Ordered Site Access and Optimal Forest Rotation

By Frederic Y. M. Wan

The ordered-site-access model of forest harvesting formulated for once-and-for-all forests in [7] is extended to the case of ongoing forests. The economic content of the corresponding optimal harvest schedule is delineated. For an infinite harvest sequence, the optimal schedule is shown to include the classical Faustmann rotation as a special case, and the effect of net revenue functions changing with harvest is studied. For the practically more important case of planning for a finite sequence of N harvests, the optimal harvest schedule is determined for a Faustmann environment with limited, and unlimited harvesting capacity, and its rapid convergence to the Faustmann rotation is shown for the case of unlimited harvesting capacity. The case of harvest cost functions varying with harvest rate is discussed. The existence of a steady-state optimal harvesting schedule (involving a pathwise uniform age distribution) for the more realistic Heaps-Neher environment and its relation to the Faustmann rotation are analyzed. The evolution of the optimal harvest schedule for a finite harvest sequence in a Heaps-Neher environment toward this steady-state (Faustmann type) rotation is demonstrated.

I. Introduction

In an "ongoing" forest, a logging company contemplates an infinite sequence of harvests with immediate replanting after each harvest. The pioneering studies on the optimal harvest policy for an ongoing forest consider situations where the

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*The research is supported in part by Canadian N.S.E.R.C. Operating Grant A9259 and by U.S.-NSF Grant No. MCS-8306592.

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initial distribution of tree age is uniform, where there is unlimited harvesting capacity so that logging may be instantaneous, and where the net revenue of the timbers depends only on tree age (see [1], [2], [3], and references therein). The optimal harvest schedule for the infinite harvest sequence under these conditions is to repeatedly clear-cut the entire forest instantaneously when it reaches the Faustmann age $A_R(\delta)$ defined by the condition [4]

$$\frac{\dot{V}(A_R)}{V(A_R)} = \frac{\delta}{1-e^{-\delta A_R}}, \tag{1.1}$$

where $V(A)$ is the net commercial value of the trees (which depends only on the tree age A), δ is the constant discount rate and $\dot{V}(A) = dV/dA$.

In reality, a logging company has only a limited harvesting capacity. An upper bound m on the harvest rate $h(t)$ requires harvesting be spread over an interval of time. The logging company must now choose the starting time x_n^l and the harvest rate $0 \le h_n(t) \le m$, for $t \ge x_n^l$ and $n = 1,2,3,\ldots$, to maximize the total discounted net timber revenue from all harvests. [Note that x_n^l and $h_n(t)$ determine the duration and therefore the time interval $x_n^l \le t \le x_n^u$ of the nth harvest.] This constrained harvesting problem was first investigated by Heaps and Neher [5]. Their optimal-control formulation of the problem (which is a natural extension of the formulation for the unlimited-harvest-capacity case [1]) involves equations of states of the form

$$\frac{d\theta_n}{dt} = \frac{h_{n-1}(t-\theta_n) - h_n(t)}{h_{n-1}(t-\theta_n)} \qquad (n = 1,2,\ldots), \tag{1.2}$$

where the state variables $\{\theta_n(t)\}$ are the age of the trees cut at time t during the nth harvest. With the state variable θ_n in the argument of the control variable h_{n-1} in (1.2), the conventional maximum principle [1] is not applicable in general. When the net timber revenue depends only on tree age (and not on the harvest rate), a special argument leads to the "optimal policy" for this substantially simplified case. Unfortunately, the same argument is not applicable to the general problem, while a Pareto-optimum consideration only yields a necessary condition for the optimal policy [5]. A partial differential equation formulation of the same problem is discussed by Davidson and Hellsten in [6]; however, the authors also did not succeed in obtaining a complete solution for the optimal harvest policy.

In this article, we extend the results of [7] for once-and-for-all forest harvesting to get a new formulation for the ongoing-forest harvesting problem which definitely yields the solution for the optimal harvest schedule. As pointed out in [7], our approach is based on the observation that in reality, tree sites are necessarily ordered for logging by practical and regulatory considerations. It may be physically necessary and economically prudent to cut trees in the order of their distance from one or more logging camps, or it may be required by law (as it is in some jurisdictions) to cut trees in the order of their age, the oldest first. We take advantage of the existence of such a logging path to develop a formulation of the forest harvesting problem to enable us to determine the optimal harvest schedule by the conventional maximum principle. For this model, the necessary conditions for the optimal schedule are straightforward consequences of this principle; for an interior maximum, they define a two-point boundary-value problem which may be solved by conventional methods. That these conditions are also sufficient is usually a consequence of the concavity and convexity of the various functionals which occur in the relevant optimal-control problem.

II. The ongoing forest

Within the framework of the continuous model for forest harvesting developed in [7], trees are cut by a single logging crew in a single array along a prescribed path winding through the entire forest. In that case, the position of any tree site can be described by the arc length s along the logging path to the site. For a continuous model, discrete tree stands are smeared out as follows: the entire forest is divided up into F_0 locations of land area, each occupied by a single tree with the stumpage of the tree distributed over its assigned area. Except for cases with sharp discontinuities, the initial tree age distribution over the logging path is replaced by a continuous (piecewise smooth) approximation of the actual distribution. The commercial value of the stumpage at different tree sites along the logging path may be different because of a nonuniform age distribution, different growth conditions, market-price differences at different cutting times, etc. We choose the length scale so that the logging path through the entire forest is normalized to unit length.

Let $T_k(s)$ be the time (measured from now) at which the tree site at location s along the logging path is harvested during the kth harvest, $k=1,2,\dots$. The initial age distribution of the trees in the forest is denoted by $-T_0(s)$, where $T_0(s)\le 0$ is the germination-time distribution of the existing trees prior to the first harvest. The tree at location s will be $A_k(s)\equiv T_k(s)-T_{k-1}(s)$ years old when it is logged during the kth harvest. By construction, $T_k'\equiv dT_k/ds$ is nonnegative along the path, with $T_k'=0$ only if instantaneous harvesting is possible (with unlimited harvest capacity), as T_k' is a measure of the time consumed in logging a particular tree site during the kth harvest, and $1/T_k'$ is therefore a measure of the kth harvest rate h_k at the location s.

Similar to the case of "once-and-for-all forests," we let $p_k\,ds$ and $c_k\,ds$ be the commercial price and harvesting cost of the timber from the kth harvest over the incremental path strip $(s, s+ds)$. For reasons already explained in [7], both p_k and c_k may vary with location s, logging time T_k, and tree age A_k, as well as current and previous harvest rates $1/T_j'$, $j=1,2,\dots,k$. The present value of the discounted future net revenue for tree stumpage along a path increment $(s, s+ds)$ from the kth harvest is $e^{-\delta_k T_k(s)}[p_k-c_k]\,ds$, where δ_k is the constant discount rate for the kth harvest. The present value of the discounted future net revenue from N harvests of the forest is

$$P_N \equiv \sum_{k=1}^{N}\left[\int_0^1 (p_k-c_k)e^{-\delta_k T_k}\,ds\right] \tag{2.1}$$

where, for simplicity, we have assumed that the forest land has no residual value after the Nth harvest. Note that we have $N=\infty$ if the forest is to be harvested indefinitely for the whole future. For this ongoing forest problem to be meaningful, P_N must remain bounded as $N\to\infty$. The management problem for the logging company is to choose a sequence of harvest schedules $\{T_1, T_2, \ldots\}$ for the forest so that P_N is a maximum. With $dT_k/ds = 1/h_k$, $\delta_k=\delta$, $p_k\equiv p$, and $C_k\equiv ch_k$, the expression (2.1) reduces to the conventional expression for the present value of discounted future net revenue [1–5] when the discount rate, unit price, and harvest cost are identical for all harvests.

