T}_j^{(0)}(z)$ by the combinations of $s_{ij}^{(0)}$ and $\bar{s}_{ij}^{(0)}$ in (2.42) can be done with the help of the relevant eigenfunction expansion theorems for the torsion and plane strain problems in section (6) for the boundary layer solution[4].

The actual implementation of the above solution process is tedious and costly (because of the amount of machine computation likely to be required by the method). Moreover, the final results, most likely in

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[4] For a solution process by way of matched asymptotic expansions, see the article by Friedrich and Dressler previously mentioned.

the form of a table of numbers for the various Fourier coefficients, are not very informative. Fortunately, some simplification is possible for the case of stress edge data (2.41). For the leading term boundary layer solution we have from (2.31a) and (2.36a,b)

$$\int_B^A\int_{-1}^{1}[(\bar{s}_{11}^{(0)}\vec{i}_1+\bar{s}_{12}^{(0)}\vec{i}_2+\bar{s}_{13}^{(0)}\vec{i}_3)_{,\bar{x}}+(\bar{s}_{31}^{(0)}\vec{i}_1+\bar{s}_{32}^{(0)}\vec{i}_2+s_{33}^{(0)}\vec{i}_3)_{,z}]dzd\bar{x}$$

$$\equiv\int_B^A\int_{-1}^{1}(\vec{\bar{s}}_{1,\bar{x}}^{(0)}+\vec{\bar{s}}_{3,z}^{(0)})dzd\bar{x}=0$$

for any fixed value of $A > B > 0$. By the divergence theorem, the double integral may be transformed into a line integral

$$\oint_C[\vec{\bar{s}}_1^{(0)}\nu_1+\vec{\bar{s}}_3^{(0)}\nu_3]ds=\int_{-1}^{1}[\vec{s}_1^{(0)}]_{\bar{x}=B}^{\bar{x}=A}dz-\int_B^A[\vec{s}_3^{(0)}]_{z=-1}^{z=1}d\bar{x}=\vec{0}$$

where $\vec{\nu}=\nu_1\vec{i}_1+\nu\vec{i}_3$ is the unit outward normal vector to the edge curve C of the area $\{|z|\le 1\ ,\ B\le\bar{x}\le A\}$. But from the boundary conditions (2.35a) and (2.39a) the integrals along $z = 1$ and $z = -1$ vanish. For a sufficiently large A, the integral along $\bar{x} = A$ can be made as small as possible by the decaying condition on $\bar{s}_{ij}^{(0)}$ at infinity. Therefore, we are left with

$$\int_{-1}^{1}[\vec{\bar{s}}_1^{(0)}]_{\bar{x}=B}dz=\vec{0}\ ,\qquad (0\le B<A)\tag{2.43}$$

In particular, we have for $B = 0$

$$\int_{-1}^{1}\bar{s}_{1j}^{(0)}(0,z)dz=0\qquad (j=1,2,3)\ .\tag{2.44a,b,c}$$

Similarly, we have from

$$0 = \int_B^A \int_{-1}^1 (\bar{s}^{(0)}_{11,x} + \bar{s}^{(0)}_{31,z}) z dz d\bar{x} = \int_B^A \int_{-1}^1 [(z\bar{s}^{(0)}_{11})_{,x} + (z\bar{s}^{(0)}_{13})_{,z} - \bar{s}^{(0)}_{13}] dz d\bar{x}$$

$$= \int_{-1}^1 [z\bar{s}^{(0)}_{11}]_{\bar{x}=B}^{\bar{x}=A} dz - \int_B^A [z\bar{s}^{(0)}_{31}]_{z=-1}^{z=1} d\bar{x} - \int_B^A \int_{-1}^1 \bar{s}^{(0)}_{13} dz d\bar{x}$$

a fourth integrated condition

$$\int_{-1}^1 \bar{s}^{(0)}_{11}(0,z) z dz = 0 \tag{2.44d}$$

as the condition

$$\int_{-1}^1 \bar{s}^{(0)}_{13}(\bar{x},z) dz = 0 \tag{2.45}$$

holds for arbitrary $\bar{x} \geq 0$ according to (2.43).

The four conditions (2.44a)-(2.44d) on $\bar{s}^{(0)}_{ij}$ may be translated into the following four boundary conditions for $s^{(0)}_{ij}$ at $x = 0$:

$$\sigma_0 \int_{-1}^1 s^{(0)}_{11}(0,z) dz = t_1 \int_{-1}^1 \bar{T}^{(0)}_1(z) dz \tag{2.46a}$$

$$\sigma_0 \int_{-1}^1 s^{(0)}_{12}(0,z) dz = t_2 \int_{-1}^1 \bar{T}^{(0)}_2(z) dz \tag{2.46b}$$

$$\sigma_0 \varepsilon \int_{-1}^1 s^{(0)}_{13}(0,z) dz + \bar{\sigma}_0 \varepsilon \int_{-1}^1 \bar{s}^{(1)}_{13}(0,z) dz = t_2 \int_{-1}^1 \bar{T}^{(0)}_2(z) dz \tag{2.47a}$$

$$\sigma_0 \int_{-1}^1 s^{(0)}_{11}(0,z) z dz = t_1 \int_{-1}^1 \bar{T}^{(0)}_1(z) z dz \quad . \tag{2.47b}$$

When $t_3 = t_1\varepsilon = t_2\varepsilon$, we take $\sigma_0 = t_1$. The two conditions (2.46a) and (2.46b) are just the two stress boundary conditions (1.50) for the problem of plate extension for the special case of a straight edge at $x_1 = 0$. The two conditions (2.47a) and (2.47b) are identical to the two stress boundary conditions (1.51) for the problem of plate bending when the edge tractions are uniform along the straight edge $x_1 = 0$.* These four conditions (which are merely a form of the Saint Venant Principle for elasticity) uncouple the determination of the leading term interior solution and leading term boundary layer solution. They make the solution process for the former considerably simpler and, more importantly, feasible; they also render the determination of the latter unnecessary if we are only interested in the plate behaviour away from the edge(s).

