Prevalent Behavior of Strongly Order Preserving Semiflows

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Abstract

Classical results in the theory of monotone semiflows give sufficient conditions for the generic solution to converge toward an equilibrium or towards the set of equilibria (quasiconvergence). In this paper, we provide new formulations of these results in terms of the measure-theoretic notion of prevalence, developed in [1, 8]. For monotone reaction-diffusion systems with Neumann boundary conditions on convex domains, we show the prevalence of the set of continuous initial conditions corresponding to solutions that converge to a spatially homogeneous equilibrium. We also extend a previous generic convergence result to allow its use on Sobolev spaces. Careful attention is given to the measurability of the various sets involved.

Keywords: strong monotonicity, prevalence, quasiconvergence, reaction-diffusion, measurability.

1 Introduction

The signature results in the theory of monotone dynamics are that certain dynamic behaviors are generic, for example, convergence to equilibrium is generic

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under suitable conditions. In order to be more precise, some notation is useful but technical definitions will be deferred to the next section. Let $\mathbb{B}$ be an ordered separable Banach space, such as $\mathbb{B} := \mathbb{R}^n$ together with the order defined by $x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1 \ldots n$. Another important example is $\mathbb{B} := C(\overline{\Omega}, \mathbb{R}^n)$, with the order defined by $u \leq v$ if and only if $u_i(x) \leq v_i(x)$ for every $x \in \overline{\Omega},\ i = 1 \ldots n$, where $\overline{\Omega} \subseteq \mathbb{R}^m$ is a smooth compact domain. Let $X \subseteq \mathbb{B}$, and consider a semiflow $\Phi : X \times \mathbb{R}_+ \to X$ which is strongly monotone with respect to a cone K with nonempty interior. For instance, in the case $\mathbb{B} = \mathbb{R}^n$ above, and denoting $u(t) := \Phi(u(0), t)$ for $t > 0$, the strong monotonicity means that whenever $u(0) \leq v(0)$, $u(0) \neq v(0)$ it must hold $u_i(t) < v_i(t)$ for every $t > 0$, $i = 1 \ldots n$.

Define E to be the set of equilibria of the semiflow Φ, and denote by $E_s \subseteq E$ the set of equilibria after excluding all linearly unstable ones (see the formal definition in Section 2). Let

$$
\begin{aligned}
&B := \{x \in X \,|\, \text{the orbit } O(x) \text{ has compact closure in } X\} \\
&Q := \{x \in X \,|\, \omega(x) \subseteq E\} \\
&C := \{x \in X \,|\, \omega(x) = \{e\} \text{ for some } e \in E\} \\
&C_s := \{x \in X \,|\, \omega(x) = \{e\} \text{ for some } e \in E_s\}.
\end{aligned}
$$

In particular, $C_s \subseteq C \subseteq Q \subseteq B$. The elements of C are said to be *convergent* or to have *convergent solution*, and those of Q are said to be *quasiconvergent*.

It was established in [4] that the generic element of B is quasiconvergent, where 'generic' is made specific in two different senses: the topological sense ($B \backslash Q$ is meager or Q is residual), and the measure theoretic sense ($\mu(B - Q) = 0$ for any gaussian measure μ). Later, Smith and Thieme [18], motivated by work of Poláčik [13], provided sufficient conditions for C to contain a set which is open and dense in B.

A drawback of topological genericity is that closed, nowhere dense subsets of X may still be quite large in terms of measure. In fact, it is well known that there exists a Cantor subset of $[0, 1]$ with positive measure whose complement is open and dense in $[0, 1]$. On the other hand, asking for a set to have measure zero in an infinite dimensional space $\mathbb{B}$ is difficult to formalize, since there doesn't exist a measure with the basic properties of the Lebesgue measure in finite dimensions. As noted in [8], the desirable properties of Lebesgue measure zero sets include: (1) a measure zero set has empty interior, (2) a subset of a measure zero set has measure zero, (3) a countable union of measure zero sets has measure zero, (4) every translate of a measure zero set has measure zero. Hunt, Sauer and Yorke [8] note that a nontrivial translation invariant Borel measure on an infinite dimensional separable Banach space assigns infinite measure to every open set. Gaussian measures, used by Hirsch as noted above, are based on the normal distributions in statistics. A Borel measure μ on $\mathbb{B}$ is Gaussian if for every nontrivial linear functional ϕ the push forward of μ by ϕ on $\mathbb{R}$ is a nontrivial Gaussian distribution. If $\mathbb{B} = \mathbb{R}^n$, then a Gaussian measure gives the same null sets as Lebesgue measure. Of course, Gaussian measures are not translation invariant.

A definition of 'sparseness' that turns out to be very useful in infinite dimensions is that of prevalence [1, 8]: a Borel set $W \subseteq \mathbb{B}$ is *shy* if there exists a nonzero compactly supported Borel measure μ on $\mathbb{B}$, such that $\mu(W + x) = 0$ for every $x \in \mathbb{B}$. A Borel set is said to be *prevalent* if its complement is shy. Given $A \subseteq \mathbb{B}$, we say here that a Borel set W is *prevalent in* A if $A - W$ is shy.

An important example of a shy set in finite dimensions is any set with Lebesgue measure zero. In fact, both concepts are equivalent for $\mathbb{B} = \mathbb{R}^n$. Thus, the complement in $\mathbb{R}^n$ of any set with measure zero is an example of a prevalent set. General properties of shy and prevalent sets include the following: (1) shy sets have empty interior (prevalent sets are dense), (2) a countable union of shy sets is shy, (3) if A is shy then so is any translate $A + x$. These and other properties can be found in [8].

The concept of prevalence seems natural to the theory of monotone systems. For instance, according to a result by Hirsch [5] in the case $B = X = \mathbb{B}$, it holds that $J - Q$ is countable, for any totally ordered arc J. This implies (Lemma 1) that Q is prevalent with respect to the uniform measure μ_v supported on the segment $J_v = \{tv : 0 \leq t \leq 1\}$, for any vector $v > 0$. Indeed, our Lemma 1 says that a set A is shy if there is some positive vector v such that A has countable intersection with every straight line parallel to v.