To apply the conventional maximum principle to the above optimal-control problem, we introduce a new set of controls by the defining equations (of state)

$$T_k' \equiv u_k \qquad (k=1,2,\ldots), \tag{2.2}$$

and write P_N as

$$P_N = \sum_{k=1}^{N} \int_0^1 e^{-\delta_k T_k} V_k(s, T_k, A_k, u_1, \ldots, u_k)\, ds, \tag{2.3}$$

where $V_k \equiv p_k - c_k$ is the net revenue per unit path length. The maximum principle [1] now requires that the sequence of (piecewise continuous, abbreviated PWC) u_k be chosen to maximize the Hamiltonian

$$\mathscr{H} \equiv \sum_{k=1}^{N} \left[e^{-\delta_k T_k} V_k + \lambda_k u_k \right] \tag{2.4}$$

subject to the equations of state (2.2), t_0 the equations for the adjoint variables $\lambda_1, \lambda_2, \ldots$ (corresponding to the discounted shadow prices for the different harvest),

$$\lambda_k' = -\frac{\partial \mathscr{H}}{\partial T_k}$$

$$= -\left[\frac{\partial V_k}{\partial T_k} + \frac{\partial V_k}{\partial A_k} - \delta_k V_k\right] e^{-\delta_k T_k} + \frac{\partial V_{k+1}}{\partial A_{k+1}} e^{-\delta_{k+1} T_{k+1}} \qquad (k=1,2,\ldots,N) \tag{2.5}$$

(with $V_{N+1}\equiv 0$), to the transversality conditions

$$\lambda_k(0) = \lambda_k(1) = 0 \qquad (k=1,2,\ldots), \tag{2.6}$$

and to the inequality constraints

$$u_k \geq \tau_k\ (\geq 0), \qquad T_1(0) \geq 0, \tag{2.7a}$$

and possibly

$$T_k(0) > T_{k-1}(1) \qquad (k=1,2,\ldots,N). \tag{2.7b}$$

By allowing τ_k to be positive, we include the possibility of a maximum feasible harvest rate for each harvest, reflecting a limited harvest capacity. If harvest capacity is unlimited, then we have $\tau_k = 0$ and (2.7) simply reflects the fact that the tree sites are ordered for the purpose of logging.

When the inequality constraints (2.7) are not binding, we have an interior solution for the optimal-control problem given by

$$\frac{\partial \mathscr{H}}{\partial u_k} = \lambda_k + \sum_{j=1}^{N} e^{-\delta_j T_j} \frac{\partial V_j}{\partial u_k} = 0 \qquad (k=1,2,\ldots,N). \tag{2.8}$$

The conditions (2.8) and (2.2) may be used to eliminate λ_k and u_k from (2.5) and (2.6) to get a system of N second-order ODE for T_k, $k=1,2,\ldots,N$, and one set of N boundary conditions at each end of the logging path. This two-point boundary-value problem may then be solved to get the optimal harvest schedule[1] for each harvest.

At the other extreme, when all of the inequality constraints on u_k in (2.7) are binding, we have then a corner solution with

$$u_k \equiv T_k' = \tau_k \quad \text{or} \quad T_k(s) = \tau_k s + t_k, \tag{2.9}$$

where the constants of integration t_k, $k=1,2,\ldots,$ are to be determined by (2.5) and (2.6).

Intermediate situations with some of the inequality constraints (on u_k) in (2.7) binding are also possible and can be treated in a straightforward manner. Regardless of whether one or more of (2.7) are binding, we get from (2.5) and the transversality conditions $\lambda_k(0) = 0$ [see (2.6)]

$$\lambda_k(s) = -\int_0^s \left[\left(\frac{\partial V_k}{\partial A_k} + \frac{\partial V_k}{\partial T_k} - \delta_k V_k \right) e^{-\delta_k T_k} - \frac{\partial V_{k+1}}{\partial A_{k+1}} e^{-\delta_{k+1} T_{k+1}} \right] ds$$

$$(k=1,2,\ldots). \tag{2.10}$$

[1]In order to focus our attention on the main issues of interest here, we assume throughout this paper the various convexity and concavity conditions are satisfied so that the necessary conditions for optimality are also sufficient.

The remaining transversality conditions, $\lambda_k(1)=0$ of (2.6), give

$$\int_0^1 \left[\left(\frac{\partial V_k}{\partial A_k} + \frac{\partial V_k}{\partial T_k} \right) e^{-\delta_k T_k} - \frac{\partial V_{k+1}}{\partial A_{k+1}} e^{-\delta_{k+1} T_{k+1}} \right] ds = \delta_k \int_0^1 V_k e^{-\delta_k T_k} \, ds. \quad (2.11)$$

Thus, under the optimal policy, the *discounted net gain* through time marginal yield of the entire forest (for not harvesting) *equals the opportunity cost* consisting now of the sum of the time marginal yield of the replanted forest and the interest earned on discounted net revenue of the harvested forest.

Observe that in our formulation of the optimal-forest-rotation problem, tree ages and harvest schedules are related in a natural way by $A_k(s) = T_k(s) - T_{k-1}(s)$. These simple relations replace the integral conditions

$$\int_{t_{s(n-1)}}^{t - A_n(t)} h_{n-1}(z) \, dz = \int_{t_{sn}}^{t} h_n(z) \, dz \qquad (n = 2, 3, \ldots) \qquad (2.12)$$

of [5], which lead to the functional-differential equations of state (1.2) for the optimal-control problem. Because of our unconventional choice of space instead of time as the independent variable, no functional-differential equations appear in our formulation, and the maximum principle is directly applicable to our new optimal-control problem.

III. Age dependent net revenue and the Faustmann rotation

3.1. *A finite sequence of harvests*

Suppose the net revenue per unit tree site for the kth harvest, $V_k \equiv p_k - c_k$, is a monotone increasing concave function of tree age only, and the constant discount rate is the same for all N harvests, so that $\delta_k = \delta$, $k = 1, 2, \ldots, N$. Then the system (2.8) reduces to

$$\lambda_k \equiv 0 \qquad (k = 1, 2, \ldots, N), \qquad (3.1)$$

and the transversality conditions (2.6) are trivially satisfied. The differential system (2.5) for λ_k now becomes an algebraic system of N simultaneous nonlinear equations for $A_k(s)$ [and therefore the harvest schedules $T_k(s) = T_{k-1}(s) + A_k(s)$], $k = 1, 2, 3, \ldots, N$:

$$\dot{V}_k(A_k) - \delta V_k(A_k) = \dot{V}_{k+1}(A_{k+1}) e^{-\delta A_{k+1}} \qquad (k = 1, 2, 3, \ldots, N), \quad (3.2)$$

with $V_{N+1}(A) \equiv 0$, where $\dot{V}_k$ indicates differentiation with respect to the argument of V_k, namely the tree age.

