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ON PLATE THEORIES AND SAINT-VENANT'S PRINCIPLE

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Abstract—It is generally known that the classical Germain-Kirchhoff plate theory is the leading term of the outer (asymptotic expansion or interior) solution in a small thickness parameter for the linear elastostatics of thin, flat, isotropic bodies. This leading term (or the actual) outer solution alone cannot satisfy arbitrarily prescribed admissible edge-data. On the other hand, the complementary inner (asymptotic expansion or boundary layer) solution is determined by a secluence of boundary value problems which are nearly as difficult to solve as the original problem. For stress edge-data, Saint-Venant's principle has been invoked to generate a set of stress boundary conditions for the classical plate theory as well as for higher order terms in the outer expansion (giving various thick plate theories) without any reference to the inner solution. Attempts to derive the corresponding boundary conditions for displacement and other types of edge-data in the literature for general shape plates have not been successful.

The present study applies a general method developed by the authors to derive the correct set of boundary conditions for arbitrarily prescribed admissible edge-data (without an explicit solution of the inner (or boundary layer) solution) for a number of special cases of general interest, including cases with displacement edge-data. Our general results also show that, to be strictly correct, Saint-Venant's principle should be applied only to the leading term outer solution, i.e. the classical plate theory.

1. INTRODUCTION

The classical Germain-Kirchhoff theory of thin elastic plates[1-3] is known to be the leading term of an interior *(or outer asymptotic) expunsion* in powers of a small thickness parameter for the linear elastostatics of thin, flat, isotropic bodies[4-7]. Neither this leading term nor *the full interior solution alone* can fit arbitrarily prescribed data along the edge of the plate. For *stress* edge-data, Saint-Venant's principle has frequently been invoked (e.g. [3], [8], [9]) to generate a set of stress boundary conditions for classical plate theory and for higher-order terms** in the interior solution, without any reference to the complementary *edge zone (or inner asymptotic expansion solution.* Previous attempts in the literature to derive the corresponding boundary conditions for displacement edge-data have not been successful (e.g. [9]; see also discussion and other references in [10]).

By a novel application of the Betti-Rayleigh reciprocal theorem, the present authors have derived a correct set of boundary conditions for the classical and higher-order plate theories for any admissible set of edge data. The special case of a semi-infinite plate in a state of plane strain, induced by edgewise uniform data, has been worked out in [10]. In this case, the stress and displacement fields generated by our boundary conditions differed from the corresponding exact solutions by only *exponentially* small terms away from the plate edge. The stress boundary conditions obtained

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** As we shall see, this use of Saint-Venant's principle gives the higher-order terms incorrectly, in general, and will, therefore, lead to incorrect solutions for thick plates. However, it is correct for the classical thin plate theory for certain classes of problems, as found in [4], [IO], and herein.

in [10] rigorously justify the application of Saint-Venant's principle for that class of problems. More importantly, correct boundary conditions for plates with displacement and mixed edge-data were obtained for the first time in [10] Our method of solution also showed that the only previous general results[9] for displacement edge-data are incorrect.

In the present study, we obtain similar results for more general edge-data and plate geometries. The case of axi-symmetric bending of a circular plate is first treated. The analysis shows that indiscriminate use of Saint-Venant's principle for plates (see [3] and [8] for examples) may lead to quantitatively and qualitatively incorrect solutions for the plate behaviour, even in the interior of the plate. In particular, the estimated level of stress and displacement may be low by orders of magnitude. The results obtained here actually delimit the range of applicability of Saint-Venant's principle for axi-symmetric bending.

The results for axi-symmetric bending of circular plates are obtained essentially by the same technique used for semi-infinite plates in a state of plane strain. The necessary conditions, deduced from the reciprocal theorem, for the edge-data to induce only a decaying elastostatic state are directly translated into appropriate boundary conditions for the plate. Once a suitable (elementary) regular state is constructed for the relevant edge-data, the translation is immediate because the plate solutions for the two classes of problems vary only in one spatial direction. This is not the situation for general edge-data. Appropriate boundary conditions which vary along the plate edge will have to be deduced from the surface integral(s) associated with the reciprocal theorem. The general method for this deduction will be illustrated in this study by way of a semi-infinite plate extending over the region, $x \geq 0, |y| < \infty$ and $|z| \leq h$ with y, z-dependent edge-data prescribed along the only edge of the plate, $x = 0$.

The plate boundary conditions obtained for the plane strain case in [10] are rigorously correct in that they have been shown to induce the correct elastostatic state in the plate interior (except for exponentially small terms). Without the needed completeness and expansion theorems for the relevant eigenfunctions, the boundary conditions for plates obtained in this study constitute only necessary conditions for the corresponding elastostatic state in the plate to be asymptotic to the exact solution away from the plate edge. However, the results for the semi-infinite plate obtained herein do reduce to the corresponding results for the plane strain case obtained in [10]

In an Appendix we apply our theory for the axi-symmetric plate bending to the particular problem of a circular plate loaded by a uniform pressure on its upper face and simply supported around its lower edge. We calculate the stresses in the plate interior to within *exponentially small* error as $h \to 0$; our results show that the existing 'solution' to this problem (e.g., Timoshenko and Goodier[8], p. 351) is not correct beyond the leading term. The corresponding problem for a point load at the center of the plate is solved in a separate communication[11]; once again the existing "solution" (e.g., Love[3], p. 475) is incorrect beyond the leading term.

Aside from the main results of correct boundary conditions for thin and thick plates with various types of edge-data, an important conclusion from the present work *is that the stresses in the interior of the plate are not in general uniquely determined solely by the stress resultants and couples acting at the edge of the plate.* In the example in Appendix 2, for instance, two different methods of 'simple support' (which have the same edge stress resultants) are found to give rise to interior solutions which differ beyond the leading term. Our work shows how the interior solution may nevertheless be calculated from a more detailed knowledge of the stresses acting at the edge of the plate; in particular, these edge stresses will be known exactly for the important case in which the edge of the plate is free and the loading is applied elsewhere. However, if the engineering origin of a plate problem is such that only the resultants of the edge stresses are known (and not their detailed distribution), then only the leading term interior solution (corresponding to the Germain-Kirchhoff plate theory) can be accurately determined in general. Similar remarks apply to the other kinds of boundary conditions.

2. DECAYING STATES IN A PLATE OF GENERAL SHAPE

In [10], the authors developed the notion of a decaying state for the particular case of the semi-infinite plate $x \geq 0, |y| < \infty, |z| \leq h$, which was traction-free on $|z| = h$ and subject to plane strain deformation independent of y. In this case, an equilibrium state $\{\mathbf{u}, \boldsymbol{\sigma}\}$ was said to be a decaying state if

$$\mathbf{u} \rightarrow \mathbf{0} \text{ as } x \rightarrow +\infty, \tag{2.1}$$

uniformly for $|z| \leq h$. It was further shown that if (2.1) is satisfied then u and $\boldsymbol{\sigma}$ must in fact satisfy the much stronger condition

$$\mathbf{u}, \boldsymbol{\sigma} = 0(e^{-\beta x/h}) \tag{2.2}$$

as $x \rightarrow +\infty$, uniformly for $|z| \leq h$, where $\beta \doteqdot 2.106$. [If the deformation was that of bending (rather than in-plane extension), then $\beta \doteqdot 3.75$.]

We now wish to extend the notion of a decaying state to a plate of general shape. In this case, the operation of letting x(or y) $\rightarrow \infty$ does not arise since the plate is of finite lateral dimensions; instead we must let the plate thickness $2h \rightarrow 0$, whilst holding the lateral dimensions constant. Figure 1 depicts the flat finite plate of general shape A and thickness $2h$, with Cartesian coordinates taken so that the mid-plane of the plate is the plane z = 0. The plate is homogeneous, isotropic, and linearly elastic. There are no body forces and the upper and lower faces of the plate are traction-free. On the curved edge E of the plate (shaded in Fig. 1) we shall limit ourselves to one of the following sets of prescribed edge data:

Case (A)

$$\sigma_{nn}(x, y, z) = \bar{\sigma}_{nn}(t, z),$$
$$\sigma_{nt}(x, y, z) = \bar{\sigma}_{nt}(t, z), \tag{2.3}$$
$$\sigma_{nz}(x, y, z) = \bar{\sigma}_{nz}(t, z),$$

Case (B)

$$\sigma_{nn}(x, y, z) = \bar{\sigma}_{nn}(t, z),$$
$$u_t(x, y, z) = \bar{u}_t(t, z), \tag{2.4}$$
$$u_z(x, y, z) = \bar{u}_z(t, z),$$

Case (C)

$$u_n(x, y, z) = \bar{u}_n(t, z),$$
$$\sigma_{nt}(x, y, z) = \bar{\sigma}_{nt}(t, z), \tag{2.5}$$
$$\sigma_{nz}(x, y, z) = \bar{\sigma}_{nz}(t, z),$$

Case (D)

$$u_n(x, y, z) = \bar{u}_n(t, z),$$
$$u_t(x, y, z) = \bar{u}_t(t, z), \tag{2.6}$$
$$u_z(x, y, z) = \bar{u}_z(t, z).$$

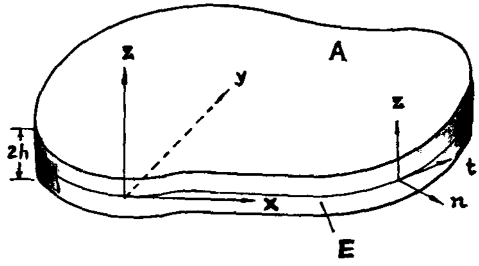

Fig. 1. The plate.

