Multivariable Calculus–Math 2D–Spring 2022

Instructor
Patrick Guidotti

Teaching Assistants

Discussions B1&B2	Matthew West.
Discussions B3&B4	Bryan Dimler.
Discussion B5	Devansh Saluja.

Learning Assistants

Discussion B1 (10am)	Timothy Cho, David Militante.
Discussion B2 (3pm)	Jiayi Fang, Boya Shao, Yuheng Tian.
Discussion B3 (5pm)	Timothy Cho, Jiayi Fang, Yuheng Tian.
Discussion B4 (6pm)	Timothy Cho, Yiting Wang, Yuheng Tian.
Discussion B5 (4pm)	Yingqi Rong, Boya Shao.

1 Prerequisites

Online material is available to help you review the prerequisites for this course.

2 Overview

The syllabus is based on and refers to the textbook *Calculus: Early Transcendentals* by James Stewart, Daniel K. Clegg, and Saleem Watson (9th Edition, Cengage).

Week/Day	Monday		Wednesday		Friday	
Week 1	**Mar 28**	Section 10.1	**Mar 30**	Sections 10.1-10.2	**Apr 1**	Section 10.2
Week 2	**Apr 4***	Section 12.1	**Apr 6**	Sections 12.1-12.2	**Apr 8**	Section 12.2
Week 3	**Apr 11***	Section 12.3	**Apr 13**	Section 12.4	**Apr 15**	Section 12.5
Week 4	**Apr 18***	Section 12.6	**Apr 20**	Review	**Apr 22**	Midterm I
Week 5	**Apr 25***	Section 13.1	**Apr 27**	Section 13.2	**Apr 29**	Section 13.3
Week 6	**May 2***	Section 14.1	**May 4**	Section 14.2	**May 6**	Section 14.3
Week 7	**May 9***	Section 14.4	**May 11**	Review	**May 13**	Midterm II
Week 8	**May 16**	Section 14.5	**May 18**	Section 14.6	**May 20**	Section 14.7
Week 9	**May 23***	Section 14.8	**May 25**	Section 15.1	**May 27**	Section 15.2
Week 10	**May 30***	Memorial Day	**Jun 1**	Review	**June 3**	Review

*** Homework due.**

3 Office Hours

Patrick Guidotti	Wed & Fri 2-2:50pm (RH 410F)
Bryan Dimler	Tue & Thu 12-2pm on Zoom
Devansh Saluja	Wed 12-2pm (RH 248) and by appointment (2 hours)
Matthew West	Tue & Thu 1-3pm (RH 440V)

4 Format

- The course is taught in a flipped format. This means that you watch the videos lectures and examples about the topics indicated in the above syllabus table before coming to class. Notice that the videos follow the textbook in terms of the topics covered but the presentation deviates from that of the book. I would recommend that you use both so as to obtain a deeper understanding of the material. Classtime is then devoted to discussing the topics of the day where I will use the questions you post on Canvas to identify the isses that prove the most challenging. Learning Assistants will be available for support in the discussion sections.
- Homework is assigned weekly through Canvas/Cengage's Webassign. No homework is due the weeks following a midterm.
- Three examinations are scheduled as indicated in the syllabus table for the midterms or in the schedule of classes for the final (Mon Jun 6 4:00-6:00pm). The final examination is comprehensive and covers the following sections of the textbook: 10.1-10.2, 12.1-12.6, 13.1-13.3, and 14.1-14.8.
- Quizzes are administered using Canvas/Respondus and in person during discussion.

5 Policies

The final gade of the course is computed as follows:

- Midterms: 20% each
- Final: 30%
- Homework: 20%
- Quizzes: 10%

There will be NO makeup midterm examinations. If any emergency were to arise which makes it impossible for you to take one of the scheduled midterms you will need to provide documentation for the cause of the emergency and obtain the instructor's approval. If one or both midterms are missed due to an approved emergency, its or their weights will be added to the weight of the final.

Cheating will be accepted under no circumstance and whoever is caught cheating will automatically receive an F in the course and his/her Dean will be notified. Only fully documented and justified absences from exams will be accepted. Unjustified no shows will result in a score of 0 in the corresponding examination. If two or more examinations are missed, the course's grade will be F.

6 Study Tips

Here is a few tips for you to consider as you develop a strategy to study the challenging material of this course.

