Real Analysis (following Folland)
Fall Quarter 2020

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Lecture 1 (pp. 19-22) Week 1

Mathematics is the quantitative science *par excellence* and measurements are at its heart. It is therefore crucial to have well-defined concepts of length, area, and volume, depending on the dimension. In general we speak of the *measure* of *subsets* of a given main set X, be it $\mathbb{R}^n$ or some more general abstract space.

> **Question 1.** What are mathematical and practical reasons to have a concept of measure? Think of the examples you are familiar with.

> **Question 2.** Given a set X, what mathematical object (such as $group, field, function, ...$) would you use to define a measure for its subsets? Provide details and reasons.

We begin with $X = \mathbb{R}^n$ and attempt to construct a measure on it, an n-dimensional volume.

> **Question 3.** What properties suggested by your real world intuition would you require such a measure to possess?

> **Presentation 1.** Introduce your team members to the example showing that any measure on $\mathbb{R}$ with the "desired" properties cannot be defined for all subsets of $\mathbb{R}$ without contradictions. See p. 20 in the textbook.

> **Question 4.** What is a σ-algebra on a set X? Why do you think we require the validity of these axioms? What are equivalent characterizations and implied properties? What are simple examples of σ-algebras? In particular, verify that the collection of countable or co-countable sets is a σ-algebra.

> **Question 5.** Why is the intersection of σ-algebras itself a σ-algebra?

> **Question 6.** What is the σ-algebra $\mathcal{A}(\mathcal{E})$ generated by a subset $\mathcal{E} \subset 2^X = \mathcal{P}(X)$?

Lecture 2 (pp. 22-23) Week 1

Skip the definition of *elementary family* right after Lemma 1.6 and Proposition 1.7 (we'll do later).

Question 1. Given a topological space (X, τ_X), what is its Borel σ-algebra $\mathcal{B}_X$?

Question 2. In the definition of G_δ sets, why are intersections of open sets considered and not of closed ones? Similarly, why are open sets not considered in the case of F_σ sets? Can you exhibit simple examples of G_δ and F_σ sets?

Presentation 1. Give a detailed and complete proof of Proposition 1.2 on p. 22.

Presentation 2. Define the concept of product σ-algebra and discuss ways in which it can be generated. Discuss the special case of product Borel σ-algebras.

Question 3. Why do we need to assume countability in Proposition 1.3 on p. 23?

Lecture 3 (pp. 24-27) Week 2

Question 1. What are measurable spaces and measures? When is a measure finite, σ-finite, and semi-finite? What are simple examples of measures?

Presentation 1. State and prove Theorem 1.8 on pp. 25-26. Include examples/nexamples that show the necessity of the conditioned used.

Presentation 2. Define the concepts of null set and of completeness for a measure μ. State and fully prove the completion theorem (Theorem 1.9 on pp. 26-27).

Lecture 4 (pp. 28-30) Week 2

Question 1. What is an outer measure? What is the concept modeled on? What is the starting point in the construction of an outer measure?

Presentation 1. State and prove Proposition 1.10 on p. 29. Why isn't an outer measure a measure in general?

Question 2. Given an outer measure μ^*, how is μ^*-measurability defined?

Presentation 2. Explain the construction of a complete measure from an outer measure (Carathéodory's Theorem on p. 29) by giving the full proof.

Lecture 5 (pp. 30-32) Week 3

Question 1. What is a premeasure on an algebra of sets? What is the relation between a premeasure and the outer measure it induces? Give a proof the statements you make.

Presentation 1. State&prove Theorem 1.14 on p. 31 about the construction of a measure starting from a premeasure.

Lecture 6 (pp. 32-35) Week 3

Question 1. What is the distribution function of a Borel measure on $\mathbb{R}$?

Presentation 1. Define the concept of elementary family and state&prove Proposition 1.7 on p. 23-24.

Presentation 2. Construction of Borel-Stieltjes measures.

- Obtain a premeasure on the algebra of finite disjoint unions of half-open intervals (Proposition 1.15 on pp. 33-34)
- Conclude using Theorem 1.14 (Theorem 1.16 on p. 35).

Question 2. What is the Lebesgue-Stieltjes measure associated to a right-continuous, non-decreasing function $F : \mathbb{R} \to \mathbb{R}$?

Lecture 7 (pp. 35-37) Week 4

Presentation 1. State&prove (complete the proof if necessary) the "regularity" properties of Lebesgue-Stieltjes measures, Theorems 1.18 and 1.19 on pp. 36-37, which require Lemma 1.17.