In each case, the above relations hold when (x, y, z) lies on the edge E of the plate, and at each point of the edge the directions of n, t, z are shown in Fig. 1; in particular, n is normal to the edge. The symbol t is also used to denote the distance measured around the edge. The remaining four admissible sets of edge-data, where $\{\bar{\sigma}_{nn}, \bar{\sigma}_{nt}, \bar{u}_z\}$, $\{\bar{\sigma}_{nn}, \bar{u}_t, \bar{\sigma}_{nz}\}$, $\{\bar{u}_n, \bar{\sigma}_{nt}, \bar{u}_z\}$ and $\{\bar{u}_n, \bar{u}_t, \bar{\sigma}_{nz}\}$ are prescribed, respectively, can also be treated by the method used for the four cases above.

The linear elastostatic boundary value problems corresponding to the edge-data *(A), (B), (C)* or (D) each have a solution which is unique except (possibly) for a rigid body displacement. Consider now in each case a class of such solutions corresponding to plates of fixed shape A and all (sufficiently small) thicknesses $2h$. The prescribed edge-data may be h dependent; for most practical purposes it is sufficiently general to allow each component of the edge-data to be a finite sum of terms of the form $h^{\alpha}\phi(t, z/h)$.

Definition: The prescribed edge-data is said to give rise to a *decaying state* within the plate if a solution $\{\mathbf{u}, \boldsymbol{\sigma}\}$ induced by it satisfies the condition

$$\mathbf{u}, \boldsymbol{\sigma} = 0(Me^{-\gamma d/h}) \text{ as } h \to 0, \tag{2.7}$$

where M is the maximum modulus of the prescribed edge-data, d is the minimum distance of the observation point from the edge of the plate, and γ is a positive constant.

For linear problems in elastostatics, we may take the maximum modulus M of this data to be unity with no loss of generality. For the bending of a semi-infinite plate in plane strain, the value of γ was found in [10] to be 3.75 ....; the same value is found in the present study for the axi-symmetric bending of a circular plate. However, the analysis in the present study is not dependent on γ taking this or any other special value.

Definition: A solution $\{\mathbf{u}, \boldsymbol{\sigma}\}$ is said to be a *regular state* if the stress and displacement fields have at worst an algebraic growth as $h \to 0$.

3. NECESSARY CONDITIONS FOR EDGE-DATA TO GIVE RISE TO A DECAYING STATE

The central step in our method of approach is to seek the answer to the following question: *What conditions must the edge-data (2.3), (2.4), (2.5),or (2.6) satisfy in order that the resulting solution in* the **plate** *should be a decaying state?* In the present section, we show how *necessary* conditions may be derived. These necessary conditions will be translated into boundary conditions for plate theories later in Sections 4 to 6. These boundary conditions completely determine the plate solution except possibly for a rigid body plate displacement.

For definiteness consider *Case (A)* in which we have the stress data $\bar{\sigma}_{nn}, \bar{\sigma}_{nt}, \bar{\sigma}_{nz}$ prescribed on the curved edge and suppose that this data does give rise to the decaying state $\{\mathbf{u}, \boldsymbol{\sigma}\}$ in the plate. We now apply the elastic reciprocal theorem

$$\oint\!\!\!\oint_S \{\sigma_{ij}^{(1)}u_i^{(2)} - \sigma_{ij}^{(2)}u_i^{(1)}\}n_j \, dS = 0. \tag{3.1}$$

where S is the surface of that part of the plate shown in Fig. 2, and $\mathbf{n} = n_j\mathbf{i}_j$ is the unit outward pointing normal to S. S consists of parts of the upper and lower faces of the plate, all of the edge E, and an inner boundary E^* whose detailed shape is not important, but whose minimum distance $\hat{d}$ from the edge is positive and independent of h. We take the state with suffix (1) to be $\{\mathbf{u}, \boldsymbol{\sigma}\}$, the decaying state arising from the prescribed data $\bar{\sigma}_{nn}, \bar{\sigma}_{nt}, \bar{\sigma}_{nz}$. For the state with suffix (2) we take any equilibrium state *regular* inside S which satisfies traction-free conditions on the upper and lower faces of S and also traction-free conditions on E. With these choices, the upper and lower faces of S

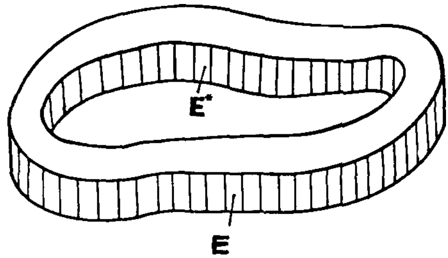

Fig. 2. The region of the plate to which the reciprocal theorem is applied.

yield no contribution to (3. 1), which then reduces to

$$\iint_E \{\bar{\sigma}_{nn}u_n^{(2)} + \bar{\sigma}_{nt}u_t^{(2)} + \bar{\sigma}_{nz}u_z^{(2)}\}\,\mathrm{d}S = -\iint_{E^*}\{\sigma_{ij}u_i^{(2)} - \sigma_{ij}^{(2)}u_i\}n_j\,\mathrm{d}S. \quad (3.2)$$

We note that by taking the suffix (2) state to be traction-free on E, only known components of the decaying state $\{\mathbf{u},\boldsymbol{\sigma}\}$ *now appear on the left in (3.2).*

We now let $h \to 0$ in (3.2). The boundary E^* is in the interior of the plate and so the decaying state $\{\mathbf{u},\boldsymbol{\sigma}\}$ which satisfies (2.6) is exponentially small on E^*. Also without loss of generality, we may take the maximum modulus of $\{\mathbf{u}^{(2)},\boldsymbol{\sigma}^{(2)}\}$ on E^* to be unity. With this normalisation we see that

$$\iint_E \{\bar{\sigma}_{nn}u_n^{(2)} + \bar{\sigma}_{nt}u_t^{(2)} + \bar{\sigma}_{nz}u_z^{(2)}\}\,\mathrm{d}S = 0(e^{-\gamma\hat{d}/h}) \text{ as } h \to 0. \quad (3.3)$$

where $\hat{d}$ (independent of h) is the minimum distance of E^* from E. The value of the left side of (3.3) is a known function of h; if this function is a power series in h (and we shall assume that this is so), then (3.3) implies that it must be the zero function. Hence we finally obtain

$$\iint_E \{\bar{\sigma}_{nn}u_n^{(2)} + \bar{\sigma}_{nt}u_t^{(2)} + \bar{\sigma}_{nz}u_z^{(2)}\}\,\mathrm{d}S = 0, \quad (3.4)$$

which is a *necessary condition* that the edge-data $\bar{\sigma}_{nn}, \bar{\sigma}_{nt}, \bar{\sigma}_{nz}$ should give rise to a decaying state. *The suffix (2) state appearing in (3.4) may be any equilibrium state regular in a neighbourhood of E which is traction-free on the upper and lower faces and also traction-free on the edge E.* In general, there will be an infinity of such (independent) suffix (2) states; but restrictions of symmetry may reduce this to a finite number (see Section 4).*

The above derivation of necessary conditions was for Case *(A)* (i.e., pure stress) edge-data. For cases (B), (C), (D), the derivation follows a similar line, the main difference being that on *E the suffix* (2) *state must be chosen to satisfy homogeneous conditions corresponding to the prescribed data.* If this is done then the necessary conditions for a decaying state are:

Case (B)

$$\iint_E \{\bar{\sigma}_{nn}u_n^{(2)} - \bar{u}_t\sigma_{nt}^{(2)} - \bar{u}_z\sigma_{nz}^{(2)}\}\,\mathrm{dS} = 0. \quad (3.5)$$

*This is assured in the plane strain case[10] because there the necessary conditions are proved to be sufficient.