- When you practice, start from a problem and try to understand what it is asking. By this I do not mean try to figure out which formula you can use, but rather try to visualize the issues and find the related concepts that can help you determine a strategy to approach the problem.
- In order to avoid the very real possibility of not having to think about what is asked, try to solve problems before you watch the corresponding video lecture or read the textbook. In a test there are inevitably problems about various topics and, unlike solving homework problems, it may not be immediately obvious which formula to use since the problem does not come with the additional information of being about a specific topic. A practical tip: screenshot a few typical problems from different sections, scramble them, and combine them into a file. Then try to solve them no longer knowing where they came from.

  Confucius said "*Learning without thought is labor lost; thought without learning is perilous*".

  I would rephrase the first part as "mindless computing/practicing is not effective".
- The video lectures are mainly focused on the important concepts and should be used as a guide to what matters and what to concentrate your attention on. They are relatively short but not easily digested. It is highly recommended that you watch them with paper and pencil and fill in the gaps in the algebraic steps and even simply redo the calculations for yourself.
- A good way to test whether you really understand something is to explain it to someone who does not know anything. That forces you to stick to the essential and not to dwell on some unimportant detail. You can try this out among yourselves if you have a study buddy or with some other friend of yours.
- Prepare a cheat-sheet or a summary of the material. This forces you to obtain an interconnected map of the important concepts. It has to read like a meaningful story (and not just a list of formulae). This makes it possible for you to structure the material you are learning and make sense of it. This is also a way to remember the material forever and be able to access it later when you need it.
- Believe or not, if you focus on the concepts and try to genuinely understand the arguments, you will end up remembering the formulae. Not because you memorized them, but because they make sense to you.
- As much as you can, try to find your own examples of the use of the important concepts because those are the ones you understand the best.

7 Weekly Schedules

Week 1

Use Desmos for plotting curves.

Section 10.1 Curves and parametrizations

Video Lecture. Parametrization of Plane Curves.

Video Example. Lines in the Plane.

Video Example. The Circle.

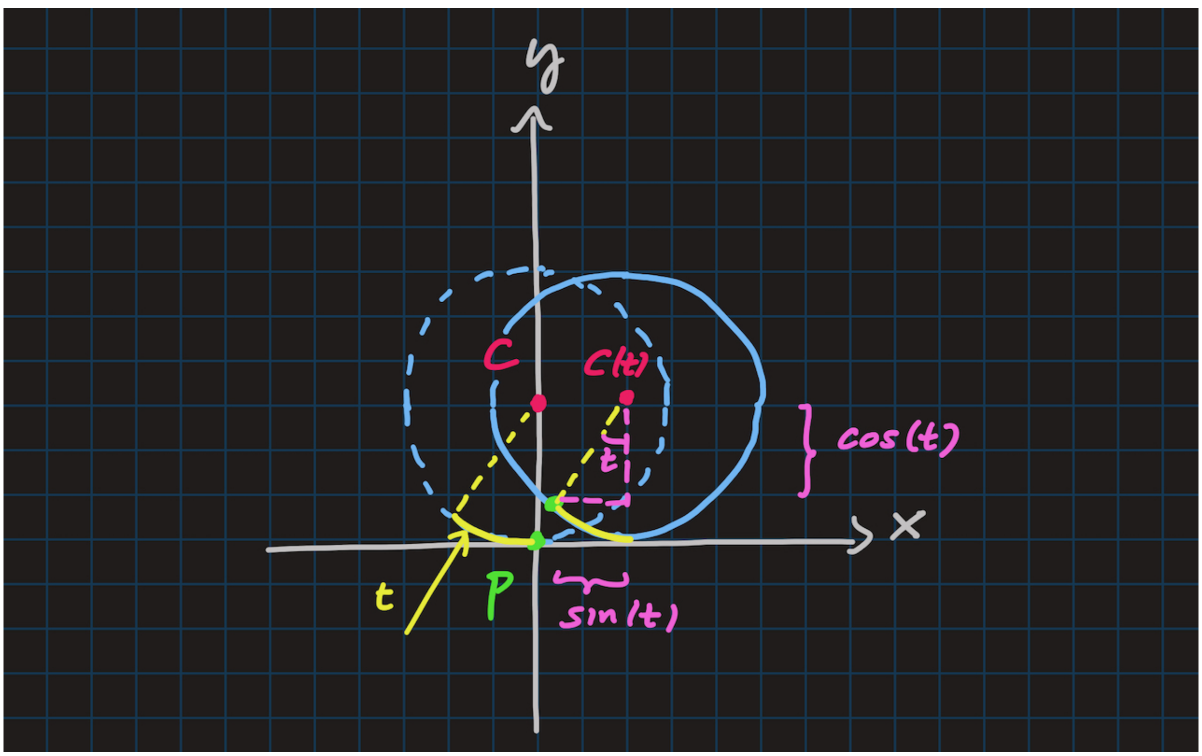
Skip *Graphing Parametric Curves with Technology.*