In this paper, we prove several results regarding the genericity of the sets Q, C, C_s, especially in the sense of prevalence. In Section 3, we obtain the counterparts of the genericity results of Hirsch, Smith, and Thieme, by proving the prevalence of the sets Q and C in B under different regularity assumptions. In Section 4, we give sufficient conditions for the prevalence of C_s. (See also Theorem 3.3 and Corollary 3.4 of Poláčik[14], where a similar result is discussed whose proof uses entirely different arguments.) The main result of Section 4 summarizes all the prevalence results in this paper under sufficient regularity assumptions, and it is stated here (see Section 2 for further definitions).

Theorem 1 *Let $\mathbb{B}$ be an ordered, separable Banach space, and let $X \subseteq \mathbb{B}$ be order convex. Consider an eventually strongly monotone semiflow Φ defined on X, with C^1 time evolution operators $\Phi_t(x)$ that have compact derivatives for $t > t_0$. Assume that $B = X$, and that every equilibrium is irreducible. Then C_s is prevalent in X.*

Using Theorem 1 and a well known result of Kishimoto and Weinberg [10], in Section 6 we give an application for strongly cooperative reaction diffusion systems of n equations with Neumann boundary conditions on a convex domain Ω. In such a system, we prove the prevalence of the set of initial data in $C(\overline{\Omega}, \mathbb{R}^n)$ corresponding to orbits that converge towards a constant equilibrium.

Note that the prevalence of Q or C in X doesn't imply *a priori* that they contain a set which is open and dense in X. As noted above, Smith and Thieme [18] gave such conditions which included strong compactness properties of the semiflow. Recently, two of us [6, 7] gave sufficient conditions for the above sets to contain open and dense sets by replacing strong compactness assumptions by the assumption that omega limit sets should have infima or suprema in the space. This result is especially useful when the state space is finite dimensional or is $C(\overline{\Omega}, \mathbb{R}^n)$ since every compact set has infima and suprema. However, the space $C(\overline{\Omega}, \mathbb{R}^n)$ is often not a convenient space, especially for partial differential equations. In Section 5 we improve the results in [6, 7] by requiring only that some possibly larger space contains infima or suprema of the limit sets. This extension facilitates the application of the theory to partial differential equations on state spaces that continuously imbed in a space of continuous functions such as Hölder and Sobelev spaces. An example of such an application is provided.

2 Definitions

Order Relations

A closed subset K of a Banach space $\mathbb{B}$ is called a *cone* if i) $x \in K$, $\alpha > 0$ implies $\alpha x \in K$; ii) $x, y \in K$ implies $x + y \in K$; iii) $x \in K$ implies $-x \notin K$. The cone K is used to define a partial relation $\leq$ by denoting $x \leq y$ whenever $y - x \in K$. For instance if $\mathbb{B} := \mathbb{R}^n$, the so-called *cooperative cone* $K = (\mathbb{R}_+)^n$ induces the order $\leq$ in which $x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1 \ldots n$.

We also write in the general case $x < y$ if $x \leq y$, $x \neq y$, and $x \ll y$ if $y - x \in \operatorname{Int} K$. In the given example, it holds $x \ll y$ if and only if $x_i < y_i$ for every i. We write $x \geq y$ to mean $y \leq x$, and similarly for $>$ and $\gg$. An *ordered Banach space* is a Banach space $\mathbb{B}$, together with a closed cone $K \subseteq \mathbb{B}$.

Given two subsets A and B of $\mathbb{B}$, we write $A \leq B$ ($A < B$) when $x \leq y$ ($x < y$) holds for each choice of $x \in A$ and $y \in B$. A set $A \subseteq \mathbb{B}$ is *unordered* if it does not contain points x, y such that $x < y$. A is *order-convex* if $x \leq z \leq y$ and $x, y \in A$ implies $z \in A$. It is *p-convex* if $x < y$ and $x, y \in A$ implies $z = tx + (1-t)y \in A$ for $0 < t < 1$.

Consider a set $X \subseteq \mathbb{B}$. A point $x \in X$ is *doubly accessible from below* (respectively, above) if in every neighborhood of x there exist y, z with $y < z < x$ (respectively, $x < y < z$). For $A \subseteq X$ define $L = \{x \in X \,|\, x \leq A\}$ be the (possibly empty) set of lower bounds for A in X. In the usual way, we define $\inf A := u$ if $u \in L$ and $L \leq u$; u is unique if it exists. Similarly, $\sup A$ is defined.

Semiflows

A *semiflow* on X is a continuous map $\Phi : X \times \mathbb{R}_+ \to X$, $(x,t) \mapsto \Phi_t(x)$, such that:

$$\Phi_0(x) = x, \quad (\Phi_t \circ \Phi_s)(x) = \Phi_{t+s}(x) \qquad (t, s \geq 0,\ x \in X)$$

We will henceforth continue to denote $\Phi(x,t)$ by $\Phi_t(x)$. The *orbit* of x is the set $O(x) = \{\Phi_t(x) : t \geq 0\}$.

An *equilibrium* is a point x for which $O(x) = \{x\}$. The set of equilibria is denoted by E. We abuse notation by calling the semiflow C^1 if the Frechet (partial) derivative $D_x\Phi(x,t)$ with respect to the state variable x exists for each x and $t > 0$ and $x \to D_x\Phi(x,t)$ is continuous. In case the domain is not open in Banach space $\mathbb{B}$, this must be interpreted with care. See pg 273 in [6]. We often use the notation $\Phi_t'(x) = D_x\Phi(x,t)$. If $e \in E$, define $\rho(e,t)$ to be the spectral radius of the Frechet derivative $D_x\Phi(e,t)$. We say that e is *linearly stable* if $\rho(e,t) < 1$ for all $t > 0$, *linearly unstable* if $\rho(e,t) > 1,\ t > 0$, and *neutrally stable* if $\rho(e,t) = 1,\ t > 0$. Under mild assumptions, a linearly stable (unstable) equilibrium is also stable (unstable) in the sense of Lyapunov. Define $E_s \subseteq E$ to be the set of equilibria that are either linearly stable or neutrally stable. The set C_s, defined in the introduction, is the set of points whose omega limit set is a point of E_s.