Case (C)

$$\iint_E \{\bar{\sigma}_{nt} u_t^{(2)} + \bar{\sigma}_{nz} u_z^{(2)} - \bar{u}_n \sigma_{nn}^{(2)}\}\, dS = 0. \tag{3.6}$$

Case (D)

$$\iint_E \{\bar{u}_n \sigma_{nn}^{(2)} + \bar{u}_t \sigma_{nt}^{(2)} + \bar{u}_z \sigma_{nz}^{(2)}\}\, dS = 0. \tag{3.7}$$

4. AXI-SYMMETRIC BENDING OF A CIRCULAR PLATE

The practical difficulty in implementing the preceding process lies in the determination of suitable suffix (2) states which satisfy the appropriate boundary conditions. However, for the case of a circular plate in axi-symmetric bending, the necessary suffix (2) states can be explicitly determined, at least for edge-data in Cases (A) and (C). Let the plate occupy the region I≤ a, $0 \leq \theta \leq 2\pi$, $|z| \leq h$ in a system of cylindrical polar co-ordinates. The assumed axi-symmetry with $\partial(\ )/\partial\theta = 0$ and $u_\theta \equiv 0$ implies that $\sigma_{r\theta} = \sigma_{z\theta} \equiv 0$ and that the remaining field components are independent of 8.

Case (A)

In this case, we have $\sigma_{rr}(a, z)$ = G,.,.(z) and $\sigma_{rz}(a, z) = \bar{\sigma}_{rz}(z)$. The condition for a decaying state (3.4) reduces to the one-dimensional integral

$$\int_{-h}^{h} [\bar{\sigma}_{rr} u_r^{(2)} + \bar{\sigma}_{rz} u_z^{(2)}]_{r=a}\, dz = 0, \tag{4.1}$$

where $\bar{\sigma}_{rr}(z)$ and $\bar{\sigma}_{rz}(z)$ are the prescribed edge-data.

First, take as the suffix (2) state a rigid body translation in the z-direction. Then $u_r^{(2)} \equiv 0$ and $u_z^{(2)} =$ constant so that (3.8) gives

$$\int_{-h}^{h} \bar{\sigma}_{rz}\, dz = 0. \tag{4.2}$$

This condition for a decaying state is not at all surprising since overall equilibrium of the plate demands that (4.2) be satisfied.

As a second choice for the suffix (2) state, take the displacement field

$$u_r^{(2)} = (1 + \nu) \frac{a}{r} z + (1 - \nu) \frac{r}{a} z, \tag{4.3a}$$

$$u_z^{(2)} = -(1 + \nu) a \log \frac{r}{a} - \frac{1}{2}(1 - \nu) \frac{r^2}{a} - \frac{\nu z^2}{a}, \tag{4.3b}$$

where v is Poisson's ratio. The corresponding values of $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ are given by

$$\sigma_{rr}^{(2)} = 2\mu(1 - \nu) \frac{z}{a}\left(1 - \frac{a^2}{r^2}\right), \tag{4.4a}$$

$$\sigma_{rz}^{(2)} = \sigma_{zz}^{(2)} = 0, \tag{4.4b}$$

where μ is the shear modulus. It may be verified that (4.3) and (4.4) do define an equilibrium state in the plate (except at I= 0), which satisfies traction-free conditions on the faces $|z| = h$, and is also traction-free on the edge I= a. The fields are singular on the axis r = 0, but are certainly *regular in a neighbourhood of the edge* and so are admissible as a suffix (2) state in the derivation leading to (3.4).

On substituting (4.3) into (4.1), we obtain, after using (4.2),

$$\int_{-h}^{h} \left\{ z\bar{\sigma}_{rr} - \frac{\nu}{2a} z^2 \bar{\sigma}_{rz} \right\} dz = 0, \tag{4.5}$$

a second necessary condition for a decaying state. The necessary condition (4.5) is rather unexpected and deserves some discussion. The question of what conditions a surface load (applied to a portion of the boundary of an elastic body) must satisfy in order that its effects should 'decay to zero most rapidly' at distances large compared with the linear dimensions of loaded surface area goes back at least to the 1855 Saint-Venant theory of torsion. Saint-Venant suggested that, in the case of cylinders loaded at their ends, the condition for most rapid decay is that the *resultant* force and moment at each end should be zero. This example of *Saint-Venant's principle* has since been proved to be correct, along with many generalizations (see the review article by Horgan and Knowles[12]). In the context of the bending of plates by edge tractions, the most rapid decay possible is exponential decay with respect to distance from the edge as in (2.7) (see the examples in Section 5). Previous attempts to decide on the conditions for rapid decay have often relied on the use of unproved extensions of the Saint-Venant principle. For instance, Timoshenko and Goodier[8] (pp. 351-352), appeal to this principle and assert (in a problem involving the axi-symmetric bending of a circular plate) that the conditions for most rapid decay consist of (4.2) and

$$\int_{-h}^{h} z\bar{\sigma}_{rr}\, dz = 0 \tag{4.6}$$

(instead of (4.5)). The same presumption is made by Love[3] in his treatment of thick plates. We now see that this particular application (or extension) of Saint-Venant's principle is incorrect. As a particular example, consider the data

$$\bar{\sigma}_{rr} = 0, \tag{4.7a}$$

$$\bar{\sigma}_{rz} = P\delta(z-h) + P\delta(z+h) - 2P\delta(z), \tag{4.7b}$$

where S(z) is the Dirac delta-function.** This data satisfies (4.2), (4.6) and so, if Saint-Venant's principle were applicable, it would generate a decaying state in the plate. However, the true condition (4.5) is *not* satisfied (except for $\nu = 0$), and so the data (4.7) do *not* generate a decaying state in the plate when $\nu > 0$, and so must instead generate a non-zero interior solution. By using the method employed in Appendix 2, this *interior* solution may be found exactly to be

$$\sigma^I_{rr} = -\frac{3\nu P}{a}\frac{z}{h}, \tag{4.8a}$$

$$\sigma^I_{rz} = \sigma^I_{zz} = 0. \tag{4.8b}$$

The appearance of the interior stress (4.8) arising from the data (4.7) may be interpreted as a Poisson's ratio effect.

Because our result (4.5) is in contradiction to that widely used in the literature, we give, in Section 5, explicit examples of decaying states in a circular plate for which (4.5) is true, whilst (4.6) is false. The difference between interior solutions associated with (4.5) and (4.6) for typical physical problems is illustrated in Appendix 2 and [11]. We note here only that, strictly speaking, it is in any case inappropriate to invoke Saint-Venant's principle for the problems of this section. As the surface tractions are distributed (uniformly) around the entire edge of the circular plate, the representative linear dimension of the loaded area (namely the plate perimeter) is not small compared with the maximum distance away from the plate edge.

Case (C)

For this case, we have $u_r(a, z) = \bar{u}_r(z)$ and $\sigma_{rz}(a, z) = \bar{\sigma}_{rz}(z)$. As in Case (A), we certainly must have (from $u_r^{(2)} \equiv 0, u_z^{(2)} \equiv 1$)

$$\int_{-h}^{h} \bar{\sigma}_{rz}\, dz = 0. \tag{4.9}$$

* Examples involving smooth data can also be constructed. The data (4.7) is chosen because it is easy to visualise.

To obtain a second condition, take the $\mathbf{u}^{(2)}$ to be

$$u_r^{(2)} = (1 - \nu)\left(\frac{rz}{a} - \frac{az}{r}\right), \tag{4.10a}$$

$$u_z^{(2)} = (1 - \nu)a \log\frac{r}{a} - \frac{1}{2}(1 - \nu)\frac{r^2}{a} - \nu\frac{z^2}{a}. \tag{4.10b}$$

The corresponding stress components are

$$\sigma_{rr}^{(2)} = \frac{2\mu z}{a}\left\{(1 - \nu)\frac{a^2}{r^2} + (1 + \nu)\right\}, \tag{4.11a}$$

$$\sigma_{rz}^{(2)} = \sigma_{zz}^{(2)} = 0. \tag{4.11b}$$

These fields are regular in a neighbourhood of the edge $r = a$; they are traction-free on $|z| = h$, and, on the edge, satisfy the homgeneous conditions

$$u_r^{(2)} = \sigma_{rz}^{(2)} = 0 \qquad (r = a). \tag{4.12}$$

On substituting (4. IO), (4.11) into (3.6) we obtain, after using (4.9),

$$\int_{-h}^{h} \{\nu z^2\bar{\sigma}_{rz} + 4\mu z\bar{u}_r\}\, \mathrm{d}z = 0 \tag{4.13}$$

as a second necessary condition for a decaying state when $\bar{u}_r$ and $\bar{\sigma}_{rz}$ are prescribed. Because (4.13) is independent of the plate radius a, one would expect the same condition to hold for the semi-infinite plate in plane strain; this is confirmed by Theorem 3 of [10].