The system (3.2) may be solved by noting that the Nth equation,

$$\dot{V}_N(A_N) - \delta V_N(A_N) = 0, \tag{3.3}$$

involves only one unknown A_N and its unique solution is the well-known Fisher age $\alpha_N(\delta) \equiv \bar{A}(\delta)$ (denoted by $A_{IF}(\delta)$ in [7]):

$$A_N(s) \equiv T_N(s) - T_{N-1}(s) = \alpha_N(\delta), \qquad \frac{\dot{V}_N(\alpha_N(\delta))}{V_N(\alpha_N(\delta))} = \delta. \tag{3.4}$$

Having determined the optimal harvest age for the last harvest, $A_N(s)$, the $(N-1)$th equation

$$\dot{V}_{N-1}(A_{N-1}) - \delta V_{N-1}(A_{N-1}) = \dot{V}_N(\alpha_N) e^{-\delta \alpha_N} \tag{3.5}$$

involves only one unknown, and may be solved to get the unique solution $\alpha^*_{N-1}(\delta)$ for $A_{N-1}(s)$. The process is repeated to get $A_{N-2} = \alpha^*_{N-2}$, $A_{N-3} = \alpha^*_{N-3}, \dots,$ with the first equation giving $A_1(s) = \alpha^*_1(\delta)$. These results for the tree age distribution during the different harvests are then used to determine the optimal harvest schedules $\{T_k(s)\}$:

$$T_1(s) = T_0(s) + \alpha^*_1(\delta), \qquad T_2(s) = T_1(s) + \alpha^*_2(\delta) = T_0(s) + \alpha^*_1(\delta) + \alpha^*_2(\delta),$$

$$T_k(s) = T_{k-1}(s) + \alpha^*_k(\delta) = T_0(s) + \sum_{j=1}^{k} \alpha^*_j(\delta) \qquad (k = 3, \dots, N-1),$$

$$T_N(s) = T_{N-1}(s) + \alpha_N(\delta) = T_0 + \sum_{j=1}^{N-1} \alpha^*_j(\delta) + \alpha_N(\delta). \tag{3.6}$$

For the case of a uniform initial age distribution and unlimited harvesting capacity, the optimal harvest policy is to clear-cut the entire forest instantaneously when the trees reach the age $\alpha^*_k(\delta)$ during the kth harvest ($k = 1, 2, \dots, N - 1$) and when they reach the Fisher age $\alpha_N(\delta) \equiv \bar{A}(\delta)$ for the last harvest [see Figure 1(a)]. [For simplicity, we will discuss only the case $T_0(s) + \alpha^*_1(\delta) \geq 0$.] This policy is optimal in view of the concavity of V_k. For a nonuniform $T_0(s)$ with $T_0'(s) \geq 0$, the harvest policy is to cut the trees when they reach the age α^*_k for the kth harvest and the Fisher age α_N for the last harvest [see Figure 1(b) with $\tau_k = 0$, $k = 1, 2, \dots$].

If $T_0'(s) \leq 0$, the harvest schedule $T_0(s) + \alpha^*_1(\delta)$ violates the inequality constraint $u_1 = T_1' \geq 0$ and we must take $T_1(s) = t_1$ instead, with t_1 determined by

$$\lambda_1' = -\left[\dot{V}_1(t_1 - T_0(s)) - \delta V_1(t_1 - T_0(s)) - \dot{V}_2(\alpha^*_2) e^{-\delta \alpha^*_2}\right] e^{-\delta t_1} \tag{3.7}$$

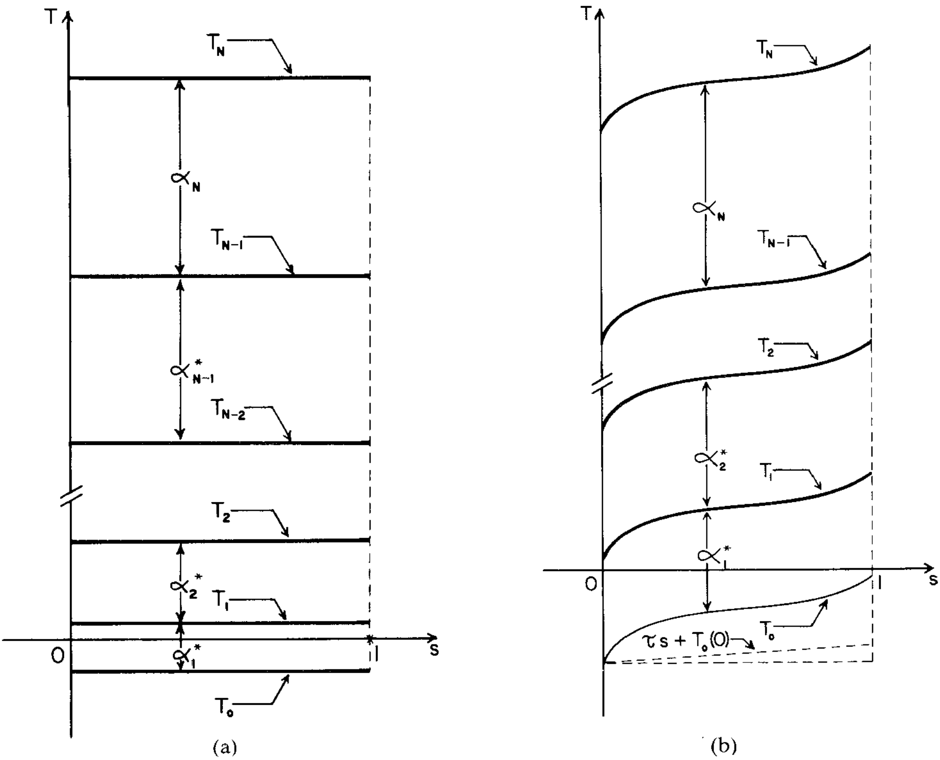

Figure 1. Optimal schedule for a finite harvest sequence with (a) a uniform initial age distribution a Faustmann environment; (b) $T_0'(s) > (\tau \geq)0$ in a Faustmann environment (and with bounded harvesting capacity of $\tau > 0$); (c) $T_0'(s) \leq 0$ in a Faustmann environment; (d) $0 \leq T_0'(s) < \tau$ in a Faustmann environment but with bounded harvesting capacity.

and $\lambda_1(0) = \lambda_1(1) = 0$. From (3.7) and $\lambda_1(0) = 0$, we get

$$\lambda_1(s) = -\int_0^s \left\{ \dot{V}_1(t_1 - T_0(s)) - \delta V_1(t_1 - T_0(s)) - \dot{V}_2(\alpha_2^*) e^{-\delta\alpha_2^*} \right\} e^{-\delta t_1}\, ds. \tag{3.8}$$

The condition $\lambda_1(1) = 0$ then gives

$$\frac{\int_0^1 \left\{ \dot{V}_1(t_1 - T_0(s)) - \dot{V}_2(\alpha_2^*) e^{-\delta\alpha_2^*} \right\} e^{-\delta t_1}\, ds}{\int_0^1 V_1(t_1 - T_0(s)) e^{-\delta t_1}\, ds} = \delta, \tag{3.9}$$

which determines t_1. Therefore, the optimal first harvest *clear-cuts the entire forest* instantaneously at the moment *when the discounted marginal yield net the dis-*