The *omega limit set* $\omega(x)$ of $x \in X$, defined in the usual way, is closed and positively invariant. When $\overline{O(x)}$ is compact in X, i.e., when $x \in B$, $\omega(x)$ is also nonempty, compact, invariant, connected, and it attracts x. Define for any set $A \subseteq X$ the *strict basin of attraction* $SB(A) := \{x \in X \,|\, \omega(x) = A \text{ and } x \in B\}$. Note the difference with the usual basin of attraction of A, $\mathcal{B}(A) = \{x \in X \,|\, \omega(x) \subseteq A\}$. For $p \in E$, define $C(p) := SB(\{p\})$. Note that $C = \bigcup_{p \in E} C(p)$.

Monotonicity

Let Φ denote a semiflow on $X \subset \mathbb{B}$. We call Φ *monotone* provided

$$\Phi_t(x) \leq \Phi_t(y) \text{ whenever } x \leq y \text{ and } t \geq 0.$$

Φ is *strongly monotone* if $x < y$ implies that $\Phi_t(x) \ll \Phi_t(y)$ for all $t > 0$ and *eventually strongly monotone* if it is monotone and whenever $x < y$ there exists $t_0 \geq 0$ such that $\Phi_t(x) \ll \Phi_t(y)$ for $t \geq t_0$. Φ is *strongly order-preserving*, SOP for short, if it is monotone and whenever $x < y$ there exist open subsets U, V of X with $x \in U$ and $y \in V$ and $t_0 \geq 0$ such that

$$\Phi_{t_0}(U) \leq \Phi_{t_0}(V).$$

Monotonicity of Φ then implies that $\Phi_t(U) \leq \Phi_t(V)$ for all $t \geq t_0$. Strong monotonicity implies eventual strong monotonicity which implies SOP [16].

We say that an equilibrium point $e \in E$ of a monotone semiflow is *irreducible* if for some $t = t_e > 0$, $\Phi'_t(e)$ is a strongly positive operator (i.e. $x > 0$ implies $\Phi'_t(e)x \gg 0$). Observe that if $\Phi'_t(e)$ is strongly positive, so is $\Phi'_s(e)$, $s \geq t$. The point e is said to be *non-irreducible* otherwise. By the well-known Krein-Rutman theorem [15], if A is compact and strongly positive then its spectral radius $\rho(A)$ is a simple eigenvalue with eigenspace spanned by a positive vector $v \gg 0$; moreover v is the unique eigenvector belonging to K, up to scalar multiple.

Consider a reaction diffusion system

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = d_i \Delta u_i + f_i(x, u), \;\; i = 1, \ldots, m,$$

defined on a smooth domain $\Omega \subseteq \mathbb{R}^n$ under Neumann, Dirichlet, or Robin boundary conditions, $d_i > 0$, $i = 1 \ldots m$. We say that this system is *cooperative and irreducible* if the C^2 function $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies i) $\partial f_i / \partial u_j(x, u) \geq 0$ for all $i \neq j$, $x \in \Omega$ and $u \in \mathbb{R}^m$, and ii) there exists $\bar{x} \in \Omega$ such that the $m \times m$ Jacobian matrix $[\partial f_i / \partial u_j(\bar{x}, u)]$ is irreducible for all u. It is well known that under these conditions the reaction diffusion system is strongly monotone with respect to the cone $C(\overline{\Omega}, \mathbb{R}^m_+)$ in $\mathbb{B} = C(\overline{\Omega}, \mathbb{R}^m)$. Moreover, this result holds true even when the operator Δ is replaced by other second order elliptic operators; see Chapter 6 of [6] for details.

Given $v \in \mathbb{B}, v \neq 0$, define the Borel measure μ_v on $\mathbb{B}$ to be the uniform measure supported on the set $S_v := \{tv \,|0 \leq t \leq 1\}$. That is, $\mu_v(A) = m\{t \in [0,1] \,|tv \in A\}$, where m is the Lebesgue measure in $[0,1]$. This family of measures will be especially important for monotone semiflows below.

3 C is Prevalent in B

Consider an SOP semiflow Φ defined on a subset X of the separable Banach space $\mathbb{B}$, ordered with respect to a cone K. We begin by quoting some fundamental properties of these semiflows from the literature; see e.g. [6, 16, 17].

Theorem 2 *Convergence Criterion: If Φ is SOP and $\Phi_T(x) > x$ ($\Phi_T(x) < x$) for some $T > 0$ then $\Phi_t(x) \to p \in E$ as $t \to \infty$.*

Theorem 3 *Nonordering of Limit Sets: Let Φ be SOP and ω be an omega limit set. Then no two points of ω are related by $<$.*

Theorem 4 *Limit Set Dichotomy: Let Φ be SOP. If $x < y$ then either*

(a) $\omega(x) < \omega(y)$, *or*

(b) $\omega(x) = \omega(y) \subseteq E$.

Smith and Thieme [18] improve part (b) of the Limit Set Dichotomy to read $\omega(x) = \omega(y) = \{e\}$ for some $e \in E$ under additional smoothness and strong monotonicity conditions. For example, this Improved Limit Set Dichotomy holds if $X \subseteq Y$ is order convex in the ordered Banach space Y with cone Y_+ having non-empty interior, $\Phi_t(x)$ is C^1 in x and its derivative is a compact, strongly positive operator. See e.g. [6, 16, 18].