Skip *Families of Parametric Curves.*

Section 10.2 Calculus with parametric curves

Video Example.

The Cycloid. Consider a circle of radius 1 initially with center at the point $(0, 1)$ as it rolls rightwardly on the x-axis. The cycloid is the curve traced by the point P on the circle initially at the origin as the circle rolls. We try to parametrize its location as a function of the length t of the arc of circle that came into contact with the x-axis, which is the same as the clockwise angle of rotation experience by the circle.



It also corresponds to distance the center has moved to the right, i.e. $C(t) = (t, 1)$ where $C(t)$ is the center's location. The point P will now have rotated to the location $\big(-\sin(t), -\cos(t)\big)$ relative to the center $C(t)$. Thus the path $\gamma(t)$ of the point P is described by

$$\gamma(t) = (t, 1) + \big(-\sin(t), -\cos(t)\big) = \big(t - \sin(t), 1 - \cos(t)\big)$$

Video Example. The Cycloid.

Video Lecture. Parametrized Plane Curves: Tangents and Length.

Correction

There is a typo in this video at minute 7:26. The point at which we would like to compute the tangent vector is $(-\sqrt{2}/2, -\sqrt{2}/2)$ *and not* $(-\sqrt{2}/2, \sqrt{2}/2)$ *as stated. With this fix everything else is correct.*

Video Example. Tangent Vectors to Parametrized Plane Curves.

Skip *Areas.*

Skip *Surface Area.*

Homework Assignment 1 **Due Apr 4 by 11:59pm**

Week 2

Use Geogebra to plot surfaces and trace curves in space.

Section 12.1 Three-Dimensional Coordinate Systems

Video Lecture. Coordinates in 3D space.

Video Example. Simple 3D Shapes.

Section 12.2 Vectors

Video Lecture. Vectors and Their Components.

Video Lecture. Properties of Vector Addition and Scalar Multiplication.

Reading assignment: *Applications of Vectors (p. 843 of the textbook).*

A comment about vectors and coordinates

Vectors have coordinates which you compute by finding the (so-called linear) combination of reference vectors. Call them e_1, e_2, e_3 and notice that they are arbitrary but usually chosen to be orthogonal to each other. This means that you can write any vector $u \in V$ in the set of all 3D vectors as

$$u = u^1 e_1 + u^2 e_2 + u^3 e_3$$

for numbers u^1, u^2, u^3. This means that we think of the vectors e_1, e_2, and e_3 as steps and the number u^i indicates the number of steps e_i ($i = 1, 2, 3$) we need to take in order to go from the tail to the tip of the vector u. We say that $\langle u^1, u^2, u^3 \rangle$ are the components of u with respect to e_1, e_2, e_3. Notice that V is the set of all vectors (arrows), while their components are triples of numbers in $\mathbb{R}^3$. In particular the set of vectors is NOT $\mathbb{R}^3$ but it can be represented by it once we have chosen e_1, e_2, e_3 and can compute components for vectors.

While we picture vectors in a 3D space, they do not have location! They only encode displacement, that is, their only give us information on how to get from they tail to their tip. We of course also imagine 3D space to be filled with points; in fact, we think of it as a set of points. In order to locate points other than pointing to them with our fingers we need three steps (or direction and size of stepping), say f_1, f_2, f_3, and a specially designated point O which we call the origin (it can be chosen anywhere). Any other point P can then be decribed in reference to O by measuring how we can get to it from O by taking steps f_1, f_2, f_3. If x^1 steps f_1, x^2 steps f_2, and x^3 steps f_3 are needed, we say that (x^1, x^2, x^3) are the coordinates of P. Again, if we think of 3D space as a set of points, by choosing a point O and steps f_1, f_2, f_3, we can describe its points by triples of numbers. In a way, we identify it with or model it by $\mathbb{R}^3$.