The situation for edgewise non-uniform edge loads is not as simple. How do the four overall equilibrium conditions (2.44a)-(2.44d) lead to the Kirchhoff contracted stress boundary conditions is not obvious (see Friedrich and Dressler). Because of the inhomogeneous terms (form lower order solutions) entering into the governing differential equations for $\bar{s}_{ij}^{(n)}$, $\bar{V}_j^{(n)}$ and $\bar{W}^{(n)}$, the four integrated conditions corresponding to (2.44a)-(2.44d) will be modified and cannot be obtained without first determining the lower order coefficients $\bar{s}_{ij}^{(k)}$, $\bar{V}_j^{(k)}$ and $\bar{W}^{(k)}$, $k > n$, in the expansions in (2.30). In other words, to go beyond the leading term

*Provided that the boundary layer solution does not contribute significantly.

of the interior solution, it appears that we must also determine the leading term layer solution, the first order correction terms for the interior solution, the first order correction terms for the layer solution so on and in that order. An explicit determination of any part of the layer solution is unpalatable at best and should be avoided if at all possible. The question is: how could we avoid it and still obtain the Kirchhoff contracted boundary conditions for the leading term interior solution as well as the higher order correction terms of the interior solution needed for plates which are not very thin. We will deal with the first question in the next section as well as section (5) of this chapter and with the second question in section (4).

3. Reissner's Plate Theory

1. Statement of the Problem

For a homogeneous and transversely rigid plate, we found earlier that the distribution across the plate thickness of the stress components in the deformed plate are known explicitly. For a plate free of body load intensities and subject only to normal stresses $\sigma_t(x_1,x_2)$ and $\sigma_b(x_1,x_2)$ at the top and bottom face, respectively; we have

$$\sigma_{ij}(x_1,x_2,x_3) = \frac{1}{h} N_{ij}(x_1,x_2) + \frac{12}{h^3} M_{ij}(x_1,x_2)x_3 = \sigma_{ji} \qquad (i,j = 1,2)$$

$$\sigma_{3j}(x_1,x_2,x_3) = \frac{3}{2h} Q_j(x_1,x_2)(1 - z^2) = \sigma_{j3} , \tag{3.1}$$

$$\sigma_{33}(x_1,x_2,x_3) = \sigma_o(x_1,x_2) + \frac{3}{4} p(x_1,x_2)(z - \frac{1}{3} z^3)$$

with

$$N_{ij,i} = 0 , \quad M_{ij,i} - Q_j = 0 \qquad (i,j = 1,2) \tag{3.2}$$
$$Q_{j,j} + p = 0$$

where $z = x_3/(h/2)$ and

$$\sigma_o = \frac{1}{2} [\sigma_t(x_1,x_2) + \sigma_b(x_1,x_2)] , \qquad p = \sigma_t - \sigma_b . \tag{3.3}$$

It turns out that for an isotropic (and homogeneous) elastic plate, a first approximation of the parametric series solution of the same plate problem in powers of a small dimensionless thickness parameter (known in singular perturbation theory as the leading term <u>outer</u> solution) also has the same stress distributions. In both cases, the independent, physically meaningful, stress boundary conditions for the bending and twisting actions of the plate cannot all be satisfied and a contraction to two independent conditions is necessary.

For an elastic plate which is not transversely rigid, it is known from singular perturbation theory that the difficulties associated with homogeneous stress boundary conditions and, more generally, inhomogeneous boundary conditions at the plate edge can be removed by the introduction of a boundary layer (or <u>inner</u> asymptotic expansion) solution to supplement the <u>outer</u> solution (3.1) and the corresponding displacement field. Unfortunately, such a boundary layer solution, even just its first approximation, is both complex in its structure and tedious to calculate. A less ambitious undertaking which nevertheless leads to more satisfactory results (including the removal of the difficulty with the boundary conditions) is definitely of interest. Reissner's theory of plates provides such an approach to approximate solutions for plate problems. The approach assumes the stress distributions (3.1), (3.2) and (3.3) (which are not exactly correct when the plate is not transversely rigid) but not the corresponding displacement distributions (1.8) and (1.9). Instead, the stress-strain relations and the strain-displacement relations are satisfied approximately by the application of the semi-direct method in calculus of variations to an appropriate variational principle.

2. Complementary Energy and Weighted Averages of Displacement Components

The approximate stress distributions (3.1) - (3.3) satisfy the three scalar equilibrium equations of three dimensional linear elasticity exactly. For an approximate solution of the displacement components (to complement (3.1) - (3.3)) by the semi-direct method of calculus variations, the principle of minimum complementary energy is appropriate. For a transversely isotropic elastic plate, we have from a previous exercise the following expression for the complementary energy density function

$$C(\sigma_{ij}) = \frac{1}{2E}\left[\sigma_{11}^2 + \sigma_{22}^2 - 2\nu\sigma_{11}\sigma_{22}\right] + \frac{1}{2G}\sigma_{12}^2 + \frac{1}{2G_3}\left[\sigma_{13}^2 + \sigma_{23}^2\right] + \frac{1}{2E_3}\sigma_{33}^2 - \frac{\nu_3}{E}\sigma_{33}(\sigma_{11} + \sigma_{22}) \tag{3.4}$$

where $G = E/2(1 + \nu)$ and E , ν , G_3 , E_3 and ν_3 are independent parameters. For simplicity, we limit our discussion to homogeneous materials so that these parameters are in fact constants for the whole plate.