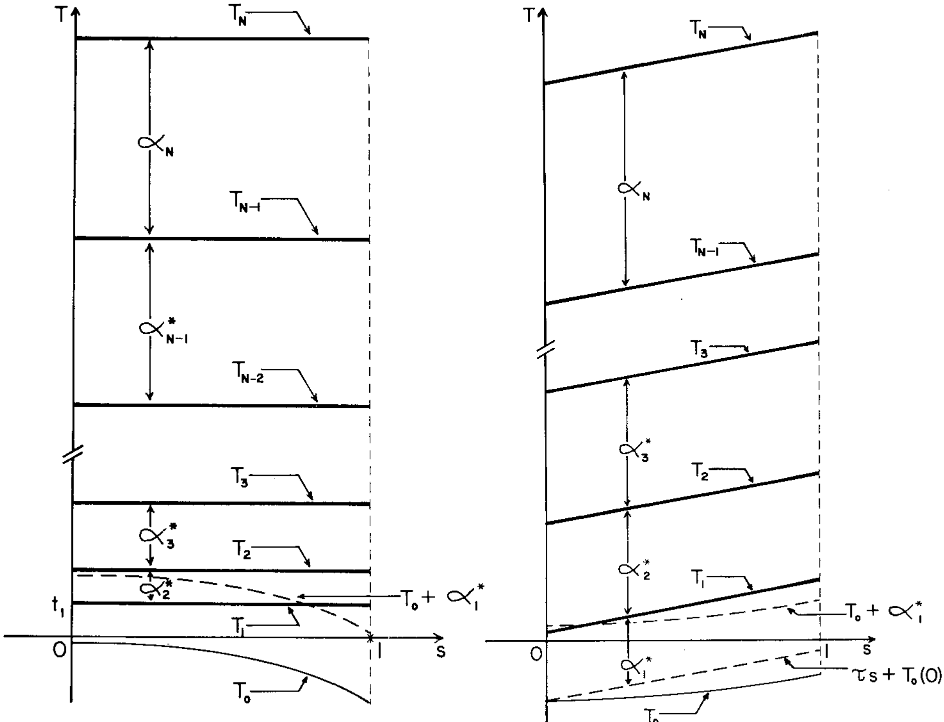

Figure 1. (*Continued*)

counted marginal yield of the second harvest, as a fraction of present value of the first harvest, equals the discount rate. The second harvest has a uniform initial age distribution, and the results of (3.6) for $k \geq 2$ now apply from there on [see Figure 1(c)], so that [after $T_1(s) = t_1$]

$$T_k(s) = t_1 + \sum_{j=2}^{k} \alpha_j^*(\delta) \qquad (k = 2, \ldots, N-1),$$

$$T_N(s) = t_1 + \sum_{j=2}^{N-1} \alpha_j^*(\delta) + \alpha_N(\delta). \tag{3.10}$$

If $T_0'(s)$ changes sign in the interval $(0,1)$, the schedule $T_0(s) + \alpha_1^*(\delta)$ again violates the inequality constraint $T_1' \geq 0$ in some segments of the interval. The optimal policy $T_1(s)$ for the first harvest is then determined by a procedure outlined in [7]. This optimal first harvest schedule serves as the germination time for the second harvest. The results of (3.6) for $k \geq 2$ now apply as $T_1' \geq 0$ in $(0,1)$.

The determination of the optimal harvest schedule for the limited-harvest-capacity case does not differ substantially from the case of unlimited harvesting capacity discussed above. We assume henceforth $\tau_k = \tau$, $k = 1, 2, \ldots$, for simplicity. The results of (3.6) again apply if $T_0'(s) \geq \tau > 0$ [see Figure 1(b) with $\tau > 0$]. Otherwise, determine $T_1(s)$ according to procedures described in [7], and apply the results of (3.6) to T_k for $k \geq 2$ [see Figure 1(d)]. As observed in [5], an ongoing forest with a uniform age distribution and unlimited harvest capacity should be clear-cut instantaneously at regular intervals, while a limited harvest capacity should give for each harvest a sustained yield over an interval of time. Should this period last beyond the start of the next in the schedules given by (3.6), say $T_k(1) = T_0(1) + \Sigma_{j=1}^{k} \alpha_j^*(\delta) > T_{k+1}(0)$, an adjustment for the two contiguous periods (and possibly beyond them) can be worked out to meet the stipulated constraints.

3.2. *An infinite harvest sequence with identical net revenue functions*

To harvest repeatedly through the entire future, we take $N = \infty$ and consider first the conventional problem with $V_k(A) \equiv V(A)$, $k = 1, 2, 3, \ldots$. The system (3.2) simplifies to

$$\dot{V}(A_k) - \delta V(A_k) = \dot{V}(A_{k+1}) e^{-\delta A_{k+1}} \qquad (k = 1, 2, \ldots). \tag{3.11}$$

A solution of this system of infinite number of coupled equations is known to be the Faustmann rotation period $A_R(\delta)$ [1–5] (denoted by $A_{MF}(\delta)$ in [7]), i.e.,

$$A_k = A_R(\delta) \qquad (k = 1, 2, 3, \ldots), \tag{3.12}$$

where $A_R(\delta)$ is a root of (1.1). This solution is unique if V is a monotone increasing concave function of its argument. Thus, the optimal policy for the case of no constraint on the harvest rate is to cut a tree when it reaches its Faustmann age A_R (or immediately if it is already older).

It is important to note that, on the basis of the results for the finite-N case, the infinitely many coupled equations (3.11) are to be solved simultaneously as a system and not as recursive relations starting with some A_1. In fact, unless A_1 is taken to be A_R, the quantities A_2, A_3, etc. obtained recursively from (3.11) either decrease monotonically without bound if $A_1 < A_R$ or increase monotonically without bound if $A_1 > A_R$ [5]. Neither situation could be optimal. For the decreasing sequence of $\{A_k\}$, the trees in all later harvests are logged immediately after replanting. For the increasing sequence, the time margin from not harvesting for all later harvests is smaller than the marginal interest from harvesting. We can do better in both cases. An immediate consequence of the above observation is the *instability* of the optimal schedule. If the trees cut during the kth harvest are not at their Faustmann age, the subsequent harvest schedules obtained from the schedule (3.11) will not be, and will not tend to, the optimal Faustmann rotation. (They actually lead to a negative-net-profit policy.)

Starting with an arbitrary schedule for the first harvest, the nonconvergence of the harvest schedule sequence generated by (3.11) toward the optimal policy has already been observed in [5]. Our contribution in this direction lies in the recognition of the fact that the determination of the optimal policy and the stability of this policy are two distinctly different issues. From the results for the finite-harvest-sequence case, we now see that the optimal harvest policy for the finite harvest sequence is a solution of the infinite number of simultaneous equations (3.11) and not determined recursively by (3.11) starting from some first harvest schedule. In fact, for a finite N, the quantities $A_1, A_2, \ldots$ are determined recursively backward starting from the last harvest schedule T_N (and not recursively forward from T_1). We will presently show that the solution for finite N tends to Faustmann rotation as $N \to \infty$.

For a finite harvest sequence, we have $V_{N+1}(A) \equiv 0$; the last (Nth) harvest should therefore take place when the trees reach their Fisher age, so that $A_N = \alpha_N(\delta) \equiv \bar{A}(\delta)$ with

$$\dot{V}(\bar{A}(\delta)) = \delta V(\bar{A}(\delta)). \tag{3.13}$$

In other words, $\bar{A}(\delta)$ is located by the intersection of the graph of the monotone increasing concave function $\delta V(A)$ and the graph of the *positive*, monotone decreasing convex function $\dot{V}(A)$ (see Figure 2). The optimal $(N-1)$th harvest schedule requires A_{N-1} to satisfy [see (3.11)]

$$\dot{V}(A_{N-1}) - \dot{V}(A_N)e^{-\delta A_N} = \delta V(A_{N-1}). \tag{3.14}$$

That is, A_{N-1} is located by the intersection of the graph of $\dot{V}(A)$, translated downward by a known distance $\dot{V}(A_N)e^{-\delta A_N}$, and the graph of $\delta V(A)$ (see Figure 2). Similarly A_k is located by the intersection of a vertically translated $\dot{V}(A)$ and