Now, while vectors and points are different objects and live in different sets, we can ask if there is any relationship between (x^1, x^2, x^3) and $\langle x^1, x^2, x^3 \rangle$. If the steps f_1, f_2, f_3 are represented by the vectors e_1, e_2, e_3 used in the description of vectors above, then the point P with coordinates (x^1, x^2, x^3), can be thought of as being at the tip of the vector u_P with components $\langle x^1, x^2, x^3 \rangle$, if we "place its tail" in the origin O. This is visually satisfactory and intuitive but does not change the fact that vectors do not have a location and, even more, do not even live in the same space (set) as points. It can be confusing that two distinct sets of different objects can be represented by $\mathbb{R}^3$ but one has to remember that the same triple of numbers can be used to describe both points and vectors. The difference between points and vectors is evident for instance in the fact that you can add vectors (and multiply them by scalars) but you can certainly not add points nor stretch or shrink them!

Homework Assignment 2 **Due Apr 11 by 11:59pm**

Week 3

Section 12.3 The Dot Product

Video Lecture. The Dot Product.

Video Example. The Distance between a Point and a Line in the Plane.
There is a typo in the above video at minute 6:20 as I point verbally out in the video. The final formula for the distance misses a square root in the denominator and should read

$$|P_1Q_1| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Reading assignment: *Application: Work (between pages 852 and 853 of the textbook).*

Section 12.4 The Cross Product

Video Lecture. The Cross Product.

Video Lecture. Properties of the Cross Product.

Reading assignment: *Application: Torque (between pages 861 and 862 of the textbook).*

Section 12.5 Equations of Lines and Planes

Video Lecture. Equations of Lines and Planes.
There is a misprint at about minute 9:45 in this video. The expression $(ta + sb) \times (a \times b) = 0$ should read

$$(ta + sb) \cdot (a \times b) = 0.$$

Video Example. Working with Lines.

Video Example. Working with Planes.

Video Example. The Distance between two Lines.

Homework Assignment 3 **Due Apr 18 by 11:59pm**

Week 4

Section 12.6 Quadric Surfaces

Video Lecture. Quadric Surfaces.
In this video at around minute 2:00 there is a typo. The final equation in the $\bar{x}, \bar{y}$ variables should read $z + \frac{1}{4}\bar{x}^2 - \frac{1}{4}\bar{y}^2$. It would be of type II with $A = \frac{1}{4}$, $B = -\frac{1}{4}$, and $I = 1$.

Video Examples using Geogebra.

8 Midterm I Solutions

Problem 1

a. Is the point $(\frac{3}{2}, 0)$ on the curve parametrized by $\Big((2-t)\cos(2\pi t), (2-t)\sin(2\pi t)\Big)$ for $t \in [0, 2]$?

Let $\Big(x(t), y(t)\Big) = \Big((2-t)\cos(2\pi t), (2-t)\sin(2\pi t)\Big)$ for $t \in [0, 2]$. Then if $(\frac{3}{2}, 0) = \Big(x(t), y(t)\Big)$ for some t, it must hold that either $(2-t) = 0$ or $\sin(2\pi t) = 0$, which only happens at $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. Then

$$x(0) = 2,\ x(\frac{1}{2}) = -\frac{3}{2},\ x(1) = 1,\ x(\frac{3}{2}) = -\frac{1}{2},\ \text{and } x(2) = 0.$$

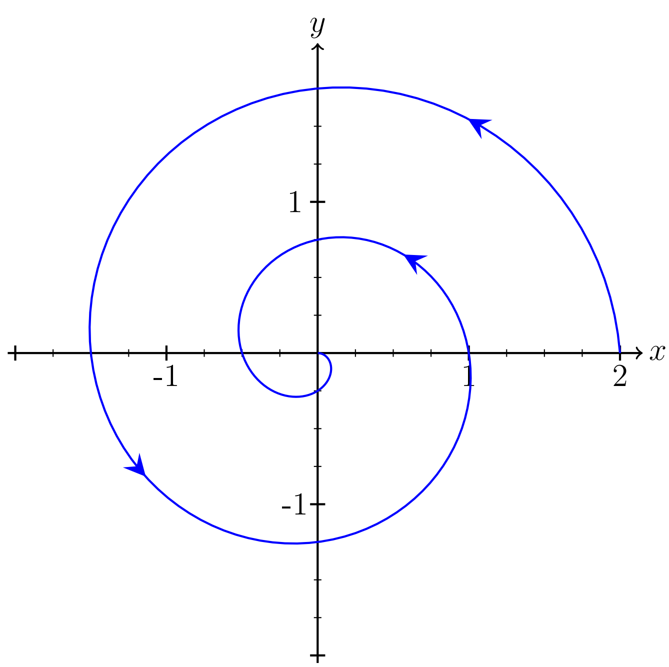
It follows that the point is not on the curve.