For the moment, we consider the special case where the prescribed edge tractions have the same x_3-dependence as the corresponding stress components in (3.1) and where the stress boundary conditions on any portion S_σ of the cylindrical boundary surface, S , are satisfied identically by the assumed stress distributions. In that case, the complementary energy of the plate is the difference between the complementary strain energy and the work done by edge stresses (on the remaining portion of the edge surface, S_d,) with the prescribed displacements:

$$P^* = \iiint_R \int_{-h/2}^{h/2} C\, dx_3 dx_2 dx_1 - \int_{\Gamma_d} \int_{-h/2}^{h/2} \vec{\sigma}_\nu \cdot \vec{u}^{\,*}\, dx_3 ds \tag{3.5}$$

where Γ_d is that portion of Γ on which edge displacements are prescribed to be $\vec{u}^{(o)}(s,x_3)$ and $\vec{\sigma}_\nu = \nu_1\vec{\sigma}_1 + \nu_2\vec{\sigma}_2$ (since $\nu_3 = 0$) . With the stress distributions (3.1), the integration across the plate thickness in the volume integral can be carried out to give

$$\int_{-h/2}^{h/2} \sigma_{ij}^2\, dx_3 = \frac{1}{h} N_{ij}^2 + \frac{12}{h^3} M_{ij}^2 \qquad \text{(no sum)}$$

$$\int_{-h/2}^{h/2} \sigma_{11}\sigma_{22} dx_3 = \frac{1}{h} N_{11}N_{22} + \frac{12}{h^3} M_{11}M_{22}$$

$$\int_{-h/2}^{h/2} \sigma_{3j}^2\, dx_3 = \frac{12}{5h} Q_j^2 \qquad \text{(no sum)}$$

$$\int_{-h/2}^{h/2} \sigma_{33}(\sigma_{11} + \sigma_{22}) dx_3 = \sigma_o (N_{11} + N_{22}) + \frac{6}{5h} p (M_{11} + M_{22})$$

and therewith

$$C_p \equiv \int_{-h/2}^{h/2} C\, dx_3 = \frac{1}{2Eh} (N_{11}^2 + N_{22}^2 - 2\nu N_{11}N_{22})$$

$$+ \frac{6}{Eh^3} (M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22}) + \frac{1}{2Gh} N_{12}^2 + \frac{6}{Gh^3} M_{12}^2 \tag{3.6}$$

$$+ \frac{3}{5G_3h} (Q_1^2 + Q_2^2) - \frac{\nu_3}{E} [\sigma_o (N_{11} + N_{22}) + \frac{6}{5h} p (M_{11} + M_{22})]$$

$$+ C_{po}(x_1,x_2)$$

where

$$C_{po}(x_1,x_2) = \frac{1}{2E_3} \int_{-h/2}^{h/2} \sigma_{33}^2 \, dx_3$$

is a known function.

For the surface integral, we use (with $z = x_3/(h/2)$)

$$\vec{\sigma}_\nu \cdot \vec{u}^* = \nu_1 \vec{\sigma}_1 \cdot \vec{u}^* + \nu_2 \vec{\sigma}_2 \cdot \vec{u}^*$$

$$= \frac{1}{h}(N_{\nu 1} u_1^* + N_{\nu 2} u_2^* + \frac{12}{h^3} (M_{\nu 1} u_1^* + M_{\nu 2} u_2^*$$

$$+ \frac{3}{2h} Q_\nu u_3^* (1 - z^2)$$

to write

$$\int_{-h/2}^{h/2} \vec{u}^* \cdot \vec{\sigma}_\nu dx_3 = N_{\nu 1} U_1^* + N_{\nu 2} U_2^* + Q_\nu W^* + M_{\nu 1} \Phi_1^* + M_{\nu 2} \Phi_2^* \qquad (3.7)$$

with

$$U_k^* (x_1,x_2) \equiv \frac{1}{h} \int_{-h/2}^{h/2} u_k^* (x_1,x_2,x_3) dx_3 \qquad (k = 1,2)$$

$$W^* (x_1,x_2) = \frac{3}{2h} \int_{-h/2}^{h/2} u_3^* (x_1,x_2,x_3)(1 - z^2) dx_3 \qquad (3.8)$$

$$\Phi_k^* (x_1,x_2) = \frac{12}{h^3} \int_{-h/2}^{h/2} u_k^* (x_1 x_2 x_3) x_3 dx_3 \qquad (k = 1,2)$$

where $z = x_3/(h/2)$ and $k = 1,2$. With (3.6) and (3.7), the expression for the complementary energy (3.5) is reduced to

$$P^* = \iint_R C_p(N_{ij}, Q_j, M_{ij})dx_1dx_2 - \int_{\Gamma_d} (\vec{N}_\nu \cdot \vec{U}^* + \vec{M}_\nu \cdot \vec{\Phi}^*)ds \quad (3.9)$$

where as in (1.60)

$$\vec{U}^* = U_1^* \vec{i}_1 + U_2^* \vec{i}_2 + W^* \vec{i}_3$$

$$\Phi^* = \vec{i}_3 \times (\Phi_1^* \vec{i}_1 + \Phi_2^* \vec{i}_2) \quad (3.10)$$

$$\vec{N}_\nu = N_{\nu 1}\vec{i}_1 + N_{\nu 2}\vec{i}_2 + \Phi_\nu \vec{i}_3$$

$$\vec{M}_\nu = \vec{i}_3 \times (M_{\nu 1}\vec{i}_1 + M_{\nu 2}\vec{i}_2)$$

The expressions in (3.8) show that U_k^*, W^* and Φ_k^* are weighted averages of the three dimensional in-plane and transverse displacement components. Their dimension and the work expression along Γ_d in (3.9) indicate that U_1^* and U_2^* are the average midplane-parallel displacement components, W^* is a weighted average measure of the transverse deflection of the plate and Φ_1^* and Φ_2^* are angles measuring the odd part u_1^* and u_2^* respectively, about the midplane (and are therefore the rotation produced by the moment resultants $M_{\nu 1}$ and $M_{\nu 2}$, respectively).

3. Euler Differential Equations and Strain-Displacement Relations

The minimum complementary energy theorem requires P^* to be a minimum for the actual deformed equilibrium configuration. With the assumed stress distributions (3.1) - (3.3), the theorem now requires that N_{ij}, Q_j and M_{ij} be chosen so that the right side of (3.9) be a minimum. As a necessary condition for a minimum, we must have $\delta P^* = 0$ with the variations of resultants and couples subject to the constraints (3.2) (in order to have force equilibrium in three dimensions). The five equality constraints may be incorporated into the variational problem by way of five Lagrange multipliers. Anticipating that they will turn out to have physical meaning as weighted averages of the displacement components $u_j(x_1,x_2,x_3)$, we take these multipliers to be $U_j(x_1,x_2)$, $\Phi_j(x_1,x_2)$ and $W(x_1,x_2)$. The modified condition of vanishing first variation then takes the form

$$\delta P^* + \iint_R [U_j \delta N_{ij,i} + \Phi_j(\delta M_{ij,i} - \delta Q_j) + W\delta Q_{j,j}]dx_1,dx_2 = 0 . \tag{3.11}$$