We now restate two standard results in terms of the concept of prevalence. The following lemma is the basis for the proofs below. Compare with Lemma 7.7 in [5].

Lemma 1 *Let $W \subseteq X$ and suppose there exists a positive vector v such that $L \cap W$ is countable for every straight line L parallel to v. Then W is shy.*

Proof. Consider $v > 0$ and the uniform measure μ_v as defined in Section 2. Let $L = \mathbb{R}v - x$ for an arbitrary $x \in \mathbb{B}$. Then

$$(W + x) \cap S_v \subseteq (W + x) \cap \mathbb{R}v = (W \cap L) + x.$$

Therefore clearly $\mu_v(W + x) = 0$, and W is shy with respect to μ_v. ∎

The proof of Hirsch's generic convergence theorem as stated in terms of prevalence becomes clear at this point. See Theorem 4.4 of [4], and [5].

Theorem 5 *Let $\mathbb{B}$ be a separable Banach space, and consider a strongly monotone system defined on $X \subseteq \mathbb{B}$. Then Q is prevalent in B.*

If the Improved Limit Set Dichotomy holds for Φ, then C is prevalent in B.

Proof. Let $N = B - Q$ be the set of states $x \in B$ such that $\omega(x) \not\subseteq E$ (see Section 7 for a proof that this set is measurable). Let $L \subseteq X$ be a straight line parallel to a vector $v > 0$. Note that if $x, y \in L \cap N$, $x < y$, then $\omega(x) < \omega(y)$, since otherwise $\omega(x) = \omega(y) \subseteq E$ by the Limit Set Dichotomy.

We can apply an argument as in Theorem 7.3 c) of Hirsch [5] to conclude that $N \cap L$ is countable: consider the set $Y = \cup_{x \in N \cap L} \omega(x)$ with the topology inherited by $\mathbb{B}$. Since no point in $\omega(x)$ can bound $\omega(x)$ from below or above (Theorem 3), no point in $\omega(x)$ can be the limit of elements in $\omega(y)$, $y \neq x$. Therefore $\omega(x)$ is open in Y, for every $x \in N \cap L$. The countability of $N \cap L$ follows by the separability of Y.

Since $N \cap L$ is countable for every strongly ordered line L, N must be shy by Lemma 1.

If the Improved Limit Set Dichotomy holds for Φ, then we can argue exactly as above to show that $N = B \setminus C$ is shy. ∎

Theorem 6 *Let $\mathbb{B}$ be a separable Banach space, and let X be p-convex in $\mathbb{B}$. If C is dense in B, then C is prevalent in B.*

Proof. Let $K = B - C$ be the set of the states $x \in B$ such that $\omega(x)$ is not a singleton (see Section 7 for a proof that this set is measurable). We will show that K is shy with respect to the measure μ_v, for every $v > 0$. Recall the definition of the strict basin of attraction $SB(A)$ of a set A (Section 2). From the assumption that C is dense in B, it holds that $SB(\omega(x))$ has empty interior for every $x \in K$.

Let L be a straight line in X parallel to a positive vector $v > 0$, $\mathcal{P}(X)$ denote the set of all subsets of X, and consider the function $\gamma : L \cap K \to \mathcal{P}(X)$ defined by $\gamma(x) = \omega(x)$. Then this function is injective. Indeed, if $x, y \in L \cap K$, $x < y$, were such that $\gamma(x) = \gamma(y) = W$, then the strong order preserving property implies that for any point $u = sx + (1-s)y$, $0 < s < 1$ there is a neighborhood U of u and $t_0 \geq 0$ such that $\Phi_t(x) \leq \Phi_t(U) \leq \Phi_t(y)$ for $t \geq t_0$ and therefore $\omega(v) = W$ for every $v \in U$ by the Limit Set Dichotomy. As U is a nonempty open subset of X, this implies that $SB(W)$ has nonempty interior in X, a contradiction to the fact that x belongs to K.

Note also that, by the Limit Set Dichotomy, the image of γ is an ordered collection of sets: $x < y$ then $\gamma(x) < \gamma(y)$. Following the same argument as in the proof of Theorem 5, it follows that $L \cap K$ is countable. By Lemma 1, K is shy with respect to μ_v, for $v > 0$. ∎

4 C_s is Prevalent in C for Smooth Φ

In this section we assume that $\mathbb{B}$ is a separable Banach space ordered by a cone K with nonempty interior and Φ is a strongly order preserving semiflow on the p-convex subset $X \subseteq \mathbb{B}$. We assume also that for every $t \geq t_0$ the time evolution operators Φ_t are compact, (Frechet) C^1 and have compact derivatives (for some fixed $t_0 \geq 0$). Recall from Section 2 the definition of E_s, C_s, and irreducible equilibria. The aim of this section is to provide sufficient conditions for C_s to be prevalent in C and in X.

The following result is well-known.

Lemma 2 *Let $T : X \to X$ be a continuous (nonlinear) operator. Let $e \in X$ be a fixed point of T, and assume that the Frechet derivative $T'(e) : T \to T$ exists and is compact. Assume also that there exists a sequence $e_1,\ e_2,\ \ldots$ of fixed points of T, $e_k \neq e$, such that $e_k \to e$ as $n \to \infty$.*

Then the unit vectors $v_k := (e_k - e)/\left|e_k - e\right|$ have a subsequence that converges towards a unit vector $w \in \mathbb{B}$, and $T'(0)w = w$.

Lemma 3 *If $(e_k)_{k\in\mathbb{N}}$ is a sequence of equilibria of the semiflow Φ such that $e_k < e_{k+1}$ ($e_k < e_{k+1}$) for all k, and if the sequence (e_k) converges towards a irreducible equilibrium $e \in E$, then $e \in E_s$.*

Proof. Let $\tau \geq t_0$ be such that $\Phi'_\tau(e)$ is strongly positive, and let $T := \Phi_\tau$. Then T satisfies the hypotheses of Lemma 2, so that defining $v_k = (e_k - e)/\left|e_k - e\right|$, there exists a subsequence v_{k_i} which converges to a unit vector $w \in \mathbb{B}$. Furthermore, $T'(e)w = w$. From the fact that $e < e_k$ for every k, we conclude that $v_k > 0$ and consequently that the unit vector $w > 0$ so $w \gg 0$.