We have not found any *simple* suffix (2) states suitable for edge-data in Cases (B), (D), but his does not mean that our approach is useless in these cases. It means that the required suffix (2) states are themselves the solutions of certain particular boundary value problems, which, *when solved once and for all,* are to be used in the appropriate decaying state conditions. We used this procedure in [10] to solve problems involving pure displacement (Case (D)) edge-data. For axi-symmetric bending of circular plates with axi-symmetric displacement edge-data, the same procedure gives the following two necessary conditions for a decaying state for (2.6):

$$\int_{-h}^{h} \{\sigma_{rr}^{BY}(a, z)\bar{u}_r(z) + \sigma_{rz}^{BY}(a, z)\bar{u}_z(z)\}\, \mathrm{d}z = 0, \qquad (Y = B, F) \tag{4.14}$$

where $\sigma_{rr}^{BB}(r, z)$ and $\sigma_{rz}^{BB}(r, z)$ may be taken as the stress fields for the boundary value problem in linear elasto-statistics with the boundary conditions

$$(B)\begin{cases} r = a:\ u_r^{BB}(a, z) = 0, \qquad u_z^{BB}(a, z) = 0 \\ r \to 0:\ \sigma_{rr}^{BB} \sim \dfrac{z}{h^2 r^2}, \qquad \sigma_{rz}^{BB} \sim 0 \end{cases} \tag{4.15}$$

while $\sigma_{rr}^{BF}(r, z)$ and $\sigma_{rz}^{BF}(r, z)$ may be taken as the stress fields for the linear elasto-static problem with

$$(F)\begin{cases} r = a:\ u_r^{BF}(a, z) = 0, \qquad u_z^{BF}(a, z) = 0 \\ r \to 0:\ \sigma_{rr}^{BF} \sim \dfrac{a(2 - \nu)z^3}{4h^3 r^2} \\ \qquad\quad \sigma_{rz}^{BF} \sim \dfrac{3a}{4h^3 r}(h^2 - z^2). \end{cases} \tag{4.16}$$

The above bending and flexure problems do not depend on the prescribed displacement edge-data along I = a and, for a fix *h/u* ratio, may be solved once and for all by some suitable numerical scheme as the corresponding semi-infinite strip problems for the plane strain case discussed in [10].

5. EXPLICIT EXAMPLES OF DECAYING STATES TO WHICH SAINT-VENANT'S PRINCIPLE DOES NOT APPLY

It is possible to represent three-dimensional axi-symmetric elastostatic fields in terms of a potential ϕ satisfying (see Chapter 13 of [8] for example)

$$\nabla^2\nabla^2\phi = 0, \qquad \nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right), \tag{5.1}$$

where the axis I = 0 of cylindrical polar coordinates has been taken to be the axis of symmetry of the deformation. In terms of $\phi(r, z)$, we have the following expressions for the nonvanishing stress components:

$$\begin{aligned} \sigma_{rr} &= [\nu\nabla^2\phi - \phi_{,rr}]_{,z} \\ \sigma_{zz} &= [(2-\nu)\nabla^2\phi - \phi_{,zz}]_{,z} \\ \sigma_{\theta\theta} &= [\nu\nabla^2\phi - r^{-1}\phi_{,r}]_{,z} \\ \sigma_{rz} &= [(1-\nu)\nabla^2\phi - \phi_{,zz}]_{,r} \end{aligned} \tag{5.2}$$

It is evident that (5.1) has solutions of the form

$$\phi = \left[A\cos\left(\xi\frac{z}{h}\right) + Bz\sin\left(\xi\frac{z}{h}\right)\right] I_0\left(\xi\frac{r}{h}\right) \tag{5.3}$$

for any ξ, where $I_n(y)$, n = 0, I, 2, ..., are modified Bessel functions of the first kind, i.e.,

$$\frac{d^2I_n}{dy^2} + \frac{1}{Y}\frac{dI_n}{dy} - \frac{n^2}{y^2}I_n - I_n = 0 \qquad (n = 0, 1, 2, \ldots). \tag{5.4}$$

The class of solutions (5.3) is regular throughout the elastic body and the corresponding radial stress fields σ_{rr} are anti-svmmetric in z. If we now chose A/B and ξ so that $\sigma_{rz} = \sigma_{zz} = 0$ on $z = \pm h$, *we* obtain (after normalization) the stress fields

$$\sigma_{zz} = \cos^2\xi\left[\frac{z}{h}\sin(\xi)\cos\left(\xi\frac{z}{h}\right) - \cos(\xi)\sin\left(\xi\frac{z}{h}\right)\right]\frac{I_0\left(\xi\frac{r}{h}\right)}{I_0\left(\xi\frac{a}{h}\right)} \tag{5.5a}$$

$$\sigma_{rz} = \xi\left[\frac{z}{h}\cos(\xi)\sin\left(\xi\frac{z}{h}\right) - \sin(\xi)\cos\left(\xi\frac{z}{h}\right)\right]\frac{I_1\left(\xi\frac{r}{h}\right)}{I_0\left(\xi\frac{a}{h}\right)} \tag{5.5b}$$

$$\sigma_{rr} = -\frac{2\nu}{\xi}\frac{h}{r}\cos(\xi)\sin\left(\xi\frac{z}{h}\right)\frac{I_1\left(\xi\frac{r}{h}\right)}{I_0\left(\xi\frac{a}{h}\right)}$$

$$-\left[\xi\frac{z}{h}\cos(\xi)\cos\left(\xi\frac{z}{h}\right)+\{\xi\sin(\xi)+\cos(\xi)\}\sin\left(\xi\frac{z}{h}\right)\right]\frac{I_1'\left(\xi\frac{r}{h}\right)}{I_0\left(\xi\frac{a}{h}\right)}. \tag{5.5c}$$

The fields (5.5) are regular in the circular plate $r \le a, |z| \le h$ and satisfy

$$\sigma_{rz} = \sigma_{zz} = 0 \tag{5.6}$$

on $z = \pm h$ for any choice of ξ; they also represent possible equilibrium stress fields provided that ξ is any root of

$$\sin(2\xi) = 2\xi. \tag{5.7}$$

In what follows, we will also assume that ξ has a positive real part.

Now it follows from the asymptotic expansion[13]

$$I_n(u) \sim \frac{e^u}{(2\pi u)^{1/2}} \tag{5.8}$$

as $|u| \to \infty\ (-\frac{1}{2}\pi < \arg u < \frac{1}{2}\pi)$ that the fields (5.5) are exponentially small (as $h \to 0$) in the interior of the plate, while the stresses on the edge $r = a$ are O(1); hence the fields (5.5) are decaying states in the plate, in the sense of Section 2. For these decaying states we now verify explicitly that the decaying state condition (4.5) is satisfied but the condition of no resultant moment (4.6) is not satisfied.

From (5.5c) it follows after a little algebra that

$$\int_{-h}^{h} z\sigma_{rr}(a, z)\, dz = -\frac{4\nu h^3 \sin^2 \xi}{\xi^2 a}\frac{I_1(\xi a/h)}{I_0(\xi a/h)}. \tag{5.9}$$

Because the right-hand side of (5.9) is not zero, this shows that (4.6) is *false,* in general. Conversely, it is evident that by taking a suitable linear combination of (5.5) and the "pure bending" field $\sigma_{rr} = z, \sigma_{rz} = \sigma_{zz} = 0$, we may obtain a field which does satisfy (4.6) but is plainly not a decaying state.

On the other hand, we get from (5.5b)

$$\int_{-h}^{h} z^2\sigma_{rz}(a, z)\, dz = -\frac{8h^3 \sin^2(\xi)}{\xi^2}\frac{I_1(\xi a/h)}{I_0(\xi a/h)}. \tag{5.10}$$

The expressions (5.10) and (5.9) together show that the decaying state condition (4.5) is satisfied by the decaying stress fields (5.5).