Upon integration by parts (with the help of divergence theorem in two dimensions), we get

$$\iint_R [(\frac{\partial C_p}{\partial N_{11}} - \varepsilon_{11})\delta N_{11} + (\frac{\partial C_p}{\partial N_{22}} - \varepsilon_{22})\delta N_{22} + (\frac{\partial C_p}{\partial N_{12}} - \gamma)\delta N_{12}$$

$$+ (\frac{\partial C_p}{\partial M_{11}} - \kappa_{11})\delta M_{11} + (\frac{\partial C_p}{\partial M_{22}} - \kappa_{22})\delta M_{22} + (\frac{\partial C_p}{\partial M_{12}} - \tau)\delta M_{12}$$

$$+ (\frac{\partial C_p}{\partial Q_1} - \gamma_1)\delta Q_1 + (\frac{\partial C_p}{\partial Q_2} - \gamma_2)\delta Q_2]dx_1,dx_2$$

$$+ \int_{\Gamma_\sigma} (\vec{U} \cdot \delta\vec{N}_\nu + \vec{\Phi} \cdot \delta\vec{M}_\nu)ds + \int_{\Gamma_d} (\Delta\vec{U} \cdot \delta\vec{N}_\nu + \Delta\vec{\Phi} \cdot \delta\vec{M}_\nu)\, ds$$

$$= 0 \tag{3.12}$$

where, for brevity, we have set

$$\varepsilon_{11} \equiv U_{1},_{1} \quad , \quad \varepsilon_{22} \equiv U_{2},_{2} \quad , \quad \gamma \equiv U_{1},_{2} + U_{2},_{1} \tag{3.13a}$$

$$\gamma_j \equiv W,_j + \Phi_j \,, \qquad (j = 1,2) \tag{3.13b}$$

$$\kappa_{ij} \equiv \Phi_{j},_{i} \quad (i,j = 1,2) \quad , \quad \tau = \kappa_{12} + \kappa_{21} \tag{3.13c}$$

and

$$\Delta\vec{U} \equiv \vec{U} - \vec{U}^{(o)}, \quad \Delta\vec{\Phi} \equiv \vec{\Phi} - \vec{\Phi}^{(o)} \tag{3.14}$$

The integral over Γ_σ vanishes because $\delta\vec{N}_2 = \delta\vec{M}_2 = \vec{0}$ (given that the stress distributions (3.1) must equal to the prescribed boundary tractions on S_σ). With δN_{ij} $(= \delta N_{ji})$, $\delta M_{ij}(= \delta M_{ji})$ and δQ_j varying independent, the condition for a stationary value of P^* requires

$$\varepsilon_{11} = \frac{\partial C_p}{\partial N_{11}}, \quad \varepsilon_{22} = \frac{\partial C_p}{\partial N_{22}}, \qquad \gamma = \frac{\partial C_p}{\partial N_{12}} \tag{3.15a}$$

$$\gamma_1 = \frac{\partial C_p}{\partial Q_1}, \qquad \gamma_2 = \frac{\partial C_p}{\partial Q_2} \tag{3.15b}$$

$$\kappa_{11} = \frac{\partial C_p}{\partial M_{11}}, \quad \kappa_{22} = \frac{\partial C_p}{\partial M_{22}}, \qquad \tau = \frac{\partial C_p}{\partial M_{12}} \tag{3.15c}$$

as Euler differential equations in R , and

$$\Delta\vec{U} = \vec{0} , \quad \Delta\vec{\Phi} = \vec{0} \tag{3.16}$$

as Euler boundary conditions along Γ_d . In addition, we have, as equality constraints, the five differential equations of equilibrium (3.2) to be satisfied by the resultants and couples in R and the five scaler stress boundary conditions corresponding to

$$\Delta\vec{N}_\nu = \vec{0} , \quad \Delta\vec{M}_\nu = \vec{0} \tag{3.17}$$

to be satisfied along Γ_σ .

We see from (3.16) that the Lagrange multipliers are equal to the various across-thickness weighted averages of the displacement components along Γ_d and, by a similar analysis, along any curve in the midplane inside Γ . We can therefore identify U_j as the average of the midplane parallel displacement u_j across the plate thickness, W as a weighted average (with weight $1 - z^2$) of the transverse displacement U_3, and Φ_j as the rotation of midplane normal turning about the x_k-axis ($k \neq j$ and $k \neq 3$) . It follows that ε_{ij} and γ_j , $i,j = 1,2$, are dimensionless and may be regarded as composite strain measures (summarizing the across-thickness effects) which are appropriate counterparts of the resultants N_{ij} and Q_j in a two dimensional version of the virtual work expression for thin plates. Similarly, κ_{ij} , $i,j = 1,2$, have the dimension of curvature and may be regarded as composite curvature change measures (summarizing the across-thickness effects) which are the appropriate counterparts of M_{ij} in the two dimensional virtual work expression for thin plates. As such, we may refer to the relations (3.13) as strain-displacement relations in Reissner's plate theory; they are the equations of definition for ε_{ij}, κ_{ij} and γ_j.

4. Stress Function Formulation of Plate Extension and Torsion

Just as in the case of transversely rigid plates, the governing differential equations and boundary conditions of our theory for elastic flat plates also naturally uncouple into two separate groups.

One group involves N_{ij}, ε_{ij} and U_j : it describes the extension and torsion action of the plate. The other group involves M_{ij}, Q_j, κ_{ij}, γ_j, Φ_j and W ; it describes the bending and twisting actions of the plate. For each of these two groups, it is possible to reduce the differential equations and arrive at a formulation similar to that of the transversely rigid plate case for the same group.