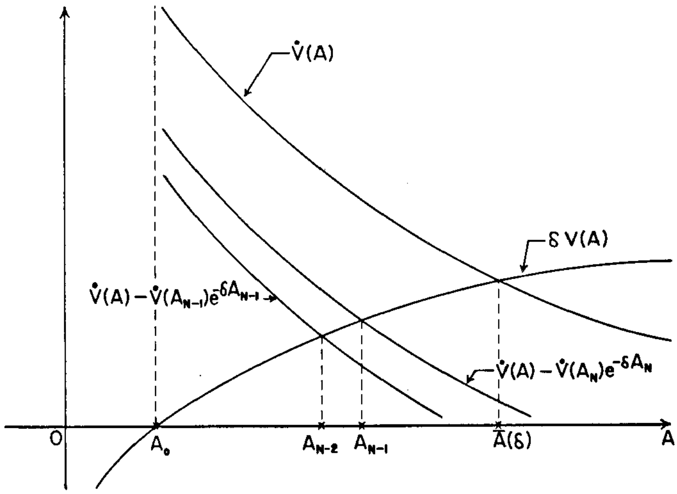

Figure 2. The convergence of a finite optimal sequence of N harvests (in a Faustmann environment) to the Faustmann rotation as $N \to \infty$.

$\delta V(A)$, with the amount of downward translation of $\dot{V}(A)$ depending on a previously determined A_{k+1}, namely, $\dot{V}(A_{k+1})e^{-\delta A_{k+1}}$. Keep in mind that $\dot{V}(A)e^{-\delta A}$ is a decreasing function of A for $A < \bar{A}(\delta)$ and that $V(A) < 0$ for $A < A_0$. Clearly, we have $A_{N-1} < A_N \equiv \bar{A}(\delta)$ and hence $\dot{V}(A_{N-1})e^{-\delta A_{N-1}} > \dot{V}(A_N)e^{-\delta A_N}$. It follows that $A_{N-2} < A_{N-1}$. Repeat the argument to get $A_k < A_{k+1}$ for $k = N-3,\ N-4,\dots,2,1$ with $A_k > A_0$, since $\dot{V}(A_{k+1})e^{-\delta A_{k+1}} < \dot{V}(A_{k+1})$ (see Figure 2). Thus, $A_{N-1}, A_{N-2},\dots, A_2, A_1$ form a monotone decreasing sequence inside the interval $(A_0, \bar{A}(\delta))$, where A_0 is the unique zero crossing of $V(A)$. Thus, starting with $A_N = \bar{A}(\delta)$, the relation (3.11) generates a monotone decreasing sequence $\{A_N, A_{N-1}, A_{N-2},\dots, A_{k+1}, A_k, A_{k-1},\dots, A_2, A_1\}$, which is bounded below by A_0 [and bounded above by $\bar{A}(\delta)$]. Therefore, we have the following theorem and its important corollary:

THEOREM 1. *As $N \to \infty$, the harvest age sequence $\{A_N,\dots, A_1\}$ generated by* (3.11) *with $A_N = \bar{A}(\delta)$ tends to a limit $\hat{A}$, i.e., $A_1 \to \hat{A}$ as $N \to \infty$.*

Proof: Let $a_k \equiv A_{N-k+1}$, $k = 1,2,\dots$. Then $\{a_k\}$ is a monotone decreasing sequence bounded from below by A_0 and therefore has a limit [8], denoted by $\hat{A}$. Given any $\varepsilon > 0$ however small, there is always an $M(<N)$, sufficiently large and depending on ε, for which $|a_k - \hat{A}| < \varepsilon$ for $k > M$. It follows that for all $N > M$, $|A_j - \hat{A}| < \varepsilon$ for all $j < N - M + 1$.

COROLLARY. *If $\delta V(A) = \dot{V}(A)$ has a unique solution, the limit $\hat{A}$ is unique and identical to the Faustmann rotation $A_R(\delta)$.*

Proof: Very briefly, we have, for a sufficiently large N, $A_{k+1} \approx A_k \approx \hat{A}$ for $k < (N - M + 1)$, so that (3.11) may be written as

$$\dot{V}(\hat{A})[1 - e^{-\delta \hat{A}}] = \delta V(\hat{A}) + \text{error}, \tag{3.15}$$

where the error term can be made as small as we wish. Hence we have, in the limit as $N \to \infty$, $\hat{A} = A_R(\delta)$ with the Faustmann age A_R given by the solution of (1.1).

With the above corollary, we have reproduced the classical solution for the forest rotation for an ongoing forest (with the specified class of net revenue functions). This policy applies to uniform or nonuniform initial distribution of tree ages as long as the constraint on the harvest rate and the requirement of ordered access to tree sites are not violated. Otherwise, a solution process similar to that described at the end of Section 3.1 applies.

3.3. An infinite harvest sequence with changing net revenue functions

With decades separating two consecutive harvests, it may be unrealistic for the net revenue functions $V_k \equiv p_k - c_k$ to remain the same for all harvests. With technical progress, cheaper and/or more attractive substitutes will continue to reduce the demand for lumber and thereby its commercial value. On the other hand, new usage of timber may create a significant increase in demand in the future. In short, there is really no way for any forest manager to know for sure what V_k (in constant dollars) will be a century from now, not to mention the

Table 1
Optimal Tree Ages α_k^* for N Harvests

k	$\delta = 0.05$	0.06	0.07	0.08	0.09	0.1
$\alpha_N \equiv A(\delta)$	44.6664	42.1129	40.2200	38.7600	37.5994	36.6545
α_{N-1}^*	43.0800	41.0794	39.5330	38.2962	37.2823	36.4356
α_{N-2}^*	42.9056	40.9950	39.4911	38.2750	37.2715	36.4299
α_{N-3}^*	42.8853	40.9878	39.4884	38.2740	37.2711	36.4298
α_{N-4}^*	42.8829	40.9872	39.4883	38.2739		
α_{N-5}^*	42.8827	40.9871				
α_{N-6}^*	42.8826					
$\vdots$						
A_R	42.8826	40.9871	39.4883	38.2739	37.2711	36.4298

entire future. Under the circumstances, a reasonable approach by the logging company would be to plan only for the period with an accurate estimate of the net revenue for each of the N harvests and to assign a residual value to the forest land beyond the Nth harvest. In that case, the result of Section 3.1 (suitably modified if there is a nonzero residual value) applies.

In view of the instability of the optimal policy for time invariant net revenue functions (see Section 3.2), it would be prudent not to plan beyond the period with a reasonably definite net revenue structure. This often means planning only for a few harvests with the same net revenue function. Fortunately, the convergence of $\{\alpha_k^*(\delta)\}$ to the Faustmann rotation is extremely rapid in all cases tested. For example, we have for

$$V(A) = 950 - 1500e^{-A/60} \tag{3.16}$$

the optimal harvest schedules given in Table 1 for several values of the discount rate δ in the interval (0.05,0.1).

Should a time-varying net revenue structure be specified (known or imposed) for each of an infinite sequence of harvests, the optimal harvest policy is determined by the solution of the simultaneous equations (3.2) with $N = \infty$. For example, with

$$V_k(A) = (1+\gamma)^k V(A), \qquad k = 1,2,3,\ldots, \tag{3.17}$$

where γ is a known constant, an exact solution of the infinite system (3.2) is a modified Faustmann rotation $A_k = A_\gamma(\delta)$, $k = 1,2,3,\ldots,$ where $A_\gamma(\delta)$ is the solution of

$$\frac{\dot{V}(A_\gamma)}{V(A_\gamma)} = \frac{\delta}{1-(1+\gamma)e^{-\delta A_\gamma}}. \tag{3.18}$$

Note that we have $A_\gamma(\delta) \le A_R(\delta)$, since future harvests now yield higher net return.