It follows from the above results that applications of Saint-Venant's principle to plate problems as done in [8] and [3] are generally inappropriate. Two correct necessary conditions for ensuring a decaying axi-symmetric bending state are (4.2) and (4.5).

6. BOUNDARY CONDITIONS FOR THE INTERIOR SOLUTION OF AXI-SYMMETRIC BENDING OF A CIRCULAR PLATE

For the circular plate $0 \le r \le a, |z| \le h$ with no body force intensities and with traction-free faces at $z = \pm h$, the outer expansion (in powers of the thickness parameter h/u) of the solution of the linear elastostatic problem can be summed to give the interior solution in a relatively simple form. All stress and displacement components for the interior solution of the axi-symmetric plate bending problem are given in terms of the mid-plane transverse deflection $w(r) \equiv u_z^I(r, z = 0)$ (where a superscript I denotes the interior solution) by the expressions recorded in Appendix I. The transverse deflection itself is the solution of the two-dimensional biharmonic equation $\nabla^2\nabla^2 w = 0$ in the plane $z = 0$.

With $\bar{\sigma}_{ij}^d \equiv \bar{\sigma}_{ij}(z) - \sigma_{ij}^I(a, z)$ and $\bar{u}_j^d(z) \equiv \bar{u}_j(z) - u_j^I(a, z)$, the necessary conditions for $\bar{\sigma}_{ij}^d$ and/or $\bar{u}_j^d$ to induce only a decaying state can be translated into a set of boundary conditions for the determination of w. For axi-symmetric bending, the edge-data are uniformly distributed along $r = a$ and this translation is immediate. The resulting plate boundary conditions for Case (A), (C), and (D) are recorded below for applications to specific problems.

Case (A)

$$\sigma_{rr}(a, z) = \bar{\sigma}_{rr}(z), \qquad \sigma_{rz}(a, z) = \bar{\sigma}_{rz}(z)$$

For this case, the condition (4.2) for a decaying state on $\bar{\sigma}_{ij}^d$ gives immediately

$$-D\frac{\partial}{\partial r}\nabla^2 w\bigg|_{r=a} = \int_{-h}^{h} \bar{\sigma}_{rz}\,dz \tag{6.1}$$

while the condition (4.5) gives

$$-D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right)\bigg|_{r=a} = \int_{-h}^{h}\left\{z\bar{\sigma}_{rr} + \left[\frac{4+\nu}{5a}h^2 - \frac{\nu}{2a}z^2\right]\bar{\sigma}_{rz}\right\}dz \tag{6.2}$$

where use has been made of (6.1) to simplify the expression and where the flexural rigidity D is given by

$$D = \frac{2Eh^3}{3(1-\nu^2)}. \tag{6.3}$$

As pointed out in Section 4, the condition (6.2) differs from the corresponding condition obtained from (4.6) which is used in [3] and [8] in an attempt to improve on Kirchhoff thin plate theory. The significance of this difference when $\bar{\sigma}_{rz} \not\equiv 0$ has already been demonstrated in Section 4.

Case (C)

$$u_r(a, z) = \bar{u}_r(z), \qquad \sigma_{rz}(a, z) = \bar{\sigma}_{rz}(z)$$

For this case, the condition (4.9) implies again (6.1). A second condition follows from (4.13) to be

$$\frac{\partial w}{\partial r}\bigg|_{r=a} = -\frac{3}{2h^3}\int_{-h}^{h} z\bar{u}_r\,dz + \frac{1}{D}\int_{-h}^{h}\left[\frac{4+\nu}{5(1-\nu)}h^2 - \frac{1}{2}\nu z^2\right]\bar{\sigma}_{rz}\,dz \tag{6.4}$$

Case (D)

$$u_r(a, z) = \bar{u}_r(z), \qquad u_z(a, z) = \bar{u}_z(z)$$

For $Y = F$, the necessary condition (4.14) for a decaying state becomes

$$\left\{w - hn_1^F\frac{\partial w}{\partial r} + \frac{\nu t_2^F h^2}{2(1-\nu)}\nabla^2 w + [(2-\nu)n_3^F - 6n_1^F]\frac{h^3}{6(1-\nu)}\frac{\partial}{\partial r}\nabla^2 w\right\}_{r=a}$$

$$= \int_{-h}^{h}\left\{\sigma_{rr}^{BF}(a, z)\bar{u}_r(z) + \sigma_{rz}^{BF}(a, z)\bar{u}_z(z)\right\}dz \tag{6.5}$$

where

$$h^j t_j^Y = \int_{-h}^{h} z^j \sigma_{rz}^{BY}(a, z)\, dz, \qquad h^j n_j^Y = \int_{-h}^{h} z^j \sigma_{rr}^{BY}(a, z)\, dz. \tag{6.6}$$

Similarly, by taking Y = B, we get from (4.14)

$$\left\{ -hn_1^B \frac{\partial w}{\partial r} + \frac{\nu t_2^B h^2}{2(1-\nu)} \nabla^2 w + [(2-\nu)n_3^B - 6n_1^B] \frac{h^3}{6(1-\nu)} \frac{\partial}{\partial r} \nabla^2 w \right\}_{r=a}$$
$$= \int_{-h}^{h} [\sigma_{rr}^{BB}(a, z)\bar{u}_r(z) + \sigma_{rz}^{BB}(a, z)\bar{u}_z(z)]\, dz \tag{6.7}$$

Not surprisingly, the form of (6.6) and (6.7) is similar to that for the plane strain case obtained in [10].

In all cases, the conditions derived above (possibly with some sign changes) also apply at the boundary of an infinite plate with a circular hole as well as at each edge of an annular disc. Furthermore, with D2D2w= 0, it is evident that each of the three sets of two boundary conditions for cases (A), (C), and (D) is sufficient to determine w(r) in the axi-symmetric plate bending problems for circular discs and annular plates, *Thus, in these cases, the necessary conditions for a decaying state which we have obtained are sufficient for the determination of the interior solution.*

7. SEMI-INFINITE PLATE WITH GENERAL EDGE DATA

Up to now, we have only obtained conditions for a decaying state for special classes of edge data which are uniformly distributed along the plate edge. For these edge-data, the conditions for a decaying state may be rewritten in a straightforward manner as the appropriate boundary conditions for classical and higher order plate theories. For edge data which are *not* uniformly distributed, a set of conditions for a decaying state may again be deduced from the reciprocal theorem of elasticity. Additional analyses are now required to transform these conditions (in the form of surface integrals over the cylindrical edge surface) into boundary conditions for plate theories along the edge curve of the plate. The transformation consists of deriving from the surface integral conditions a corresponding set of local conditions (involving only integration across the plate thickness) for points along the edge curve of the midplane of the plate; these local conditions may then be translated into boundary conditions for plate theories. In this section, we describe the method for deriving the local necessary conditions for a decaying state by working out the details for the *bending* of a semi-infinite plate. The same method applies to plates of other shapes as wel! as to plate extension and torsion.

Consider the semi-infinite plate $x \geq 0, |y| < \infty, |z| \leq h$, which is traction-free on its faces $|z| = h$, while on the end $x = 0$ we have some prescribed "anti-symmetric" data resulting in a state of plate bending. The special case of edgewise uniform data leading to an elastostatic state of plane strain has been extensively discussed by Gregory and Wan[lO], We consider here edge-wise nonuniform data for Cases (B) and (C) for which simple suffix (2) states may be obtained.