Reflection Question. What are the steps in the construction of a measure and why are they all necessary?

Lecture 8 (pp. 36-39) Week 4

Presentation 1. Define the Lebesgue measure on $\mathbb{R}$ and verify its translation invariance.

Question 1. Are there other translation invariant Lebesgue-Stieltjes measures on $\mathbb{R}$?

Presentation 2. Discuss the examples of pp. 38-39 about the relationship between topological and measure theoretic properties of subsets of the real line. Include a discussion of the Cantor sets and its generalization.

Lecture 9 (pp. 43-46) Week 5

Question 1. What is a measurable function $f : X \to Y$ between measurable spaces $(X, \mathcal{M})$ and $(X, \mathcal{N})$? How do you check measurability of a function? What about the case of topological spaces with their Borel σ-algebras?

Presentation 1. Discuss the properties of measurability of functions. State&prove Propositions 2.4, 2.6, and 2.7. on pp. 44-45.

Lecture 10 (pp. 46-48) Week 5

Question 1. What are the positive f^+ and the negative part f^- of a real-valued function $f : X \to \mathbb{R}$ on a set X? How do they relate to f and to $|f|$? What are simple functions $f : X \to \mathbb{R}$? If $(X, \mathcal{M})$ is a measurable space, when is a simple function $f : X \to \mathbb{R}$ measurable? What is the standard representation of a simple function?

Presentation 1. State&prove Theorem 2.10 as well as Propositions 2.11 and 2.12 on pp. 47-48.

Lecture 11 (pp. 49-51) Week 7

Presentation 1. Definition of integral for measurable, nonnegative simple functions and nonnegative measurable functions and basic properties with proof.

Presentation 2. State&prove Theorems 2.14, 2.15, Proposition 2.16, and Corollary 2.17.

Question 1. What property of measures is the most important ingredient of the Monotone Convergence Theorem 2.14?

Question 2. Is the monotonicity assumption in the Monotone Convergence Theorem necessary? Provide evidence for your answer.

Lecture 12 (pp.52-53) Week 7

Presentation 1. State&prove Fatou's Lemma, Corollary 2.19, and Proposition 2.20.

Question 1. What is the definition of integral for measurable real- or complex-valued functions? Why is this definition chosen? Discuss alternative definitions and drawbacks.

Presentation 2. Show that the space of integrable functions, denoted by $\mathcal{L}^1(X, \mu)$, is a vector space.

Question 2. Is $\|f\|_1 = \int |f| \, d\mu$ a norm on $\mathcal{L}^1(X, \mu)$? Explain why and discuss possible fixes.

Lecture 13 (pp. 53-55) Week 8

Presentation 1. State&prove Propositions 2.22 and 2.23.

Question 1. Why can we identify $\mathrm{L}^1(X,\mu)$ and $\mathrm{L}^1(X,\overline{\mu})$?

Presentation 2. State&prove Lebesgue's Dominated Convergence Theorem (LDCT) and Theorem 2.25.

Lecture 14 (pp. 55-58) Week 8

Presentation 1. State&prove that integrable simple functions are dense in $\mathrm{L}^1(X,\mu)$. If $X=\mathbb{R}$ and μ is a Lebesgue-Stieltjes measure, continuous functions with compact support are also dense in $\mathrm{L}^1(\mathbb{R},\mu)$, where the support of a function $f : X \to \mathbb{R}$ on a topological space X is given by

$$\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}}$$

Presentation 2. Discuss the application of LDCT to the differentiability of parameter dependent integrals.

Discussion. Consider the relation between the Riemann integral and Lebesgue integration.

Lecture 15 (pp. 60-62) **Week 9**

Presentation 1. Discuss pointwise, uniform, L^1, and in measure convergence and their relations to each other by proving implications or giving counterexamples.

Presentation 2. State&prove Egoroff's Theorem 2.33.

Lecture 16 (pp. 64-66) **Week 9**

Presentation 1. Give a detailed description of the construction of the product measure $\mu_1 \times \mu_2$ on $\mathcal{M}_1 \otimes \mathcal{M}_2$ of two measures μ_i on $(X_i, \mathcal{M}_i)$, $i = 1, 2$.

Presentation 2. Introduce and prove the Monotone Class Lemma 2.35.

Lecture 17 (pp. 66-68) — Week 10

Presentation 1. State&prove Theorems 2.36 and 2.37

Lecture 18 (Recap) — Week 10

Presentation 2. Give an overview of the steps we followed in the construction of an integral pointing out the main issues and the important outcomes.