IV. Harvest cost dependent on harvest rate

4.1. A finite sequence of harvests

In this section we consider the unit (site) harvest cost functions (instead of being a constant or only tree age dependent) to depend only on the harvest rate for that harvest with the conventional U-shaped graph. Within the framework of our formulation, we have for the kth harvest, $c_k = c_k(T_k') > 0$ convex in T_k' with a minimum at $T_k' = \mu_k > 0$, so that $\min[c_k(T_k')] = c_k(\mu_k)$. This class of cost functions includes both the effect of a fixed cost component (important at low harvest rate) and an overload cost component (important at high harvest rate).

With p_k depending only on tree age as before and with $T_k' = u_k$, we have from (2.8)

$$\lambda_k = e^{-\delta_k T_k} \dot{c}_k(u_k) \qquad (k=1,2,\ldots,N) \tag{4.1}$$

for an interior maximum. When the constraints $T_{k-1}(1) < T_k(0)$ and $T_k(1) < T_{k+1}(0)$ are not binding, the transversality conditions $\lambda_k(0) = \lambda_k(1) = 0$ [see (2.6)] are satisfied by the interior solution with

$$T_k'(0) = T_k'(1) = \mu_k > 0 \qquad (k=1,2,\ldots,N). \tag{4.2}$$

No other possibility exists, as each c_k has a unique stationary point and $e^{-\delta_k T_k}$ never vanishes. With (4.2), we have reproduced and *extended* property (A) obtained in [5]: *Each harvest should start and end with the harvest rate* $1/\mu_k$ *which gives a minimum unit site harvest cost.* Thus, property (A), which has as its economic content that the discounted shadow prices of the first and last tree cut are both zero, applies to the optimal policy for both finite and infinite sequences of harvests.

From (4.1) and the fact that $\dot{c}_k \to -\infty$ as u_k tends to zero from above, we see that T_k' remains positive along the entire logging path for the interior solution, independent of the initial age distribution of the forest. Provided that $T_k'(s)$ is not less than its lower bound $\tau \geq 0$ (with $1/\tau$ the upper bound on harvest rate), the optimal harvest policy is determined by the system (4.1), the equations of states $T_k' = u_k$, and the equations for the adjoint variables (or discounted shadow prices) (2.5), which simplify to read

$$\begin{aligned} \lambda_k' &= -\frac{\partial \mathscr{H}}{\partial T_k} \\ &= -\left[\dot{p}_k(A_k) - \delta_k\{p_k(A_k) - c_k(T_k')\}\right] e^{-\delta_k T_k} + \dot{p}_{k+1}(A_{k+1}) e^{-\delta_{k+1} T_{k+1}} \end{aligned}$$
$$(k=1,2,\ldots,N), \tag{4.3}$$

where $p_{N+1}(A) \equiv 0$ and $A_k = T_k(s) - T_{k-1}(s)$, along with the boundary conditions (4.2). The $(2N)$th-order system (4.1), (2.2), (4.3), and (4.2) is a simultaneous system for the $2N$ unknowns $T_k(s)$ and $\lambda_k(s)$, $k=1,2,\ldots,N$.

In planning for a finite number of harvests in Section II, we have in effect adopted the extreme position that timbers (and the forest land) have no commercial or residual value beyond the last harvest. In that case, we have $p_{N+1}(A) \equiv 0$, so that (4.3) with $k = N$ simplifies to read

$$\lambda'_N = -\left[\dot{p}_N(A_N) - \delta_N p_N(A_N) + \delta_N c_N(T'_N)\right] e^{-\delta_N T_N}. \tag{4.4}$$

Upon using (4.1) and (2.2), the above equation may be written as a second-order ODE for T_N,

$$\ddot{c}_N(T'_n) T''_N = -\dot{p}_N(A_N) + \delta_N\left[p_N(A_N) + T'_N \dot{c}_N(T'_N) - c_N(T'_N)\right], \tag{4.5}$$

while (4.2) gives

$$T'_N(0) = T'_N(1) = \mu_N. \tag{4.6}$$

Unlike the case with $\partial c_k / \partial T'_k \equiv 0$ treated in Section III, the differential equation (4.5) and boundary conditions (4.6) governing the optimal schedule for the last harvest are now coupled to those of the earliest harvests. The coupling is through the tree age $A_N \equiv T_N(s) - T_{N-1}(s)$, which involves the schedule for the $(N-1)$th harvest. In the same way, we see also that all earlier harvest schedules but the first are coupled in both the backward and forward directions by $A_k \equiv T_k - T_{k-1}$ and $A_{k+1} \equiv T_{k+1} - T_k$, respectively. For the first harvest, the quantity $T_0(s)$ in $A_1 \equiv T_1 - T_0$ is known, so that the only coupling is through $A_2 \equiv T_2 - T_1$. The N second-order ODE for the N unknowns $\{T_1, \ldots, T_N\}$ (or, alternatively, the $2N$ first-order ODE for T_k and λ_k, $k = 1, 2, \ldots, N$), supplemented by the $2N$ boundary conditions (4.2) [or (2.6)], must be solved simultaneously. For large N, this is at best an expensive computational problem. However, it is difficult (if not impossible) to anticipate developments in the distant future. Given the instability of the optimal policy relative to a small perturbation for the simpler situations analyzed in Section III, it would be inappropriate and unrealistic to plan for a period of more than a few harvests. For the planning of a sequence of three or four harvests, the computational problem associated with the determination of the optimal harvest policy is definitely manageable once we know $p_k(\cdot)$, $c_k(\cdot)$, and δ_k.

4.2. Identical discount rates and pathwise uniform harvest age distributions

We now introduce the undiscounted shadow prices

$$\tilde{\lambda}_k(s) = e^{\delta_k T_k(s)} \lambda_k(s) \qquad (k = 1, 2, \ldots) \tag{4.7}$$

and write (4.1) as $\dot{c}_k(u_k) = \tilde{\lambda}_k$ or, in view of the monotonicity of $\dot{c}_k(\cdot)$,

$$u_k = \dot{c}_k^{-1}(\tilde{\lambda}_k) \qquad (k = 1, 2, \ldots). \tag{4.8}$$

The differential equations for the adjoint variables $\{\lambda_k\}$ may then be written as

$$\tilde{\lambda}'_k = \delta_k \dot{c}_k^{-1}(\tilde{\lambda}_k)\tilde{\lambda}_k - \left[\dot{p}_k(A_k) - \delta_k p_k(A_k) + \delta_k c_k\left(\dot{c}_k^{-1}(\tilde{\lambda}_k)\right)\right] + \dot{p}_{k+1}(A_{k+1})e^{-(\delta_{k+1}T_{k+1}-\delta_k T_k)} \tag{4.9}$$

for $k=1,2,\ldots,N$, with $p_{N+1}(\cdot)\equiv 0$. If the discount rates are identical for all harvests, so that $\delta_k = \delta$, $k=1,2,\ldots,N$, the right-hand side of (4.9) may be expressed in terms of the undiscounted shadow price and the tree age distribution for two consecutive harvests, $A_k(s)$ and $A_{k+1}(s)$:

$$\tilde{\lambda}'_k = \delta\tilde{\lambda}_k \dot{c}_k^{-1}(\tilde{\lambda}_k) - \left[\dot{p}_k(A_k) - \delta p_k(A_k) + \delta c_k\left(\dot{c}_k^{-1}(\tilde{\lambda}_k)\right)\right] + \dot{p}_{k+1}(A_{k+1})e^{-\delta A_{k+1}} \qquad (k=1,2,\ldots,N). \tag{4.9$'$}$$