By the Krein Rutman theorem, the fact that $T'(e)$ has a positive eigenvector with eigenvalue 1 implies that in fact $\rho(T'(e)) = 1$. Therefore $e \in E_s$, and this concludes the proof.

The case $e_{k+1} < e_k$ for every $k \in \mathbb{N}$ can be treated similarly. ∎

We introduce the property (P):

(P) Every set of equilibria $\hat{E} \subseteq E$ which is totally ordered by $<$ has at most countably many non-irreducible points.

For instance, this condition holds if all equilibria in X are irreducible (see condition (S) in [16], p. 19). It also holds if every totally ordered subset of E has at most countably many points.

Lemma 4 *Let property (P) be satisfied. If $\hat{E} \subseteq E$ is totally ordered by $<$, and if every element of $\hat{E}$ is linearly unstable, then $\hat{E}$ is countable.*

Proof. Suppose that $\hat{E}$ is not countable. Then the set $\tilde{E} \subseteq \hat{E}$ of irreducible elements in $\hat{E}$ is also uncountable, by property (P). Let $e \in \tilde{E}$ be an accumulation point of $\tilde{E}$, which exists by the separability of the Banach space $\mathbb{B}$ (otherwise $\mathbb{B}$ would contain an uncountable set of pairwise disjoint open balls). Then there exists a monotone sequence of elements in $\tilde{E}$ which converges towards e. By the previous lemma it holds that $e \in E_s$, contradicting $e \in \tilde{E}$. ∎

Lemma 5 *If $e \in E \setminus E_s$, then $SB(e)$ is unordered and hence shy.*

Proof. The same argument as Lemma 2.1 in [19] shows that $SB(e)$ is unordered. This implies that it is shy with respect to μ_v for any $v > 0$, by Lemma 1, since any line parallel to v meets $SB(e)$ at most once. ∎

Our next result is similar to Theorem 4.4 in [4] in finite dimensions, and to a lesser extent to Theorem 10.1 in [5] but it drops the assumptions of finiteness or discreteness for the set E.

Theorem 7 *In addition to the assumptions of this section, let property (P) be satisfied. Then C_s is prevalent in C.*

Proof. We follow a very similar argument as in the proof of Theorem 5. Let $N = C - C_s$ be the set of $x \in C$ such that $\omega(x)$ is a linearly unstable equilibrium. It will be shown in Section 7 that this set is Borel. Let $v > 0$ and let L be a line parallel to v. Then we can define the function $\sigma : L \cap N \to X$ by $\sigma(x) = \lim_{t\to\infty} \Phi(x,t)$. If $x_1, x_2 \in L \cap N$, $x_1 < x_2$, then necessarily $\sigma(x_1) < \sigma(x_2)$ by the Limit Set Dichotomy since $SB(\omega(x_1))$ is unordered by Lemma 5. Thus σ is injective. As $\hat{E} =$ range σ is totally ordered, it is countable by Lemma 4, and so is $L \cap N$ by injectivity. By Lemma 1, N is shy. ∎

See also Theorem 4.4 and Theorem 4.1 of [4]. We are ready now to prove the result stated in the introduction.

Proof of Theorem 1: *Proof.* In this case, the Improved Limit Set Dichotomy holds so C is prevalent in X by Theorem 5. As C_s is prevalent in C by Theorem 7, the result follows since $X \setminus C_s = (X \setminus C) \cup (C \setminus C_s)$ is the union of two subsets, each shy relative to the same μ_v, $v > 0$. ∎

5 Q Contains an Open and Dense set

In this section, we do not require a Banach space structure of our state space. It suffices for X to be a partially ordered metric space such that the order relation is closed; we call such a space and ordered metric space. Let Φ be an SOP semiflow on the ordered metric space X, having compact orbit closures. In this section we improve a result in [7] by weakening the conditions for Q to contain an open and dense set. We introduce the following hypothesis:

(K) $X \subseteq Z$ where Z is an ordered metric space with order relation $\leq_Z$, the inclusion $i : X \to Z$ is continuous and X inherits its order relation from Z. Φ extends to a mapping $\Psi : \mathbb{R}_+ \times Z \to Z$, not necessarily continuous, where

(a) $\Psi|_{\mathbb{R}_+ \times X} = \Phi$,

(b) Ψ is monotone on Z.

(c) For every $z \in Z$, there exists $t_z \geq 0$ such that $\Psi_{t_z}(z) \in X$.

Observe that if $z \in Z$, then $\Psi_t(z) \in X$ and $\Psi_t(z) = \Phi_t(z)$ for $t \geq t_z$. Consequently, the omega limit set of the orbit through z exists in both X and in Z and they agree.

Recall (Section 2) the definition of the set $C(p)$ for an equilibrium $p \in E$, and that of a set X which is doubly accessible from below. All topological properties used hereafter are relative to the space X.