b. What is a vector tangent to this curve at its endpoint? Show your calculation.

The tanget to the curve at the point $\Big(x(t), y(t)\Big)$ is given by $\Big(x'(t), y'(t)\Big)$ so that

$$\Big(x'(2), y'(2)\Big) = (2-2)2\pi\Big(\sin(4\pi), \cos(4\pi)\Big) - \Big(\cos(4\pi), \sin(4\pi)\Big) = (-1, 0).$$

c. Sketch the curve. Indicate with an arrow the direction in which the parametrization traverses the curve.



Problem 2

a. What is a vector perpendicular to the plane $\mathcal{P}$ with equation $x + 2y - 3z + 5 = 0$?

One is given by $\vec{n} = \langle 1, 2, -3\rangle$.

b. Explain why your chosen vector is perpendicular to $\mathcal{P}$ using mathematical identities/equations and not English sentences.

Any vector parallel to the plane can be obtained as the displacement vector $\overrightarrow{P_0P_1}$ determined by any two points P_0, P_1 in the plane. If (x_0, y_0, z_0) and (x_1, y_1, z_1) are the coordinates of P_0 and P_1, then

$$\begin{aligned}\vec{n} \cdot \overrightarrow{P_0P_1} &= \langle 1, 2, -3\rangle \cdot \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle = x_1 + 2y_1 - 3z_1 - (x_0 + 2y_0 - 3z_0) \\ &= -5 - (-5) = 0\end{aligned}$$

c. How far is the point P with coordinates $(1, 1, 1)$ from the plane $\mathcal{P}$?

First we compute the orthogonal projection Q of P onto the plane by finding $t \in \mathbb{R}$ such that

$$Q = P + t\vec{n} \in \mathcal{P}.$$

This gives $1 + t + 2(1 + 2t) - 3(1 - 3t) + 5 = 0$, so that $t = -\frac{5}{14}$. Then the desired distance d is given by

$$d = \left|\overrightarrow{PQ}\right| = \frac{5}{14}\sqrt{1 + 4 + 9} = \frac{5}{\sqrt{14}}.$$

Problem 3

a. Let $\big(x(t), y(t), z(t)\big)$, $t \in [0, 3]$, be a parametrization of a space curve $\mathcal{C}$. What is the parametrization of $\mathcal{C}$ that goes through it in the opposite direction and at twice the speed?

It is given by

$$\Big(x(3 - 2t), y(3 - 2t), z(3 - 2t)\Big),\ t \in [0, \frac{3}{2}].$$

b. Let $\big(x(t), y(t)\big)$, $t \in [0, 1]$, be a parametrization of a curve $\mathcal{C}$ in the plane. Give a parametrization of the line orthogonal to $\mathcal{C}$ at the point where $t = \frac{1}{2}$.

The tangent to the curve at $t = \frac{1}{2}$ is given by $\big\langle x'(\frac{1}{2}), y'(\frac{1}{2})\big\rangle$ and the vector $\big\langle -y'(\frac{1}{2}), x'(\frac{1}{2})\big\rangle$ is orthogonal to it so that the desired line can be parametrized by

$$\Big(x(\frac{1}{2}), y(\frac{1}{2})\Big) + t\Big\langle -y'(\frac{1}{2}), x'(\frac{1}{2})\Big\rangle,\ t \in \mathbb{R}.$$

Problem 4

a. In which point does the line parametrized by $(1, 0, 1) + t\langle 1, 1, 1\rangle$ for $t \in \mathbb{R}$ intersect the plane with equation $x - y + z = 1$? Show your work.