For extension and torsion plate actions, we have the equilibrium equations

$$N_{ij,i} = 0 \quad (i,j = 1,2) \tag{3.18}$$

the stress-strain relations

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial C_p}{\partial N_{11}} = A(N_{11} - \nu N_{22}) - \frac{\nu_3}{E}\sigma_o \\ \varepsilon_{22} &= \frac{\partial C_p}{\partial N_{22}} = A(N_{22} - \nu N_{11}) - \frac{\nu_3}{E}\sigma_o \\ \varepsilon_{12} &= \varepsilon_{21} = \frac{1}{2}\frac{\partial C_p}{\partial N_{12}} = \frac{1}{2Gh}N_{12} \end{aligned} \tag{3.19}$$

and the strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \qquad (i,j = 1,2) \tag{3.20}$$

For these differential equations, we have the stress boundary conditions

$$N_{\nu\nu} = N^*_{\nu\nu}(s) \ , \quad N_{\nu t} = N^*_{\nu t}(s) \tag{3.21}$$

along Γ_σ , and the displacement conditions

$$U_\nu = U^*_\nu(s) \ , \quad U_t = U^*_t(s) \tag{3.22}$$

along Γ_d . Of course, two independent conditions involving other combinations of N_{ij} and U_j are also possible. For example, in the case of an elastically supported edge, we have

$$N_{\nu\nu} + \alpha_\nu U_2 = 0 \ , \quad N_{\nu t} + \alpha_t U_t = 0 \tag{3.23}$$

along Γ .

For this class of problems, we may again introduce the Airy stress function by setting

$$N_{11} = F,_{22} \ , \quad N_{22} = F,_{11} \quad \text{and} \quad N_{12} = N_{21} = -F,_{12} \tag{3.24}$$

so that the two equilibrium equations are satisfied identically. To determine F , we again make use of the compatibility equation $\varepsilon_{11.22}+\varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$ which is also satisfied by the strain measure (3.20) of our theory. Upon expressing ε_{ij} in terms of F by the stress strain relations (3.19) and the stress function representation (3.24), this compatibility equation becomes a fourth order PDE for F :

$$\boxed{A \nabla^2\nabla^2 F = \frac{\nu_3}{E} \nabla^2\sigma_o \ , \quad A = \frac{1}{Eh}} \tag{3.25}$$

where $\sigma_o = \frac{1}{2}(\sigma_t + \sigma_b)$ is the average of the two face pressure distributions. Note that the governing PDE for F is identical to that for the transversely rigid plate (1.57) and reduced to the latter if we set $\nu_3 = 0$ (which is required for transverse rigidity). In fact, transverse shear rigidity $(1/G_3 = 0)$ has no effect on the problem of extension and torsion in the present theory, and transverse in extensibility $(1/E_3 = \nu_3 = 0)$ only eliminates the forcing term in (3.25) and an inhomogeneous term in the expression for ε_{11} and ε_{22}.

For the fourth order PDE (3.25), we have typically two stress boundary conditions (3.21) or two displacement boundary conditions (3.22). The stress conditions may be written as two independent conditions for F and its derivatives. These two conditions and the PDE (3.25) define a well-posed BVP for F. If displacement boundary conditions (3.22) are prescribed, we may either (i) use the two coupled second order PDE formulation for U_1, and U_2 (see equations (1.32) of Chapter (1)), or (ii) insert the general solution of (3.25) into (3.20) by way of (3.19) and (3.24), and then solve the resulting system of three first order PDE (and the displacement boundary conditions) for U_1 and U_2. Both of these approaches have been and are still being used in practice and both have some obvious undesirable features. It will be seen in a later section that a third approach is possible. This new approach has all the advantages of the other two approaches and avoids the shortcomingsof both.

5. Reduction of Equations for Bending and Twisting

For bending and twisting plate actions, we have the equilibrium equations

$$M_{ij,i} - Q_j = 0 , \quad Q_{j,j} + p = 0 , \quad (j,j = 1,2) , \quad (3.26a,b,c)$$

the stress-strain relations

$$\kappa_{11} = \frac{\partial C_p}{\partial M_{11}} = \frac{(M_{11} - \nu M_{22})}{D(1 - \nu^2)} - \frac{6\nu_3}{5Eh} p$$

$$\kappa_{22} = \frac{\partial C_p}{\partial M_{22}} = \frac{M_{22} - \nu M_{11}}{D(1 - \nu^2)} - \frac{6\nu_3}{5Eh} p \qquad (3.27)$$

$$\tau = \frac{\partial C_p}{\partial M_{12}} = \frac{12M_{12}}{Gh^3}$$

$$\gamma_j = \frac{6}{5G_3 h} Q_j \qquad (j = 1,2) \qquad (3.28)$$

and the strain-displacement relations

$$\gamma_j = W_{,j} + \Phi_j \qquad (3.29a)$$

$$(i,j = 1,2)$$

$$\kappa_{ij} = \Phi_{j,i} \quad , \quad \tau = \kappa_{ij} + \kappa_{ji} \qquad (3.29b)$$

For these differential equations, we have the stress boundary conditions

$$M_{\nu\nu} = M^*_{\nu\nu}(s), \quad M_{\nu t} = M_{\nu t} = M^*_{\nu t}(s), \quad Q_\nu = Q^*_\nu(s) \qquad (3.30)$$

along Γ_σ , and the displacement conditions

$$\Phi_\nu = \Phi^*_\nu(s), \quad \Phi_t = \Phi^*_t(s), \quad W = W^*(s) \qquad (3.31)$$

along Γ_d . Three independent conditions involving other combinations of M_{ij}, Q_j , Φ_j and W are also possible. For example, in the case of an elastically supported edge, we have

$$M_{\nu\nu} + \beta_\nu \Phi_\nu = 0 , \quad M_{\nu t} + \beta_t \Phi_t = 0$$
$$Q_\nu + kW = 0 \tag{3.32}$$

along Γ where $\beta_\nu(s)$ and $\beta_t(s)$ are known functions.

For a transversely rigid plate, we have $1/G_3 = 0$ so that $\gamma_j = 0$ and therewith $\Phi_j = -W_{,j}$ and $\kappa_{ij} = -W_{,ij}$. The stress strain relations (3.27) and moment equilibrium equations (3.26a,b) allow us to express M_{ij} and Q_j in terms of W. The transverse force equilibrium equation (3.26c) then gives an inhomogeneous biharmonic equation for W. Note that this reduction may be carried out whether we have the property of transverse inextenisbility. However, only when we have $\nu_3 = 0$ does the PDE for W coincide with the one previously obtained in Chapter (1) (see equation (1.24)).