Lemma 6 *Let (K) hold. Suppose $x \in X \setminus Q$ and $a = \inf \omega(x) \in Z$ exists. Then $\omega(a) = \{p\}$ where $p \in X$ satisfies $p < \omega(x)$, and $x \in \overline{\operatorname{Int} C(p)}$ provided x is doubly accessible from below.*

Proof. Fix an arbitrary neighborhood M of x. Note that $a <_Z \omega(x)$ because $\omega(x) \subseteq X$ is unordered (Theorem 3). By invariance of $\omega(x)$ and (K) we have $\Psi_t a \leq_Z \Psi_t \omega(x) = \Phi_t \omega(x) = \omega(x)$, hence $\Psi_t a \leq_Z a, t \geq 0$. It follows from (K) that $\Phi_t(w) \leq w := \Psi_{t_a}(a)$ for $t \geq 0$ and therefore the Convergence Criterion Theorem implies that $\omega(a)$ is an equilibrium $p \in X$ with $p \leq a$. Because $p < \omega(x)$, SOP yields a neighborhood N of $\omega(x)$ and $s \geq 0$ such that $p \leq \Phi_t N$ for all $t \geq s$. Choose $r \geq 0$ with $\Phi_t x \in N$ for $t \geq r$. Then $p \leq \Phi_t x$ if $t \geq r + s$. The set $V := (\Phi_{r+s})^{-1}(N) \cap M$ is a neighborhood of x in M with the property that $p \leq \Phi_t V$ for all $t \geq r + 2s$. Hence:

$$u \in V \Rightarrow p \leq \omega(u) \tag{1}$$

Now assume x doubly accessible from below and fix $y_1, y \in V$ with $y_1 < y < x$. By the Limit Set Dichotomy $\omega(y) < \omega(x)$, because $\omega(x) \nsubseteq E$. By SOP we fix a neighborhood $U \subseteq V$ of y_1 and $t_0 > 0$ such that $\Phi_{t_0} u \leq \Phi_{t_0} y$ for all $u \in U$. The Limit Set Dichotomy implies $\omega(u) = \omega(y)$ or $\omega(u) < \omega(y)$; as $\omega(y) < \omega(x)$, we therefore have:

$$u \in U \Rightarrow \omega(u) < \omega(x) \tag{2}$$

For all $u \in U$, (2) implies $\omega(u) \leq \omega(a) = \{p\}$, while (1) entails $p \leq \omega(u)$. Hence $U \subseteq C(p) \cap M$, and the conclusion follows. ∎

An analogous result holds if "$a = \inf \omega(x) \in Z$ exists" is replaced by "$b = \sup \omega(x) \in Z$ exists", in which case $\omega(b) = \{q\}$ where $q > \omega(x)$. Furthermore, the conclusion $x \in \overline{Int\, C(p)}$ holds provided x is doubly accessible from above.

We introduce an additional condition on the semiflow Φ similar to the one in [7]:

(L) Either every omega limit set $\omega(x)$, $x \in X$, has an infimum in Z and the set of points that are doubly accessible from below has dense interior in X, or every omega limit set has a supremum in Z and the set of points that are doubly accessible from above has dense interior in X.

Theorem 8 *Let Φ be an SOP semiflow on the ordered metric space X, having compact orbit closures, and satisfying axioms* (L) *and* (K)*. Then $X \setminus Q \subseteq \overline{\mathrm{Int}\, C}$, and* $\mathrm{Int}\, Q$ *is dense.*

Proof. To fix ideas we assume the first alternative in (L), the other case being similar. Let X_0 denote a dense open set of points doubly accessible from below. Lemma 6 implies $X_0 \subseteq Q \cup \overline{\mathrm{Int}\, C} \subseteq Q \cup \overline{\mathrm{Int}\, Q}$, hence the open set $X_0 \setminus \overline{\mathrm{Int}\, Q}$ lies in Q. This prove $X_0 \setminus \overline{\mathrm{Int}\, Q} \subseteq \mathrm{Int}\, Q$, so $X_0 \setminus \overline{\mathrm{Int}\, Q} = \emptyset$. Therefore $\overline{\mathrm{Int}\, Q} \supset X_0$, hence $\overline{\mathrm{Int}\, Q} \supset \overline{X_0} = X$. ∎

Axiom (L) is a restriction on both the space X (order and topology) and the semiflow (limit sets). If X continuously embeds in $Z = C(A, \mathbb{R})$, the Banach space of continuous functions on a compact set A with the usual ordering, Φ extends to a monotone mapping on Z with the smoothing property (c), then axiom (L) holds. This is true because every compact subset of $C(A, \mathbb{R})$ has a supremum and infimum (see Schaefer [15], Chapt. II, Prop. 7.6). In particular, X may be a Euclidean space $\mathbb{R}^n$, a Hölder space $C^{k+\alpha}(\Omega, \mathbb{R}^n)$, $0 \leq \alpha < 1, k = 0, 1, 2, \cdots$, for Ω a compact smooth domain in R^m, or a Sobolev space $H^{k,p}(\Omega)$ for $k - \frac{n}{p} \geq 0$ where the usual functional ordering is assumed. These cases cover ordinary, delay, and parabolic partial differential equations under suitable hypotheses.

Theorem 8 extends the corresponding result in [7], where it was assumed that $Z = X$, by allowing X to be imbedded in a larger space Z in which it is more likely that omega limit sets have infima and suprema. This extension is important for partial differential equations for the reasons mentioned above.

We show how Theorem 8 can be used to improve Theorem 6.17 of [6] concerning the system of reaction diffusion equations given by

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= A_i u_i + f_i(x,u),\ i = 1\ldots m,\ x \in \Omega,\ t > 0 \\ B_i u_i &= 0,\ x \in \partial\Omega,\ t > 0 \end{aligned}$$

where A_i are uniformly elliptic second order differential operators and B_i are boundary operators of Dirichlet, Robin, of Neumann type and f is cooperative and irreducible in the sense of Section 2 (see Section 6.2 of [6] for an additional discussion of this system). These conditions could be formulated with respect to an alternative orthant order with no change in conclusions.

Let $\Gamma \subseteq \mathbb{R}^m$ be a rectangle, i.e., product of m nontrivial intervals, and $k = 0, 1$. Consider the sets

$$X_\Gamma^k := \{u \in C_B^k(\overline{\Omega}, \mathbb{R}^m) : u(\overline{\Omega}) \subseteq \Gamma\}, \quad X_\Gamma := \{u \in L^p(\Omega, \mathbb{R}^m) : u(\Omega) \subseteq \Gamma\}$$

See [6] for details on the notation; the subscript B indicates the boundary conditions are accommodated. We assume that the system above generates a semiflow Φ on X_Γ and semiflows $\Phi^k,\ k = 0, 1$ on X_Γ^k. See [6] for such conditions.