We determine the parameter t by enforcing the plane equation

$$(1 + t) - (t) + (1 + t) = 1, \text{ so that } t = -1,$$

and the desired point is $(0, -1, 0)$.

b. Explain why the line $\mathcal{L}$ parametrized by $(0, 0, 1) + t\langle 2, 1, 3\rangle$ for $t \in \mathbb{R}$ is parallel to the plane $\mathcal{P}$ with equation $2x + 5y - 3z = 4$.

This is due to the fact that the direction vector $\langle 2, 1, 3\rangle$ of $\mathcal{L}$ is orthogonal to the normal vector of $\mathcal{P}$ as follows from

$$\langle 2, 1, 3\rangle \cdot \langle 2, 5, -3\rangle = 4 + 5 - 9 = 0.$$

c. What distance are the line $\mathcal{L}$ and the plane $\mathcal{P}$ (from b.) from one another? Explain how you arrive at your answer.

It is enough to choose any point on the line $\mathcal{L}$, say $P = (0, 0, 1)$, obtained for $t = 0$, and compute the distance d from it to $\mathcal{P}$. Using the formula learned in the course

$$d = \frac{|-3-4|}{\sqrt{4+25+9}} = \frac{7}{\sqrt{38}}$$

Problem 5

a. What is a normal vector to the plane containing the points with coordinates $(1, 2, 3)$, $(2, 3, 1)$,and $(3, 2, 1)$? Explain your steps.

We use the points (call them P, Q, and R) to find two vectors parallel to the plane

$$\overrightarrow{PQ} = \langle 1, 1, -2 \rangle \text{ and } \overrightarrow{PR} = \langle 2, 0, -2 \rangle.$$

Their cross product yields a vector that is normal to the plane

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, -2, -2 \rangle$$

b. Are the planes with equations $x + y + z = 7$ and $x - 2y + z = 0$ orthogonal? Justify your answer.

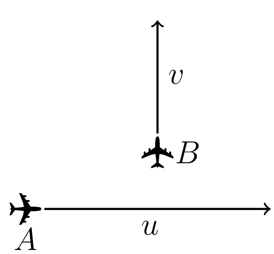
Taking the inner product of the normal vectors gives

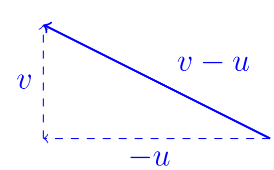
$$\langle 1, 1, 1 \rangle \cdot \langle 1, -2, 1 \rangle = 1 - 2 + 1 = 0.$$

Thus the two planes are orthogonal since their normal vectors are.

Bonus Problem

Imagine that you are seated on plane A that is moving with velocity u and looking out of the window. You see plane B, which is flying with velocity v (at a different altitude!). At what velocity does plane B appear to be moving to you? Draw your answer as a velocity vector and explain how you obtain it.





Since you are moving away at velocity u from plane B, the latter will appear to you as moving with velocity $-u$. Adding this to its cruising velocity yields its apparent velocity $v - u$.

Week 5

Section 13.1 Vector Functions and Space Curves

Skip *Using Technology to Draw Space Curves.*

Video Lecture. Vector Functions.

Video Lecture. Continuity of Vector Functions.

Video Examples. Vector Functions: Computing Intersection Curves.
There is a typo in this video at around time 5:50. The y-slice of the cylinder $(x-2)^2+z^2=4$ is parametrized by $\big(2+2\cos(t), y, 2\sin(t)\big)$ and not by $\big(2+\cos(t), y, \sin(t)\big)$ as stated in the video since the radius of the (base) circle is 2 and not 1.

Section 13.2 Derivatives and Integrals of Vector Functions

Video Lecture. Derivatives and Integrals.

Video Examples. Calculus with Vector Functions.

Section 13.3 Arc Length and Curvature

Video Lecture. Arc Length.

Video Examples. Computing Arc Length.

Video Lecture. Curvature.

Video Examples. Computing Curvature.

Skip *The Normal and Binormal Vectors* as well as *Torsion*.

Homework Assignment 5 **Due May 2 by 11:59pm**

Week 6

Section 14.1 Functions of Several Variables

Video Lecture. Functions of two or more Variables.

Video Examples. Visualizing Functions of Two Variables.