The equations of state (2.2) may also be written in terms of $\tilde{\lambda}_k$ and A_k alone by forming their consecutive differences to get

$$A'_1 = u_1 - u_0 = \dot{c}_1^{-1}(\tilde{\lambda}_1) - T'_0,$$

$$A'_k = u_k - u_{k-1} = \dot{c}_k^{-1}(\tilde{\lambda}_k) - \dot{c}_{k-1}^{-1}(\tilde{\lambda}_{k-1}) \qquad (k=2,\ldots,N). \tag{4.10}$$

With $p_{N+1}(\cdot)\equiv 0$, the system of $2N$ equations (4.9′) and (4.10), together with the $2N$ transversality conditions (2.6) written in terms of $\tilde{\lambda}_k$ as

$$\tilde{\lambda}_k(0) = \tilde{\lambda}_k(1) = 0 \qquad (k=1,2,\ldots,N), \tag{4.11}$$

determine the $2N$ unknowns $\{A_1, A_2,\ldots, A_N\}$ and $\{\tilde{\lambda}_1,\ldots,\tilde{\lambda}_N\}$. This system admits a *pathwise uniform solution* for the tree age distributions of the different harvests (so that $A'_k \equiv 0$) provided that

$$T'_0(0) = T'_0(1) = \mu_k = \mu \qquad (k=1,\ldots,N). \tag{4.12}$$

For such a solution, we have from (4.10) $\dot{c}_1^{-1}(\tilde{\lambda}_1) = \mathrm{T}'_0$ or $\tilde{\lambda}_1 = \dot{c}_1(T'_0)$, so that $\tilde{\lambda}_1(0) = \tilde{\lambda}_1(1) = 0$ require $T'_0(0) = T'_0(1) = \mu_1 \equiv \mu$. Similarly, $A'_k = 0$ $(k \geq 2)$ requires $u_k = \dot{c}_k^{-1}(\tilde{\lambda}_k) = \dot{c}_{k-1}^{-1}(\tilde{\lambda}_{k-1}) = u_{k-1}$ $(=T'_0)$, and the transversality conditions $\tilde{\lambda}_k(0) = \tilde{\lambda}_k(1) = 0$ require $\mu_k = \mu_{k-1} = \mu$ and $u_k(0) = u_k(1) = \mu$. With $\tilde{\lambda}_k(s) = \dot{c}_k(u_k) = \dot{c}_k(T'_0)$, the system of N first-order ODE (4.9′) with $p_{N+1}(\cdot)\equiv 0$ and with $A_1,\ldots,A_N$ as unknown parameters can be satisfied [for general $c_k(\cdot)$ and $p_k(\cdot)$] only by

$$\tilde{\lambda}_k(s) \equiv 0, \quad T'_0(s) = \mu, \quad u_k(s) = \mu \qquad (k=1,2,\ldots,N), \tag{4.13}$$

with the pathwise uniform age distributions $\{A_k\}$ determined by

$$\frac{\dot{p}_N(A_N)}{p_N(A_N)-c_N(\mu)} = \frac{\dot{V}_N(A_N)}{V_N(A_N)} = \delta,$$

$$\frac{\dot{p}_k(A_k)-\dot{p}_{k+1}(A_{k+1})e^{-\delta A_{k+1}}}{p_k(A_k)-c_k(\mu)} = \delta \qquad (k=N-1, N-2,\ldots,2,1). \quad (4.14)$$

This solution is identical to the one in Section 3.1.

Thus, a pathwise uniform solution for A_k requires $\mu_k=\mu$ and $T_0'(s)=\mu$. The condition on the initial age distribution is rather stringent and unlikely to be met. It is of some interest to note, however, that for the special class of p_k and c_k considered here, we do not need the price and cost functions to be identical for all harvests for a pathwise uniform solution; we only need all the U-shaped $c_k(T_k')$ to have the same minimum-cost harvest rate. More importantly, the results of this section suggest the possibility of a pathwise uniform steady state solution.

4.3. Identical price and cost functions for all harvests and a steady-state solution

Suppose in addition to $\delta_k=\delta$, $k=1,2,\ldots,N$, we have also $p_k=p(A_k)$ and $c_k=c(T_k')$, $k=1,2,3,\ldots,N$. Upon using (4.1) to eliminate λ_k from (4.3) for this case, we get

$$\ddot{c}(T_k')T_k'' = -p(A_k)+\delta\left[\dot{p}(A_k)+T_k'\dot{c}(T_k')-c(T_k')\right] + \dot{p}(A_{k+1})e^{-\delta A_{k+1}}$$

$$(k=1,2,3,\ldots,N-1),$$

$$\ddot{c}(T_N')T_N'' = -\dot{p}(A_N)+\delta\left[p(A_N)+T_N'\dot{c}(T_N')-c(T_N')\right]. \quad (4.15)$$

With h_k and $C(h_k)$ of [5] identified as $1/T_k'$ and $c(1/T_k')/T_k'$, respectively, in our formulation and with

$$\frac{dh_k}{dt} = \frac{d}{ds}\left(\frac{1}{T_k'}\right)\frac{ds}{dT_k} = -\frac{T_k''}{\left(T_k'\right)^3}, \quad (4.16)$$

it is straightforward to verify that (4.15) is identical to the corresponding condition (21) in [5] for a Pareto optimum. Therefore, all properties correctly deduced in [5] from the Pareto optimal solution also hold for the actual optimal policy of our model.

The case of an ongoing forest ($N=\infty$) with identical discount rates, price functions, and cost functions for all harvests admits a *steady-state solution* for tree age distributions, so that $A_k(s)=A(s)$, $k=1,2,3,\ldots$. For such a steady-state solution, we have from (4.10), with $c_k(u_k)=c(u_k)$,

$$u_k = u_{k-1}+A' = u_0+kA' = T_0'(s)+kA' \qquad (k=1,2,\ldots,N) \quad (4.10')$$

with

$$\tilde{\lambda}_k(s) = \dot{c}(u_k) = \dot{c}\big(T_0'(s)+kA'(s)\big), \tag{4.17}$$

so that $\tilde{\lambda}_k$ and $u_k(s)$ generally change from harvest to harvest unless $A' \equiv 0$. Equation (4.9′), with $p_k = p(A_k)$ and $A_k = A(s)$, becomes

$$\tilde{\lambda}_k' = \delta \dot{c}^{-1}(\tilde{\lambda}_k)\tilde{\lambda}_k + \delta\big[p(A) - c\big(\dot{c}^{-1}(\tilde{\lambda}_k)\big)\big] - \dot{p}(A)[1-e^{-\delta A}]$$

$$(k=1,2,\dots) \quad (4.18)$$

with $\tilde{\lambda}_k(0) = \tilde{\lambda}_k(1) = 0$ as before. For general $c(\cdot)$ and $p(\cdot)$, the single unknown function $A(s)$ in (4.17) cannot satisfy all the ODE in (4.18) unless $T_0'(s) \equiv \mu$. For this special case, the *unique* solution for (4.18) and $\tilde{\lambda}_k(0) = \tilde{\lambda}_k(1) = 0$ is the Faustmann rotation A_R determined by

$$\frac{\dot{p}(A_R)}{p(A_R) - c(\mu)} = \frac{\delta}{1-e^{-\delta A_R}}, \tag{4.19}$$

where $1/\mu$ is the minimum-cost harvest rate for the U-shaped unit cost function $c(\cdot)$ [so that $\dot{c}(\mu) = 0$]. From (4.10′) and (4.17), we have also $\tilde{\lambda}_k(s) \equiv 0$ and $u_k(s) = T_0'(s) = \mu$. Thus, a steady-state (optimal) harvest age distribution is also pathwise uniform. This is not altogether surprising in view of (4.10′). The harvest rate at location s is monotone increasing with k if $A'(s) > 0$ [and monotone decreasing if $A'(s) < 0$]; the harvest rates of later harvests would be too slow [too fast] and therefore too costly to yield any profit if $A(s)$ should not be pathwise uniform.