Finally, assuming that these semiflows have compact orbit closures, it is observed in [6] that:

- X_Γ^1 is dense in X_Γ^0 and in X_Γ.
- Φ and Φ^0 agree on X_Γ^0, and Φ, Φ^0 and Φ^1 agree on X_Γ^1
- Φ_t (respectively, Φ_t^0) maps X_Γ (respectively, X_Γ^0) continuously into X_Γ^1 for $t > 0$
- Φ, Φ^1 and Φ^0 have the same omega limit sets, compact attractors and equilibria.

Theorem 6.17 in [6] concludes, among other things, that the set of quasi-convergent points for each of the semiflows is residual in the appropriate space. In particular, $Q(\Phi^k)$ is residual in X_Γ^k for $k = 0, 1$. In fact, $Q(\Phi^k)$ is open and dense in X_Γ^k by Theorem 8. To see this, we need only note that X_Γ^1 imbeds continuously in $Z = X_\Gamma^0$, that axiom (K) holds by virtue of the properties itemized above, and that axiom (L) holds in Z for the reasons noted following the proof of Theorem 8.

6 Reaction-Diffusion Systems with No-Flux Conditions on a Convex Domain

Consider a reaction-diffusion system

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= d_i \Delta u_i + f_i(u), \quad i = 1 \ldots m,\ x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{3}$$

for a smooth domain $\Omega \subseteq \mathbb{R}^n$, a C^2 function $f : \mathbb{R}^m \to \mathbb{R}^m$, $d_i > 0,\ i = 1 \ldots m$, and an initial condition $u_0 \in C(\overline{\Omega}, \mathbb{R}^m)$. Kishimoto and Weinberger [10] showed that if Ω is convex, and assuming that $\partial f_i / \partial u_j > 0$ for all $i \neq j$, then any nonconstant equilibrium $\overline{u}$ is linearly unstable. A careful reading of the proof in that paper will show that in fact it is sufficient that f is cooperative and irreducible (Section 2). By making a linear change of variables, we may extend the following result to any system which is monotone with respect to one of the other orthants K. See [16].

Equilibria of (3), solutions of the associated elliptic boundary value problem, are known for having multiple and sometimes unexpected solutions. Not only is it possible for a strongly monotone reaction diffusion system to have several spatially nonhomogeneous equilibria, but it is in fact possible that there is a continuum of them [3]. The following application of Theorem 1 shows that the generic solution converges to a uniform (constant) solution.

Theorem 9 *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^2 function, and consider the dynamical system*

$$\frac{\partial u}{\partial t} = f(u). \tag{4}$$

Assume that this system is cooperative and irreducible and that all initial value problems have bounded solutions for $t \geq 0$. If the domain $\Omega \subseteq \mathbb{R}^n$ is convex, then the set of initial conditions u_0 of (3) whose solution $u(x,t)$ converges towards a uniform equilibrium is prevalent in $C(\overline{\Omega}, \mathbb{R}^m)$.

Proof. We need to show that all the general assumptions of the previous section are satisfied, as well as the hypotheses of Theorem 1. Clearly $X = C(\overline{\Omega}, \mathbb{R}^m)$ is a separable Banach space under the uniform norm with cone given by $C(\overline{\Omega}, \mathbb{R}^n_+)$. The fact that the time evolution operators generate a semiflow of compact operators with compact derivatives on X is well known in the literature; see for instance [6, 11, 12, 14]. The fact that $B = X$ follows from comparison with solutions of the ordinary differential equations (4); see e.g. Theorem 7.3.1 in [16]. To see that the system (3) has no non-irreducible equilibria, let $\hat{u}$ be an equilibrium of the system, and recall that the linearization around $\hat{u}$ is of the form

$$\frac{\partial u}{\partial t} = D\Delta u + M(x)u,$$

together with Neumann boundary conditions, where $M(x) = \partial f/\partial u(\hat{u}(x))$ and $D = \operatorname{diag}(d_1, \ldots d_m)$. According to Theorem 7.4.1 of [16], to prove that this system is strongly monotone it is enough to verify that the associated finite-dimensional system with no diffusion is monotone for every fixed value of $x \in \Omega$, and strongly monotone for at least one value of x. This therefore follows from the irreducibility assumption on the linearizations of (4).

By Theorem 1, C_s is prevalent in $B = X$. But by the main theorem in [10], any initial condition in C_s has a solution which converges towards an equilibrium which is uniform in space. This completes the proof. ∎

7 Appendix: Measurability

It is important to observe that in order to apply measure-theoretic arguments on Theorems 5, 6, and 7, one needs to prove first that the sets involved in each result are Borel measurable. This is carried out in the present section, where we assume throughout that X is a Borel subset of a *separable* Banach space $\mathbb{B}$, that $\Phi : \text{Dom}\ \Phi \to X$ is a continuous local semiflow defined on the open subset Dom Φ of $X \times R_+$ containing $X \times \{0\}$. For each $x \in X$, $\{t \geq 0 : (x,t) \in \text{Dom}\ \Phi\} = [0, \sigma_x)$. The set

$$Ext = \{x \in X : \sigma_x = +\infty\} = \cap_q \{x \in X : \sigma_x > q\},$$

where the intersection is taken over all positive rational q, is Borel since it is the countable intersection of open subsets of X. Therefore, we may as well rename $X = Ext$ and consider the global semiflow $\Phi : X \times \mathbb{R}_+ \to X$ where X is Borel. Given $p \in \mathbb{B}$ and $r > 0$, let $B_r(p) = \{x \in \mathbb{B} : |x - p| < r\}$.