Section 14.2 Limits and Continuity

Video Lecture. Limits and Continuity.
There is a typo in this video at about minute 6:58 when the function $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is rewritten in polar coordinates. The expression $\frac{\sin(r)}{r}$ should read $\frac{\sin(r^2)}{r^2}$. The limit is not affected.

Video Examples. Computing Limits and Checking Continuity.

Video Lecture. Properties of Limits.

Section 14.3 Partial Derivatives

Video Lecture. Directional and Partial Derivatives.

Video Examples. Computing Partial and Directional Derivatives.

Skip *Partial Differential Equations.*

An example where $\partial_{xy} f \neq \partial_{yx} f$

Take the function defined by

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Then we have that (check for yourself):

$$f_{xy}(0, 0) = 1 \text{ and } f_{yx}(0, 0) = -1.$$

Homework Assignment 6 **Due May 9 by 11:59pm**

Week 7

Section 14.4 Tangent Planes and Linear Approximations

Video Lecture. Tangent Planes and Linear Approximations.

Video Examples. Differentiability and linear Approximation.

Reading assignment *Differentials*.

Midterm Examination

The midterm will cover all the material presented in the video lectures from Week 4 (included) through Week 6 (included). In terms of textbook sections, it will cover 12.6, 13.1-13.3, and 14.1-14.3.

While the midterm will test both your understanding of the concepts and ability to perform calculations, it won't require you to demonstrate an ability to give proofs. You are, however, expected to be able to justify your steps and calculations. Here are a few questions to be pondered in preparation for the test.

- What is a quadric surface? What are simple examples? What are corresponding two dimensional sets (curves)?
- What is a plane or space curve? How can it be described mathematically? When is a curve smooth?
- What is a parametrization of a curve? How can the tangent vector to a smooth curve at one of its points be computed? How does it relate to the line tangent to the curve at the same point?
- How is the length of a smooth curve computed? What is the arc length parametrization of a curve? How do you know if a curve is arc length parametrized?
- What is the curvature of a smooth curve? How does it relate to arc length and the unit tangent vector? How can it be computed either given the unit tangent vector or given a parametrization?
- What is a function of 2 or 3 variables? If it is given by some expression, what is its domain?
- What are limits and what are they useful for? How do you go about determining whether they exist or not and what they are if they do?
- What does it mean for a function of several variables to be continuous at an argument and how do you check continuity at that argument? Think of "real life" examples of functions of several variables and what their continuity or discontinuity mean.
- What are partial and directional derivatives of a function? What information do they provide about the behavior of the function? How are they computed? Think of "real life" examples and relate them to the concepts.

Try to answer these questions both by fleshing out the corresponding concepts with concrete examples and abstracting from the examples. Always try to see the structure behind an example, in particular as it relates to the concepts it exemplifies. Make an effort to distinguish between example specific features and more fundamental features that are valid for a whole class of examples and illustrate the abstract concepts.

Solution to Some Problems

There was not enough time to go over all the problems I had selected in review and I am posting the solutions here.

Problem
Parametrize the set obtained as the intersection of the surface with equation $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.
Solution: If a point belongs to both sets it satisfies both equations, so we plug the scond equation into the first to obtain

$$1 + y = \sqrt{x^2 + y^2}.$$

Next we square both side of the identity to see that

$$1 + 2y + y^2 = x^2 + y^2 \text{ or } y = \frac{1}{2}x^2 - \frac{1}{2}.$$

This yields a parametrization of the curve in the form

$$\left(x, \frac{1}{2}x^2 - \frac{1}{2}, \frac{1}{2}x^2 + \frac{1}{2}\right),\ x \in \mathbb{R},$$

using again that $z = 1 + y$.

Problem
Find a parametric representation of the line tangent to the curve of intersection of the ellipsoid $9x^2 + 2y^2 + z^2 = 58$ and the plane $y = 3$ at the point $(2, 3, 2)$.
Solution: The intersection is an ellipse with equation

$$9x^2 + z^2 = 40,$$

using that $y = 3$ and the ellipsoid's equation. We can use implicit differentiation to obtain the slope of the tangent to the point $(2, 2)$ in the xz-plane

$$18x + 2z\frac{dz}{dx} = 0 \text{ or } \frac{dz}{dx} = -\frac{18x}{2z},$$

which yields $\frac{dz}{dx} = -\frac{36}{4} = -9$ at $(x, z) = (2, 2)$. Thus a tangent vector is $\langle 1, -9\rangle$ in the xz-plane or $\langle 1, 0, -9\rangle$ in 3D considering that the curve lies on the $y = 3$ plane. Thus the desired parametric representation is given by

$$\Big(x(t), y(t), z(t)\Big) = \Big(2 + t, 3, 2 - 9t\Big),\ t \in \mathbb{R}.$$

9 Midterm II Solutions

Problem 1

Consider the curve $\mathcal{C}$ parametrized by $r(t) = \big(t\cos^2(2\pi t), -t, 2t\sin^2(2\pi t)\big)$ for $t \in [-1, 1]$.