When $1/G_3 \neq 0$, it would seem at first that a single biharmonic equation for W is not possible. It is rather remarkable that, for the present more general theory, the governing differential equations can in fact be reduced to two uncoupled equations: a biharmonic equation for W alone (but now with a different forcing term) and a second order PDE of the Helmholtz type for an auxiliary variable χ alone, with

$$\chi \equiv A_Q(Q_{2,1} - Q_{1,2}) , \quad A_Q = \frac{6}{5G_3h} \tag{3.33}$$

We begin the reduction to a biharmonic equation for W by writing (3.29a) as

$$\Phi_j = A_Q Q_j - W_{,j} , \tag{3.34}$$

where we have used (3.28) for γ_j (with A_Q as defined in (3.33)), and then (3.29b) as

$$\tau = A_Q (Q_{i,j} + Q_{j,i}) - 2W,_{ij}$$

The stress-strain relations (3.27) in their inverted form are then used to express M_{ij} in terms of W and Q_j:

$$M_{11} = - D[W,_{11} + \nu W,_{22}] + DA_Q[Q_{1,1} + \nu Q_{2,2}] + \frac{3\nu_3 D}{5Gh} p \qquad (3.35a)$$

$$M_{22} = - D[W,_{22} + \nu W,_{11}] + DA_Q [\nu Q_{1,1} + Q_{2,2}] + \frac{3\nu_3 D}{5Gh} p \qquad (3.35b)$$

$$M_{12} = M_{21} = -D(1 - \nu) W,_{12} + \frac{1}{2} DA_Q (1 - \nu)[Q_{2,1} + Q_{1,2}] \qquad (3.35c)$$

The two moment equilibrium equations (3.26a,b) are then expressed in terms of W and Q_j alone:

$$Q_1 = - D \nabla^2 W,_1 + DA_Q \nabla^2 Q_1 + \frac{1}{2} (1 + \nu) D \chi,_2 + \frac{3\nu_3 D}{5Gh} p,_1$$

$$Q_2 = - D \nabla^2 W,_2 + DA_Q \nabla^2 Q_2 - \frac{1}{2} (1 + \nu) D \chi,_1 + \frac{3\nu_3 D}{5Gh} p,_2 \qquad (3.36a,b)$$

where χ is as given by (3.33). The transverse force equilibrium equation (3.26c) then becomes

$$- D \nabla^2 \nabla^2 W + DA_Q \nabla^2 (Q_{1,1} + Q_{2,2}) + \frac{3\nu_3 D}{5Gh} \nabla^2 p + p = 0$$

With $Q_{1,1} + Q_{2,2} = - p$, we get from the above equation the following inhomogeneous biharmonic equation for W alone:

$$\boxed{- D \nabla^2 \nabla^2 W + [1 - D (A_Q - A_3) \nabla^2] p = 0 \ , \ A_3 = \frac{3\nu_3}{5Gh}} \qquad (3.37)$$

Note that, with $A_Q = A_3 = 0$, equation (3.37) reduces to equation (1.24) for a transversely rigid plate.

The combination $Q_{1,1} + Q_{2,2}$ needed in the transverse force equilibrium equation to give us (3.37) uses up only a portion of the contents of (3.36). With $\vec{Q} = Q_1\vec{i}_1 + Q_2\vec{i}_2$, we have $Q_{1,1} + Q_{2,2} = \vec{\nabla}\cdot\vec{Q}$; therefore, an independent portion of the content of (3.36) is $(Q_{2,1} - Q_{1,2}) = (\chi/A_Q)$. From (3.36), we get

$$A_Q(Q_{2,1} - Q_{1,2}) = DA_Q \nabla^2\chi - \frac{1}{2}(1 + \nu) DA_Q \nabla^2\chi$$

and therefore a second order PDE for χ alone:

$$\boxed{\frac{1}{2} DA_Q(1 - \nu) \nabla^2\chi - \chi = 0} \tag{3.38}$$

Equation (3.38) is a Helmholtz equation with the wrong sign; we will call such a equation the modified Helmholtz equation. Note that, for an isotropic plate, we have DA_Q proportional to h^2 so that the solution of (3.38) generally has a boundary layer behaviour.

The two PDEs (3.37) and (3.38) form an uncoupled sixth order system and the general solution of this system can satisfy three independent boundary conditions along an edge of the plate. The Kirchhoff contraction of the stress boundary conditions is not necessary. For these stress boundary conditions, we need to have Q_ν , $M_{\nu\nu}$ and $M_{\nu t}$ expressed in terms of W and χ . To get these expressions, we recall from (3.26c) and (3.33)

$$Q_{1,1} + Q_{2,2} = -p , \qquad Q_{2,1} - Q_{1,2} = \frac{1}{A_Q}\chi$$

From these, we get

$$\nabla^2 Q_1 = - p,_1 - \frac{1}{A_Q} \chi,_2 \, , \quad \nabla^2 Q_2 = - p,_2 + \frac{1}{A_Q} \chi,_1 \qquad (3.39)$$

We then use (3.39) to eliminate $\nabla^2 Q_j$ from (3.36) to get

$$Q_1 = - D[\nabla^2 W + (A_Q - A_3)p],_1 - \frac{1}{2} D(1 - \nu)\chi,_2$$
$$(3.40)$$
$$Q_2 = - D[\nabla^2 W + (A_Q - A_3)p],_2 + \frac{1}{2} D(1 - \nu)\chi,_1$$

These expressions of Q_j in terms of W and χ alone are then used to eliminate $Q_{i,j}$ from (3.35) leaving us with*

$$M_{11} = - D[(1 + DA_Q \nabla^2) W,_{11} + \nu (1 - DA_Q \nabla^2) W,_{22} + \frac{1}{2} DA_Q(1 - \nu)^2 \chi,_{12}$$

$$- A_3 p + DA_Q(A_Q - A_3)(p,_{11} - \nu p_{22})]$$

$$M_{22} = - D[(1 + DA_Q \nabla^2) W,_{22} + \nu(1 - DA_Q \nabla^2)W,_{11} - \frac{1}{2} DA_Q(1 - \nu)^2\chi,_{12}$$

$$- A_3 p + DA_Q(A_Q - A_3)(p,_{22} - \nu p,_{11})] \qquad (3.41)$$

$$M_{12} = M_{21} = - D(1 - \nu)[(1 + DA_Q \nabla^2) W,_{12} + \frac{1}{4} DA_Q(1 - \nu)(\chi,_{22} - \chi,_{11})$$

$$+ DA_Q(A_Q - A_3)p,_{12}]$$

The expressions in (3.40) and (3.41) also reduce to those obtained for the same quantities in Chapter (1) (see equations (1.33) and (1.34)) when the plate is transversely rigid so that $A_Q = A_3 = 0$ and therewith $\chi \equiv 0$ from (3.38).