Let $D \subseteq X$ be a closed set in X and $r \in R_+$, and consider the set

$$W(D,r) = \{x \in X \mid \Phi_t(x) \in D, \text{ for all } t \geq r\} = \bigcap_{q \in Rat,\ q > r} \Phi_q^{-1}(D),$$

where Rat denotes the rationals. The equality holds from the continuity of Φ. Since each operator Φ_q is continuous, $W(D,r)$ is a Borel measurable set.

In the following we assume only that Φ is a continuous semiflow on the closed set X.

Lemma 7 *If X is closed in $\mathbb{B}$, then the set B of the elements $x \in X$ with precompact orbit is Borel measurable.*

Proof. Note that B is the set of $x \in X$ such that $O(x)$ is totally bounded, and that a set S is totally bounded if and only if for every $\epsilon > 0$ there exists a finite collection of *closed* balls of radius less than ϵ, whose union contains S. Let $\{p_i\}_{i\in\mathbb{N}}$ be a countable dense set in $\mathbb{B}$ and let $\mathfrak{F}$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathfrak{F}$ is countable and

$$B = \bigcap_{n\in\mathbb{N}} \bigcup_{F\in\mathfrak{F}} W(S_{F,n}, 0),$$

where $S_{F,n} = \cup_{i\in F}\overline{B}_{1/n}(p_i)$. It is easy to see from this expression that B must be Borel measurable.

∎

Lemma 8 *Let $D \subseteq X$ be a closed set, and let $C(D) = \{x \in B \,|\, \omega(x) \subseteq D\}$. Then $C(D)$ is Borel measurable.*

Proof. Given a set $A \subseteq X$ and $\epsilon > 0$, let

$$A_\epsilon = \{x \in X \,|\, d(A, x) \leq \epsilon\},$$

which is a closed set by continuity of the function $d(\cdot, A)$. Then we can write

$$\{x \in B \,|\, \lim_{t\to\infty} d(\Phi_t(x), A) = 0\} = \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} W(A_{\frac{1}{m}}, k).$$

Finally, note that for any closed set $D \subseteq X$ and for any $x \in B$, it holds that

$$\omega(x) \subseteq D \Leftrightarrow \lim_{t\to\infty} d(\Phi_t(x), D) = 0.$$

The first statement follows. ∎

Corollary 1 *The set Q of quasiconvergent elements is Borel measurable.*

Proof. The proof follows immediately from the above result by noting that E is a closed set. ∎

It follows that the set $N = B - Q$ involved in the proof of Theorem 5 is measurable, by the previous Corollary and Lemma 7.

Lemma 9 *The set C of convergent elements is Borel measurable.*

Proof. Let (b_i) be a dense enumerable collection of elements of X. Then the statement follows from the equation

$$C = \bigcap_{k=1}^{\infty} \bigcup_{i=1,r=1}^{\infty} W(\overline{B}_{\frac{1}{k}}(b_i), r).$$

To see this, let first $x \in C$. Note that for any fixed k, there exists some b_i within $1/(2k)$ of $\omega(x)$, and that therefore $x \in W(\overline{B}_{\frac{1}{k}}(b_i), r)$ for some large enough r. Therefore for every fixed k, C is contained in the union of the RHS, and thus one direction is proven. Conversely, let x be in the RHS term. For every k, let a_k, r_k be such that $x \in W(\overline{B}_{\frac{1}{k}}(a_k), r_k)$; such sequences exist by hypothesis. Note that for $k_1 \neq k_2$, it must hold

$$W(\overline{B}_{\frac{1}{k_1}}(a_{k_1}), r_{k_1}) \cap W(\overline{B}_{\frac{1}{k_2}}(a_{k_2}), r_{k_2}) \neq \emptyset$$

In particular the sequence (a_k) is Cauchy, and it therefore converges towards a point $a \in \mathbb{B}$. Given $\epsilon > 0$, let k large enough that $1/k < \epsilon/2$ and that a_k lies within $\epsilon/2$ of a. By definition, $|x(t) - a| < \epsilon$ for $t \geq r_k$; we conclude that $x(t) \to a$, and therefore that $a \in X$ and $x \in C$. ∎

Lemma 10 *Assume that the time evolution operator is continuously differentiable. Then the set C_s is Borel measurable.*

Proof. Note first that the spectral radius function $\rho(T)$, though not a continuous function of the linear bounded operator $T : L(\mathbb{B}, \mathbb{B}) \to \mathbb{R}$ (see Kato [9]), is nevertheless a measurable function. To see this, simply write it as the pointwise limit of continuous functions as $\rho(T) = \lim_n \|T^n\|^{1/n}$. Fix now $t > t_0$, and define $\beta : X \to \mathbb{R}$, $\beta(z) := \rho(\Phi'_t(z))$. Since $z \to \Phi'_t(z)$ is a continuous function by hypothesis, it follows that β is measurable.

The next step is to note that the function $z \to \omega(z)$ (defined on C) is also measurable. To see this, write it as the pointwise limit of the continuous functions $\omega(z) = \lim_n \Phi_n(z)$. Thus, the function $z \to \beta(\omega(z))$ is itself measurable. But

$$C_s = \{z \in C \,|\, \beta(\omega(z)) \leq 1\},$$

and the proof is complete. ∎

Note that the continuous differentiability of the time evolution operators was only used to show that β is measurable; it would be sufficient to assume $z \to \Phi'_t(z)$ to be measurable, which should be satisfied in very large generality.

Lemma 11 *Let $A \subseteq B$ be compact. Then $SB(A)$ is Borel measurable.*

Proof. For every $\epsilon > 0$, there exists a finite collection R_ϵ of open balls of radius ϵ, such that i) each ball intersects A, and ii) the union of all balls contains A. Let $R = \cup_{n \in \mathcal{N}} R_{1/n}$. Then the set $\bigcup_{V \in R} W(V^C, 0)$ consists of the vectors $x \in X$ such that $a \notin \omega(x)$ for some $a \in A$. Consequently,

$$SB(A) = C(A) - \bigcup_{V \in R} W(V^C, 0),$$

and this set is also Borel measurable. ∎

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