Case B

$$\sigma_{xx}(0, y, z) = \bar{\sigma}_{xx}(y, z), \qquad u_y(0, y, z) = \bar{u}_y(y, z), \qquad u_z(0, y, z) = \bar{u}_z(y, z).$$

First let us suppose that the deformation and the data are proportional to e^{iky}, and that we write in all cases $\bar{\sigma}_{ij}(y, z) = \hat{\sigma}_{ij}(z)e^{iky}$ and $\bar{u}_j(y, z) = \hat{u}_j(z)e^{iky}$. The argument leading to the necessary condition (3.5) for a decaying state is almost the same as in Section 3, except that in the present case the edge E is of infinite extent and would in general give a nonconvergent integral in (3.5). However, if we choose the suffix (2) state to satisfy

$$\sigma_{xx}^{(2)}(0, y, z) = u_y^{(2)}(0, y, z) = u_z^{(2)}(0, y, z) = 0 \tag{7.1}$$

and also to be proportional to e^{-iky}, then the argument of section 3 may be applied to the bounded region $0 \leq x \leq d, |y| \leq 1, |z| \leq h$. The contributions from the faces $y = \pm 1$ cancel and we obtain

$$\iint_{E_1} [\bar{\sigma}_{xx} u_x^{(2)} - \bar{u}_y \sigma_{xy}^{(2)} - \bar{u}_z \sigma_{xz}^{(2)}] \, dy \, dz = 0 \tag{7.2}$$

as a necessary condition for a decaying state, where E_1 is $x = 0, |y| \leq 1, |z| \leq h$. However, because of our choice of y-dependences, the integrand in (7.2) is, in fact, independent of y and so we obtain the condition

$$\int_{-h}^{h} [\bar{\sigma}_{xx} u_x^{(2)} - \bar{u}_y \sigma_{xy}^{(2)} - \bar{u}_z \sigma_{xz}^{(2)}]_{x=0} \, dz = 0. \tag{7.3}$$

Our first choice for the suffix (2) field is

$$u_x^{(2)} = -kz \cosh(kx) e^{-iky}, \tag{7.4a}$$

$$u_y^{(2)} = ikz \sinh(kx) e^{-iky}, \tag{7.4b}$$

$$u_z^{(2)} = \sinh(kx) e^{-iky}. \tag{7.4c}$$

The corresponding values of σ_{xy}, σ_{xz} are

$$\sigma_{xy}^{(2)} = 2i\mu k^2 z \cosh(kx) e^{-iky}, \tag{7.5a}$$

$$\sigma_{xz}^{(2)} = 0 \tag{7.5b}$$

On substituting (7.4), (7.5) into (7.3) we obtain the first condition

$$\int_{-h}^{h} [z\hat{\sigma}_{xx}(z) + 2ik\mu z \hat{u}_y(z)] \, dz = 0$$

or, upon multiplying through by the factor e^{iky},

$$\int_{-h}^{h} [z\bar{\sigma}_{xx} + 2ik\mu z \bar{u}_y] \, dz = 0. \tag{7.6}$$

Our second choice for the suffix (2) field is

$$u_x^{(2)} = \frac{-z}{1-\nu} \left\{ (1-\nu) [\cosh(kx) + kx \sinh(kx)] + \left(h^2 - \frac{2-\nu}{6} z^2 \right) 2k^2 \cosh(kx) \right\} e^{-iky} \tag{7.7a}$$

$$u_y^{(2)} = \frac{z}{1-\nu} \left\{ (1-\nu) ikx \cosh(kx) + \left(h^2 - \frac{2-\nu}{6} z^2 \right) 2ik^2 \sinh(kx) \right\} e^{-iky} \tag{7.7b}$$

$$u_z^{(2)} = \left\{ x \cosh(kx) + \frac{\nu}{1-\nu} kz^2 \sinh(kx) \right\} e^{-iky}. \tag{7.7c}$$

The corresponding values of σ_{xy}, σ_{xz} on x = 0 are

$$\sigma_{xy}^{(2)}(0, y, z) = \frac{2ik\mu z}{1 - \nu}\left[(1 - \nu) + \left(h^2 - \frac{2 - \nu}{6} z^2\right) 2k^2\right] e^{-iky}, \tag{7.8}$$

$$\sigma_{xz}^{(2)}(0, y, z) = -\frac{2k^2\mu}{1 - \nu}(h^2 - z^2)e^{-iky}. \tag{7.9}$$

On substituting (7.7)-(7.9) into (7.3), we obtain the second condition

$$\int_{-h}^{h}\left\{(h^2 - z^2)\bar{u}_z + \frac{2 - \nu}{3} z^3 ik\bar{u}_y + \frac{2 - \nu}{6\mu} z^3\sigma_{xx}\right\} dz = 0, \tag{7.10}$$

after multiplying through by e^{iky} and using (7.6) to simplify the result.

Since

$$ik\bar{u}_y = \frac{\partial \bar{u}_y}{\partial y}, \tag{7.11}$$

the conditions (7.6) and (7.10) may be written in the form

$$\int_{-h}^{h}\left\{z\bar{\sigma}_{xx} + 2\mu z \frac{\partial \bar{u}_y}{\partial y}\right\} dz = 0, \tag{7.12}$$

$$\int_{-h}^{h}\left\{(h^2 - z^2)\bar{u}_z + \frac{2 - \nu}{3} z^3 \frac{\partial \bar{u}_y}{\partial y} + \frac{2 - \nu}{6\mu} z^3\bar{\sigma}_{xx}\right\} dz = 0. \tag{7.13}$$

Now k does not appear in (7.12) and (7.13) (except in edge data) and so these decaying state conditions must also apply to any data which is a sum or integral of the e^{iky} type data considered above. It follows that (7.12) *and (7.13) are necessary* conditions **for** *a decaying state for any data of the type (B) which has a Fourier transform in y.*

We can determine the interior solution for plate problems corresponding to this Case (B) data by requiring that the *difference* between σ_{xx}, u_y, u_z on x = 0 due to the interior solution and the data $\bar{\sigma}_{xx}, \bar{x}_y, \bar{u}_z$ should satisfy (7.12) and (7.13). If we express these conditions in terms of $w(x\, y)$, the transverse midplane displacement, (7.12) gives

$$[\nabla^2 w]_{x=0} = -\frac{1}{D}\int_{-h}^{h}\left[z\bar{\sigma}_{xx} + 2\mu z \frac{\partial \bar{u}_y}{\partial y}\right] dz, \tag{7.14a}$$

while (7.13) gives

$$[w - \tfrac{1}{5}h^2\nabla^2 w]_{x=0} = \frac{1}{h^3}\int_{-h}^{h}\left[\tfrac{3}{4}(h^2 - z^2)\bar{u}_z + \frac{2 - \nu}{4} z^3 \frac{\partial \bar{u}_y}{\partial y} + \frac{2 - \nu}{8\mu} z^3\bar{\sigma}_{xx}\right] dz. \tag{7.14b}$$

Appropriate boundary conditions for plate theories of various order may be obtained from (7.14a) and (7.14b) by retaining terms up to an appropriate power of h on both sides of these conditions.

For the special case in which $\bar{\sigma}_{xx} \equiv \bar{u}_y \equiv \bar{u}_z \equiv 0$, the conditions (7.14a) and (7.14b) reduce to

$$[w]_{x=0} = [\nabla^2 w]_{x=0} = 0, \tag{7.15}$$

which are equivalent to the conditions at a 'freely hinged edge' given in the review paper of Gol'denveizer (p. 709 of [14]) for Case (B) edge-data, The conditions given in [14] are *approximate* (correct to order h) but they coincide with the exact conditions (7.15) for a straight edge.

Case (C)

$$u_x(0, y, z) = \bar{u}_x(y, z), \qquad \sigma_{xy}(0, y, z) = \bar{\sigma}_{xy}(y, z), \qquad \sigma_{xz}(0, y, z) = \bar{\sigma}_{xz}(y, z).$$

The method follows closely that used above for Case (B). First consider those deformations and data which are proportional to e^{iky}. If we take the suffix (2) state in (3.6) to be proportional to e^{-iky}, then (3.6) reduces to

$$\int_{-h}^{h} [\bar{\sigma}_{xy}u_y^{(2)} + \bar{\sigma}_{xz}u_z^{(2)} - \bar{u}_x\sigma_{xx}^{(2)}]_{x=0}\, dz = 0 \tag{7.16}$$

as a necessary condition for a decaying state.