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*Simpler, but less symmetric, alternative forms are also possible.

For displacement boundary conditions (3.31), we need expressions of Φ_j in terms of χ and W. But with (3.34) and (3.40), these expressions are readily calculated to be

$$\begin{aligned} \Phi_1 &= -(1 + DA_Q\nabla^2)W,_1 - \frac{1}{2}DA_Q(1-\nu)\chi,_2 - DA_Q(A_Q - A_3)p,_1 \\ \Phi_2 &= -(1 + DA_Q\nabla^2)W,_2 + \frac{1}{2}DA_Q(1-\nu)\chi,_1 - DA_Q(A_Q - A_3)p,_2 \end{aligned} \qquad (3.42)$$

6. A Variational Principle and Contracted Boundary Conditions

The system of differential equations (3.2), (3.13) and (3.15), and the typical boundary conditions (3.16) on Γ_d and (3.17) on Γ_σ completely characterize deformed plates in equilibrium. These differential equations and boundary conditions are the two dimensional approximations of the differential equations and boundary conditions for the three dimensional theory of linear elasticity. Our experience with three dimensional elasticity theory suggests that the governing equations and boundary conditions for plates are the Euler differential equations and Euler boundary conditions of a variational principle for stresses and displacements similar to Reissner's variational principle for three dimensional elasticity theory.

It is not difficult to verify that with the strain-displacement relations (3.13) as equality constraints and with δN_{ij}, δM_{ij}, δQ_j, δU_j and δW varying independently, the condition $\delta I_p = 0$ where

$$I_p = \iint\{\varepsilon_{ij}N_{ij} + \kappa_{ij}M_{ij} + \gamma_j Q_j - C_p(N_{ij},M_{ij},Q_j) + G_p(U_j,W,\Phi_j\}dx_1dx_2$$

$$- \int_{\Gamma_\sigma}(\vec{N}_\nu^* \cdot\vec{U} + \vec{M}_\nu^* \cdot\vec{\Phi})ds - \int_{\Gamma_\alpha}[\Delta\vec{U}\cdot\vec{N}_\nu+\Delta\vec{\Phi}\cdot\vec{M}_\nu]ds \qquad (3.43)$$

gives the equilibrium equations (3.2) with $p_k = -\partial G/\partial U_k$, etc. being surface loads omitted in (3.2) and the stress-strain relation (3.15) as Euler differential equations and the displacement boundary conditions (3.16) on Γ_d and the stress boundary conditions (3.17) on Γ_σ as Euler boundary conditions. In (3.43), we have again $\Delta\vec{f} = \vec{f} + \vec{f}^{(o)}$ where $\vec{f}^{(o)}$ is a prescribed vector function of the arc length variable s.

Aside from its usefulness for a Rayleight-Ritz type of approximate solutions (including the use of finite elements) for plate problems, this variational principle is also useful in theoretical developments. One result in this direction is the Kirchhoff contracted stress boundary conditions (1.47) and (1.48). Suppose $1/G_3 = 0$ (so that $A_Q = 0$) in the complementary energy density function for plate, (3.6), two of the Euler differential equations corresponding the stress strain relations for γ_j and Q_j become

$$\gamma_j = 0 \qquad (j=1,2) \ . \tag{3.44}$$

From the equality constraints, we have

$$W,_j + \Phi_j = 0 \quad \text{or} \quad \Phi_j = -W,_j \qquad (j=1,2) \tag{3.44}$$

and Φ_1 and Φ_2 can no longer vary independently.

Consider first the variation of I_p along Γ_σ written in terms of the components of the relevant vectors. In view of (3.44), $\delta\Phi_\nu$, $\delta\Phi_t$ and δW do not vary independently along Γ_σ . Instead, we have

$$\int_{\Gamma_\sigma} [\Delta N_{\nu\nu}\delta U_\nu + \Delta N_{\nu t}\delta N_{\nu t} + \Delta M_{\nu\nu}\delta\Phi_\nu + \Delta M_{\nu t}\delta\Phi_t + \Delta Q_\nu \delta W]ds$$

$$= \int_{\Gamma_\sigma} [\Delta N_{\nu\nu}\delta U_\nu + \Delta N_{\nu t}\delta u_t + \Delta M_{\nu\nu}\delta(-W,_\nu) + \Delta M_{\nu t}\delta(-W,_s) + \Delta Q_\nu \delta W]ds$$

$$= [\Delta M_{\nu t}\delta(-W)]_{s_1}^{s_2} + \int_{\Gamma_\sigma} [-\Delta M_{\nu\nu}\delta W,_\nu + (\Delta Q_\nu + \Delta M_{\nu t,s})\delta W$$

$$+ \Delta N_{\nu\nu}\delta U_\nu + \Delta N_{\nu t}\delta U_t]ds \tag{3.45}$$

where we have integrated by parts the term involving $\delta W,_s$. Now

δW, $\delta W,_{\nu}$, δU_{ν} and δU_t may be varied independently and the Euler stress boundary conditions along Γ_{σ} are what we asserted previously, namely

$$\Delta N_{\nu\nu} = \Delta N_{\nu t} = 0 \tag{1.50}$$

for extension and torsion and

$$\Delta M_{\nu\nu} = \Delta Q^e_{\nu} = 0 \;, \qquad Q^e_{\nu} \equiv Q_{\nu} + M_{\nu t,s} \tag{1.52}$$

for bending and twisting. If Γ_{σ} is the entire (single) closed edge curve Γ with a continuously changing normal vector, then we have

$$[\Delta M_{\nu t}\delta W]_{s_1}^{s_2} = [\Delta M_{\nu t}\delta W]_{s_1+}^{s_1-} = 0 \;. \tag{3.46}$$