The existence of a (unique) steady-state solution is extremely significant even when it requires the very stringent condition $T_0'(s) = \mu$ [with $\dot{c}(\mu) = 0$] on the initial age distribution. While this requirement on $T_0(s)$ is unlikely to be met in general, we now have the possibility that *the optimal harvest schedule for an arbitrary $T_0(s)$ evolves toward this steady-state solution, which we now know to be pathwise uniform and identical to the classical Faustmann rotation for the minimum cost $c(\mu)$*. We demonstrate that this is the case for a typical price and cost function in the next section.

4.4. Convergence to the Faustmann rotation

To gain some insight into the possibility of convergence to a steady-state solution which is pathwise uniform in both harvest age and undiscounted shadow price, we consider as in [7] the case of

$$p(A) = p_0F_0(1-\sigma e^{-A/\tilde{A}_0}), \qquad c(u) = C_f u + C_0F_0 + \nu C_0F_0^2u^{-1} \tag{4.20}$$

with

$$\mu = F_0\sqrt{\frac{\nu C_0}{C_f}}, \qquad c(\mu) = F_0\Big[C_0 + 2\sqrt{\nu C_0 C_f}\Big]. \tag{4.21}$$

Table 2

$F_0 = 0.10\text{E}+06$	$\nu = 0.50\text{E}-04$	$C_0 = 200.00$	$C_f = 0.10\text{E}+06$
$P_0 = 1500.00$	$\sigma = 1.25$	$\tilde{A}_0 = 60.00$	$\delta = 0.0500$

$T_0(s) = 0.0$ $A_R = 40.18185992$

6 harvests CPU time = 10.102 sec

s	$A_1(s)$	$A_3(s)$	$A_5(s)$	$A_6(s)$
0.0	31.93121	39.30747	40.20492	42.09915
0.050	33.36941	39.33107	40.20891	42.10115
0.100	34.61423	39.39510	40.22083	42.10722
0.150	35.74149	39.48975	40.24038	42.11733
0.200	36.78954	39.60691	40.26701	42.13134
0.250	37.78128	39.74039	40.30001	42.14900
0.300	38.73198	39.88556	40.33854	42.16997
0.350	39.65242	40.03896	40.38173	42.19382
0.400	40.55128	40.19792	40.42870	42.22010
0.450	41.43540	40.36038	40.47853	42.24829
0.500	42.31094	40.52465	40.53032	42.27784
0.550	43.18353	40.68927	40.58314	42.30818
0.600	44.05889	40.85283	40.63605	42.33867
0.650	44.94307	41.01388	40.68805	42.36868
0.700	45.84291	41.17074	40.73807	42.39749
0.750	46.76675	41.32130	40.78493	42.42435
0.800	47.72564	41.46261	40.82730	42.44864
0.850	48.73477	41.59051	40.86368	42.46894
0.900	49.81955	41.69850	40.89231	42.48483
0.950	51.01962	41.77658	40.91118	42.49514
1.000	52.43285	41.80784	40.91802	42.49882

The Fisher age for this class of $V = p(a) - c(u)$ is

$$A = \tilde{A}_0 \ln\left[\frac{\sigma(1+1/\tilde{A}_0\delta)}{1-(1/p_0)\left(C_0+2\sqrt{\nu C_0 C_f}\right)}\right], \tag{4.22}$$

while the Faustmann rotation is the root of

$$\left\{1-\frac{1}{p_0}\left[C_0+2\sqrt{\nu C_0 C_f}\right]\right\}e^{(\delta\tilde{A}_0+1)A/\tilde{A}_0} - \frac{\sigma_0}{\tilde{A}_0}(1+\delta\tilde{A}_0)e^{\delta A} + \frac{\sigma_0}{\tilde{A}_0} = 0. \tag{4.23}$$

We show in Tables 2 and 3 the solution of the boundary-value problem (4.9′), (4.10), (4.11) for two finite harvest sequences (one with $N = 6$ and the other with

Table 3

$F_0 = 0.10\text{E}+06$ $\nu = 0.50\text{E}-04$ $C_0 = 200.00$ $C_f = 0.10\text{E}+06$
$P_0 = 1500.00$ $\sigma = 1.25$ $\tilde{A}_0 = 60.00$ $\delta = 0.0500$

$T_0(s) = 0.0$ $A_R = 40.18185992$

9 harvests CPU time = 25.713 sec

s	$A_1(s)$	$A_2(s)$	$A_3(s)$	$A_5(s)$	$A_7(s)$	$A_8(s)$	$A_9(s)$
0.0	31.93113	38.08111	39.30286	39.94763	40.14016	40.39477	42.20256
0.050	33.36933	38.18305	39.32646	39.95163	40.14127	40.39540	42.20291
0.100	34.61415	38.40374	39.39050	39.96359	40.14467	40.39731	42.20399
0.150	35.74133	38.67919	39.48515	39.98321	40.15038	40.40056	42.20584
0.200	36.78954	38.98165	39.60230	40.00993	40.15838	40.40512	42.20844
0.250	37.78117	39.29815	39.73580	40.04303	40.16857	40.41097	42.21178
0.300	38.73184	39.62121	39.88098	40.08170	40.18079	40.41803	42.21583
0.350	39.65235	39.94710	40.03438	40.12505	40.19483	40.42619	42.22053
0.400	40.55121	40.27359	40.19334	40.17219	40.21043	40.43531	42.22579
0.450	41.43532	40.59945	40.35582	40.22221	40.22730	40.44521	42.23153
0.500	42.31084	40.92393	40.52010	40.27419	40.24510	40.45570	42.23762
0.550	43.18343	41.24658	40.68473	40.32722	40.26346	40.46655	42.24394
0.600	44.05884	41.56701	40.84829	40.38033	40.28198	40.47752	42.25033
0.650	44.94299	41.88483	41.00936	40.43254	40.30021	40.48833	42.25664
0.700	45.84276	42.19930	41.16625	40.48275	40.31771	40.49871	42.26269
0.750	46.76671	42.50900	41.31678	40.52979	40.33399	40.50835	42.26831
0.800	47.72499	42.81205	41.45816	40.57233	40.34853	40.51694	42.27331
0.850	48.73641	43.10050	41.58588	40.60884	40.36081	40.52416	42.27751
0.900	49.81776	43.37069	41.69414	40.63758	40.37026	40.52970	42.28071
0.950	51.01963	43.59156	41.77211	40.65652	40.37633	40.53324	42.28276
1.000	52.43260	43.69917	41.80334	40.66339	40.37847	40.53448	42.28347

$N = 9$) and for a typical set of price and cost parameters. The optimal last harvest schedule is seen to approach the Fisher age, while earlier (optimal) harvest schedules are seen to approach the Faustmann rotation. Similar results have also been obtained for other input parameter values, with the rate of convergence depending on the discount rate δ.

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(Received January 22, 1985)