Our first choice for the suffix (2) field is

$$u_x^{(2)} = -kz\sinh(kx)e^{-iky}, \tag{7.17a}$$

$$u_y^{(2)} = ikz\cosh(kx)e^{-iky}, \tag{7.17b}$$

u$*) = $\cosh(kx)e^{-iky}$. (7.17c)

The corresponding value of σ_{xx} on $x = 0$ is

$$\sigma_{xx}^{(2)}(0, y, z) = -2\mu k^2 z e^{-iky}. \tag{7.18}$$

On substituting (7.17) and (7.18) into (7.16), we obtain (after multiplying through by e^{iky})

$$\int_{-h}^{h} [\bar{\sigma}_{xz} + ikz\bar{\sigma}_{xy} + 2\mu k^2 z\bar{u}_x]\, dz = 0. \tag{7.19}$$

Our second choice for the suffix (2) field is

$$u_x^{(2)} = -\frac{z}{1-\nu}\left\{(1-\nu)[\sinh(kx) + kx\cosh(kx)] + \left(h^2 - \frac{2-\nu}{6}z^2\right)2k^2\sinh(kx)\right\}e^{-iky}, \tag{7.20a}$$

$$u_y^{(2)} = \frac{z}{1-\nu}\left\{(1-\nu)ikx\sinh(kx) + \left(h^2 - \frac{2-\nu}{6}z^2\right)2ik^2\cosh(kx)\right\}e^{-iky}, \tag{7.20b}$$

$$u_z^{(2)} = \left\{x\sinh(kx) + \frac{\nu}{1-\nu}z^2 k\cosh(kx)\right\}e^{-iky}. \tag{7.20c}$$

The corresponding value of σ_{xx} on x = 0 is

$$\sigma_{xx}^{(2)}(0, y, z) = -\frac{2Ez}{1-\nu^2}\left[k + \left(h^2 - \frac{2-\nu}{6}z^2\right)k^3\right]e^{-iky}. \tag{7.21}$$

On substituting (7.20) and (7.21) into (7.16) we obtain (after multiplying through by e^{iky})

$$\int_{-h}^{h}\left[4\mu z\left\{1+\left(h^2-\frac{2-\nu}{6}z^2\right)k^2\right\}\bar{u}_x + 2ikz\left(h^2-\frac{2-\nu}{6}z^2\right)\bar{\sigma}_{xy}+\nu z^2\bar{\sigma}_{xz}\right]dz = 0. \quad (7.22)$$

With k appearing in (7.19) and (7.22) only through e^{iky} in the edge-data, these two conditions may alternatively be written in the form

$$\int_{-h}^{h}\left[\bar{\sigma}_{xz}+z\frac{\partial\bar{\sigma}_{xy}}{\partial y}-2\mu z\frac{\partial^2\bar{u}_x}{\partial y^2}\right]dz = 0, \quad (7.23)$$

$$\int_{-h}^{h}\left[4\mu z\bar{u}_x - 4\mu z\left(h^2-\frac{2-\nu}{6}z^2\right)\frac{\partial^2\bar{u}_x}{\partial y^2} + 2z\left(h^2-\frac{2-\nu}{6}z^2\right)\frac{\partial\bar{\sigma}_{xy}}{\partial y}+\nu z^2\bar{\sigma}_{xz}\right]dz = 0. \quad (7.24)$$

As before, we may assert that (7.23) *and* (7.24) *are necessary conditions for a decaying state for any Case* (*C*) *data which has a Fourier transform in y.*

The conditions (7.23) and (7.24) may be used to derive the conditions which the interior midplane displacement $w(x, y)$ must satisfy at the edge $x = 0$ with edge-data $\bar{\sigma}_{xz}$, $\bar{\sigma}_{xy}$, and $\bar{u}_x$. The details are omitted and the results are

$$\left[\frac{\partial}{\partial x}\nabla^2 w\right]_{x=0} = -\frac{1}{D}\int_{-h}^{h}\left[\bar{\sigma}_{xz}+z\frac{\partial\bar{\sigma}_{xy}}{\partial y}-2\mu z\frac{\partial^2\bar{u}_x}{\partial y^2}\right]dz, \quad (7.25)$$

which follows from (7.23), and

$$\left[\frac{\partial w}{\partial x}+\frac{4+\nu}{5(1-\nu)}h^2\frac{\partial}{\partial x}\nabla^2 w\right]_{x=0} = -\frac{3}{2h^3}\int_{-h}^{h}\left\{z\bar{u}_x - z\left(h^2-\frac{2-\nu}{6}z^2\right)\frac{\partial^2\bar{u}_x}{\partial y^2}+\frac{1}{2\mu}z\left(h^2-\frac{2-\nu}{6}z^2\right)\frac{\partial\bar{\sigma}_{xy}}{\partial y}+\frac{\nu}{4\mu}z^2\bar{\sigma}_{xz}\right\}dz. \quad (7.26)$$

which follows from (7.24). Again, appropriate boundary conditions for plate theories of various order may be obtained from (7.25) and (7.26) by retaining terms up to an appropriate power of h on both sides of these conditions.

The above results for Cases (B) and (C) of a semi-infinite plate in bending illustrate the general method for deriving local necessary conditions (along a generator of the cylindrical edge of the plate) for a decaying state and therewith appropriate boundary conditions for plate theories.

Since w must satisfy the equation $\nabla^4 w = 0$, it is evident that conditions of the type (7.14) and (7.15) or (7.25) and (7.26) uniquely determine w (possibly only up to a rigid body displacement). *Thus the necessary conditions for a decaying state are sufficient for the determination of the interior solution.*

Note on the Kirchhoff contracted boundary condition

If only terms of leading order (as $h \to 0$) are retained in (7.26), we obtain the approximate condition

$$\left[\frac{\partial w}{\partial x}\right]_{x=0} = -\frac{3}{2h^3}\int_{-h}^{h} z\bar{u}_x\,dz. \quad (7.27)$$

If we now eliminate $\bar{u}_x$ from (7.25), (7.27) we obtain

$$-D\left[\frac{\partial}{\partial x}\nabla^2 w + (1-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=0} = \int_{-h}^{h}\left\{\bar{\sigma}_{xz} + z\frac{\partial \bar{\sigma}_{xy}}{\partial y}\right\} dz, \tag{7.28}$$

which is the well-known Kirchhoff contracted boundary condition of thin plate theory.

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APPENDIX I. INTERIOR SOLUTION FOR PLATES WITH TRACTION-FREE FACES

The parametric series representation (in powers of a thickness parameter) of the interior solution for plate bending has been obtained by Friedrichs and Dressler[4] and Gol'denveizer and Kolos[7]. For the case in which the plate is free of body force intensities and its faces $|z| = h$ are traction-free, their results are summarized by Gregory and Wan[10], Section 6. In this case the series may be summed to give

$$u_x^I = -\frac{z}{(1-\nu)}\frac{\partial}{\partial x}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{I.1}$$

$$u_z^I = \left[1 + \frac{\nu}{2(1-\nu)} z^2\nabla^2\right] w, \tag{I.2}$$

$$\sigma_{xx}^I = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{I.3}$$

$$\sigma_{xy}^I = -\frac{Ez}{1-\nu^2}\frac{\partial^2}{\partial x \partial y}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{I.4}$$

$$\sigma_{xz}^I = -\frac{E}{2(1-\nu^2)}(h^2 - z^2)\frac{\partial}{\partial x}\nabla^2 w, \tag{I.5}$$

$$\sigma_{zz}^I = 0, \tag{I.6}$$

where $w(x, y)$ satisfies

∇^2 being the two-dimensional Laplacian. The formulae for $u_y^I, \sigma_{yy}^I, \sigma_{yz}^I$ are obtained from $u_x^I, \sigma_{xx}^I, \sigma_{xz}^I$ by interchanging x and y.

In cylindrical coordinates (AI.1-AI.6) become

$$u_r^I = -\frac{z}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{I.8}$$

$$u_\theta^I = -\frac{z}{1-\nu}\frac{1}{r}\frac{\partial}{\partial \theta}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{I.9}$$

$$u_z^I = \left[1 + \frac{\nu}{2(1-\nu)} z^2\nabla^2\right] w, \tag{1.10}$$

$$\sigma_{rr}^I = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left\{\nu - \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right\}\right] w, \tag{1.11}$$

$$\sigma_{\theta\theta}^I = -\frac{Ez}{1-\nu^2}\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2}\left\{\nu - \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right\}\right] w, \tag{1.12}$$

$$\sigma_{r\theta}^I = \frac{Ez}{1-\nu^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w, \tag{1.13}$$

$$\sigma_{rz}^I = -\frac{E}{2(1-\nu^2)}(h^2 - z^2)\frac{\partial}{\partial r}\nabla^2 w, \tag{I. 14}$$

$$\sigma_{\theta z}^I = -\frac{E}{2(1-\nu^2)}(h^2 - z^2)\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2 w, \tag{I.15}$$

$$\sigma_{zz}^I = 0. \tag{I. 16}$$

The formulae have also been obtained in a different manner by Lur'e[15], and more recently by Cheng[16].