Since W is single-valued, the condition (3.46) requires $\Delta M_{\nu t}$ to be single-valued also:

$$[\Delta M_{\nu t}]_{s_1+}^{s_1-} = 0 \tag{3.47}$$

Consider next the case of prescribed displacement boundary data for the single smooth, edge curve so that $\Gamma_d = \Gamma$. With the restriction $A_Q = 0$, the variation of I_p along Γ_d is

$$-\int_{\Gamma_d} \{\Delta U_{\nu}\delta N_{\nu\nu} + \Delta U_t \delta N_{\nu t} + \Delta\Phi_{\nu}\delta M_{\nu\nu} + \Delta\Phi_t \delta M_{\nu t} + \Delta W \delta Q_{\nu}\} ds$$

$$= -\int_{\Gamma_d} \{\ldots . + [-W,_{\nu} - \Phi^*_{\nu}\;]\delta M_{\nu\nu} + [-W,_s - \Phi^*_t\;]\delta M_{\nu t} + [W - W^*\;]\delta Q_{\nu}\} ds. \tag{3.48}$$

For the two Euler boundary conditions $W = W^*$ and $W,_s = -\Phi^*_t$ along

Γ_d to be compatible, we must have $\Phi_t^* = -W,_s^*$ for consistency. Thus, the restriction of no transverse shear deformation requires that Φ_t^* not be prescribed independently.

In many applications, the single edge curve of the plate has one or more corners where the normal of the edge curve changes direction abruptly. A simple example is a rectangular plate while another often encountered configuration is an annular plate slit along a radial line. For plates with corners, the restriction of no transverse shear deformability introduces additional fictitious concentrated forces at the plate corners. To obtain the magnitude and direction of these corner forces we consider a rectangular plate with $a \leq x_1 \leq b$ and $c \leq x_2 \leq d$ free of edge resultants and couples along its edges. The calculation is similar for more general plate shapes. For the rectangular plate, we have

$$\delta I_p = \int_c^d\int_a^b \{[-N_{ij,i} + \frac{\partial G_p}{\partial U_j}]\delta U_j + [Q_j - M_{ij,i} + \frac{\partial G_p}{\partial \Phi_j}]\delta\Phi_j + [-Q_{i,i} + \frac{\partial G_p}{\partial W}]\delta W\} dx_1 dx_2 \tag{3.49}$$

$$+ \int_a^b [N_{2j}\delta U_j + M_{2j}\delta\Phi_j + Q_2\delta W]_c^d dx_1 + \int_c^d [N_{1j}\cdot\delta U_j + M_{1j}\delta\Phi_j + Q_1\delta W]_a^b dx_2$$

where we have satisfied the Euler differential equations for the stress strain relations (3.15). The conditions of no transverse shearing strains

$$\gamma_j = W,_j + \Phi_j = 0 \qquad (j=1,2)$$

give $\Phi_j = -W,_j$ so that $\delta\Phi_j$ and δW are not independent variations.

Integration by parts transforms (3.49) into

$$I_p = \int_c^d \int_a^b \{[-N_{ij,i} + \frac{\partial G_p}{\partial U_j}]\delta U_j + [-M_{ij,ij} + (\frac{\partial G_p}{\partial \Phi_j})_{,j} + \frac{\partial G_p}{\partial W}]\delta W\} dx_1 dx_2$$

$$+ \int_a^b [N_{2j}\delta U_j - M_{22}\delta W_{,2} + (M_{i2,i} - \frac{\partial G_p}{\partial \Phi_2} + M_{21,1})\delta W]_c^d dx_1$$

$$+ \int_c^d [N_{ij}-\delta U_j - M_{11}\delta W_{,1} + (M_{i1,i} - \frac{\partial G_p}{\partial \Phi_1} + M_{12,2})\delta W]_a^b dx_2$$

$$- [(M_{12} + M_{21})\delta W]_{(b,d)} + [(M_{12} + M_{21})\delta W]_{(b,c)} - [(M_{12} + M_{21})\delta W]_{(a,c)}$$

$$+ [(M_{12} + M_{21})\delta W]_{(a,d)} \qquad (3.50)$$

From (3.50), we see that the remaining Euler differential equations associated with the independent variations of δU_1, δU_2 and δW are three (force) equilibrium equations. The two equilibrium equations in the $\vec{i}_1$ and $\vec{i}_2$ directions are as before; the third is effectively as before but now with Q_j eliminated, essentially by way of the previously obtained moment equilibrium equations. The four Euler boundary conditions along a plate edge obtained from (3.50) are effectively the four Kirchhoff contracted stress boundary conditions, now with Q_j eliminated (by way of moment equilibrium conditions). In addition to the Euler differential equations in the plate interior and boundary conditions along a plate edge, we have from the four integrated terms in (3.50) four new conditions, $M_{12} + M_{21} = 0$, one at each of the four corners of the plate. However, the boundary value problem defined by the Euler differential equations (equilibrium equations and stress-strain relations), equality constraints (strain-displacement relations) and Euler boundary conditions of the variational principle completely determines M_{12} and M_{21}. Consequently, the four homogeneous point conditions at the plate corners cannot be satisfied except in special cases. Thus for plates with corners, the correct variational principle should include work terms associated with a fictitious corner force and the normal displacement at each of the corners of the plate. Such a corner force should be equal to $M_{12} + M_{21}$ in magnitude at the corner and opposite in direction to the corresponding term in (3.50). Fictitious corner forces are the results of the Kirchhoff contracted stress boundary condition in which $M_{\nu t}$ along an edge is fictitiously converted to two transverse shear resultants which produce the same twisting couple over an incremental distance ds. The fictitious shear resultants from neighboring edge elements cancel except at corners of the plate. They must be balanced by the fictitious corner forces introduced above. It is important in classical plate theory that the contributions of these fictitious corner forces be included in overall force and moment equilibrium calculations. By the static geometric duality, similar fictitious corner displacements of the form $\varepsilon_{12} + \varepsilon_{21}$ will contribute to, and should be included in overall compatibility calculations.