APPENDIX II. A CIRCULAR PLATE UNDER UNIFORM PRESSURE ON ITS UPPER FACE AND SIMPLY SUPPORTED AT ITS LOWER EDGE

With the loading and support of the plate as shown in Fig. (3), we may regard the support as supplying a uniform vertical line load around the lower edge of the very short circular cylinder.* For the plate (or cylinder) to be in overall equilibrium, this line load must have magnitude $\frac{1}{2}pa$ per unit length where a is the radius of the circular plate, and p is the uniform pressure at the top face. Thus, the relevant elastostatic boundary value problem is one of ***prescribed surface tractions***

We first consider the bending part of the problem. If we subtract away the particular stress fields

$$\sigma_{rr}^p = \frac{pz}{32h^3}[-4(2+\nu)z^2 + 3(3+\nu)r^2 + 12h^2], \tag{II.1a}$$

$$\sigma_{rz}^p = \frac{3pr}{8h^3}(h^2 - z^2), \tag{II.1b}$$

$$\sigma_{zz}^p = \frac{pz}{4h^3}(z^2 - 3h^2), \tag{II.1c}$$

which satisfy $\sigma_{zz}(r, \pm h) = \mp\frac{1}{2}p, \sigma_{rz}(r, \pm h) = 0$, then the *residual* bending problem involves only edge

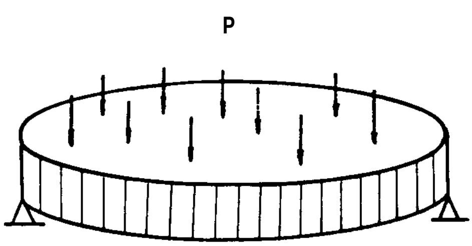

Fig. 3. The circular plate under uniform pressure p and simply supported around its lower edge.

† It is important to specify the precise nature of the support; as the example will show, other forms of 'simple support' do not lead to the same solution even *in the inferior of the plate.*

tractions and is of the type treated in Section 3, Case (A). The prescribed edge-data is

$$\bar{\sigma}_{rr}(z) = \frac{pz}{32h^3}[4(2+\nu)z^2 - 3(3+\nu)a^2 - 12h^2], \tag{II.2a}$$

$$\bar{\sigma}_{rz}(z) = \frac{3pa}{8h^3}(z^2 - h^2) + \tfrac{1}{4}pa[\delta(z-h) + \delta(z+h)], \tag{II.2b}$$

where the Dirac &functions in (II.2b) arise from the supporting line load.

We are not able to find the elastostatic field in the plate which fits the edge-data (11.2) exactly. However, in the spirit of plate theory, we will determine the *interior solution* $\{\mathbf{u}^I, \boldsymbol{\sigma}^I\}$, which differs from the exact solution only by a decaying state; this difference is thus *exponentially small* as $h \to 0$ in any interior region of the plate. For the case in which the plate is traction-free on its faces $|z| = h$, this interior solution has a simple form which is characterised by the single function $w(x, y)$, the transverse deflection of the midplane of the plate. All the field quantities are simply derived from w, the formulae being given for reference in Appendix I.

Taking into account the axi-symmetry and the fact that this interior solution must be regular throughout the plate, it follows from (I.7–I.16) that $w(r, \theta)$ (apart from a rigid body translation in the z direction which does not affect the stress fields) must be proportional to r^2 and that the interior stresses must have the form

$$\sigma^I_{rr} = Kz, \tag{II.3a}$$

$$\sigma^I_{zz} = \sigma^I_{rz} = 0, \tag{II.3b}$$

where K is a constant to be determined. We determine K by requiring that the *difference* between the prescribed edge-tractions (11.2) and those generated from (11.3) should give rise to a decaying state in the plate; in particular the necessary conditions (4.2) and (4.5) must be satisfied by $\bar{\sigma}_{rr}(z) - \sigma^I_{rr}(a, z)$ and $\bar{\sigma}_{rz}(z) - \sigma^I_{rz}(a, z)$. In fact (4.2) is satisfied identically and (4.5) determines the value of K to be

$$K = -\frac{p}{32h^3}\left[3(3+\nu)a^2 + \frac{36}{5}(1+\nu)h^2\right] \tag{II.4}$$

If we now substitute (11.4) into (11.3) and then add on the stress fields (II. 1), we obtain the full interior solution to the bending part of our problem

$$\sigma^I_{rr} = \frac{pz}{32h^3}\left[-3(3+\nu)(a^2 - r^2) + \frac{12}{5}(2-3\nu)h^2 - 4(2+\nu)z^2\right], \tag{II.5a}$$

$$\sigma^I_{rz} = \frac{3pr}{8h^3}(h^2 - z^2), \tag{II.5b}$$

$$\sigma^I_{zz} = \frac{pz}{4h^3}(z^2 - 3h^2). \tag{II.5c}$$

The stress fields (11.5) differ from the exact bending part only by a decaying state.

We now need to find the interior solution of the plate extension part of the problem. In this case, if we subtract away the particular stress fields

$$\sigma^P_{rr} = \sigma^P_{rz} = 0, \tag{II.6a}$$

$$\sigma^P_{zz} = -\tfrac{1}{2}p, \tag{II.6b}$$

then we are left with the residual problem with traction-free conditions on the faces $z = \pm h$, together with the edge-data

$$\bar{\sigma}_{rr}(z) = 0. \tag{II.7a}$$

$$\bar{\sigma}_{rz}(z) = \tfrac{1}{4}pa[-\delta(z-h) + \delta(z+h)]. \tag{II.7b}$$

The interior solution of this residual problem must have the form

$$\sigma^I_{rr} = T, \tag{II.8a}$$

$$\sigma^I_{rz} = \sigma^I_{zz} = 0, \tag{II.8b}$$

where I is a constant be determined. I is determined by requiring that the difference between the prescribed edge-traction (11.7) and those generated by (11.8) should give rise to a decaying state in the plate. The condition for a decaying state in the in-plane extension case is derived in [11] to be

$$\int_{-h}^{h}\left\{\bar{\sigma}^d_{rr} - \frac{\nu}{a} z\bar{\sigma}^d_{rz}\right\} dz = 0. \tag{II.9}$$

If we now substitute $\bar{\sigma}^d_{rr} \equiv \bar{\sigma}_{rr} - \sigma^I_{rr}, \sigma^d_{rz} \equiv \bar{\sigma}_{rz} - \sigma^I_{rz}$ into (II.9), T is found to be

$$T = \tfrac{1}{4}\nu p. \tag{II. 10}$$

It follows from (II.6) (II.8), (II.10) that the interior solution of the plate extension part of our problem is given by

$$\sigma^I_{rr} = \tfrac{1}{4}\nu p, \quad \text{(II.11a)}$$

$$\sigma^I_{rz} = 0, \quad \text{(II.11b)}$$

$$\sigma^I_{zz} = -\tfrac{1}{2}p. \quad \text{(II.11c)}$$

The complete interior stress field in our problem is the sum of (II.5) and (II.11). In particular, then the total value of σ^I_{rr} is given by

$$\sigma^I_{rr} = \frac{p}{32h^3}\left[-3(3+\nu)(a^2-r^2)z + \frac{12}{5}(2-3\nu)h^2z - 4(2+\nu)z^3 + 8\nu h^3\right], \quad \text{(II.12)}$$

which differs from the expression obtained by Timoshenko and Goodier[8], p. 351. As $h \to 0$ both (II.12) and the expression in [8] have as their leading term

$$\sigma^I_{rr} \sim -3(3+\nu)pz(a^2-r^2)/32h^3. \quad \text{(II.13)}$$

which is the value predicted by classical Kirchhoff plate theory. However, we see that, apart from the leading term (II.13), the expression in [8] is not correct; the reason for this is the use of the incorrect decay condition (4.6) instead of the correct condition (4.5). Thus application of Saint-Venant's principle to this problem gives the correct leading term interior solution; the difference from the exact solution is only of higher order in h/a and not exponentially small as one would hope or (erroneously) expect.

We note also that the precise method of support employed has an important influence, even on the interior solution. If the plate were supported by a vertical line load around its *upper* edge (rather than around its lower edge as before), then the bending part of the interior solution would be unaltered, but the symmetric part would become

$$\sigma^I_{rr} = -\tfrac{1}{4}\nu p, \quad \text{(II.14a)}$$

$$\sigma^I_{rz} = 0, \quad \text{(II.14b)}$$

$$\sigma^I_{zz} = -\tfrac{1}{2}p, \quad \text{(II.14c)}$$

and (II.12) would be replaced by

$$\sigma^I_{rr} = \frac{p}{32h^3}\left[-3(3+\nu)(a^2-r^2)z + \frac{12}{5}(2-3\nu)h^2z - 4(2+\nu)z^3 - 8\nu h^3\right] \quad \text{(II.15)}$$

The difference between these two interior solutions is not exponentially small as we might expect, though it is of higher order in h/a.