a. In which plane does $\mathcal{C}$ lie? Give the plane's equation and explain why.

b. Where does $\mathcal{C}$ intersect the plane with equation $2x + z + 1 = 0$? Show and expain your work.

c. Compute the unit tangent vector to $\mathcal{C}$ at the point with cordinates $(0, 0, 0)$.

Solution:

a. If the curve lied in the plane $ax + by + cz + d = 0$, then, for all $t \in [-1, 1]$, we would have that

$$at\cos^2(2\pi t) - bt + c2t\sin^2(2\pi t) + d = 0.$$

While we could derive a system for a, b, c, d by evaluating the equation at various $t \in [-1, 1]$, we can guess that $a = b = 1$, $c = 1/2$, and $d = 0$ is a solution. Thus the plane has equation $x + y + z/2 = 0$.

b. The coordinates (x, y, z) of the point of intersection have to be on the curve, i.e. $x = t\cos^2(2\pi t)$, $y = -t$, and $z = 2t\sin^2(2\pi t)$ and satisfy the plane equation as well

$$2t\cos^2(2\pi t) + 2t\sin^2(2\pi t) + 1 = 0.$$

Thus $t = -1/2$ and the intersection point is $(-1/2, 1/2, 0)$.

c. The tangent vector to the curve at $t = 0$ is given by

$$\begin{aligned} r'(0) &= 0 \cdot \frac{d}{dt}\big(\cos^2(2\pi), -1, \sin^2(2\pi t)\big)\Big|_{t=0} + \big(\cos^2(2\pi 0), -1, 2\sin^2(2\pi 0)\big) \\ &= (1, -1, 0), \end{aligned}$$

which has length $\sqrt{2}$. The unit tangent vector is thus $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle$.

Problem 2

Consider the curve $\mathcal{C}$ parametrized by $r(t) = \big(\sqrt{2}t, e^t, e^{-t}\big)$ for $t \in [0, 1]$.

a. Compute the arc length $s(t)$ of $\mathcal{C}$ starting at the point $(0, 1, 1)$.

b. Compute the unit tangent vector $T(t)$ along $\mathcal{C}$.

c. Compute the curvature κ of $\mathcal{C}$ where $t = 0$. Indicate which representation of curvature you are using (recall that curvature is defined as $\left|\frac{dT}{ds}\right|$ where s is arc length).

Solution:

a. We first compute

$$r'(t) = \big(\sqrt{2}, e^t, -e^{-t}\big) \text{ and } |r'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t},\ t \in [0, 1].$$

Then we have that arc length is given by

$$s(t) = \int_0^t (e^s + e^{-s})\, ds = e^t - 1 - (e^{-t} - 1) = e^t - e^{-t},\ t \in [0, 1]$$

b. The unit tangent vector is given by

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{e^t + e^{-t}}\big(\sqrt{2}, e^t, -e^{-t}\big),\ t \in [0, 1].$$

c. Curvature is given by

$$\Big|\frac{dT}{ds}\Big| = \Big|\frac{dT}{dt}\Big|\Big|\frac{dt}{ds}\Big|,$$

where $\frac{dt}{ds} = 1/\frac{ds}{dt} = 1/|r'(t)|$ and

$$\begin{aligned}\frac{dT}{dt} &= \frac{1}{(e^t + e^{-t})^2}\Big[\big(0, e^t, e^{-t}\big)(e^t + e^{-t}) - \big(\sqrt{2}, e^t, e^{-t}\big)\big)(e^t - e^{-t})\Big] \\ &= \frac{1}{(e^t + e^{-t})^2}\big(-\sqrt{2}(e^t - e^{-t}), 2, 2\big).\end{aligned}$$

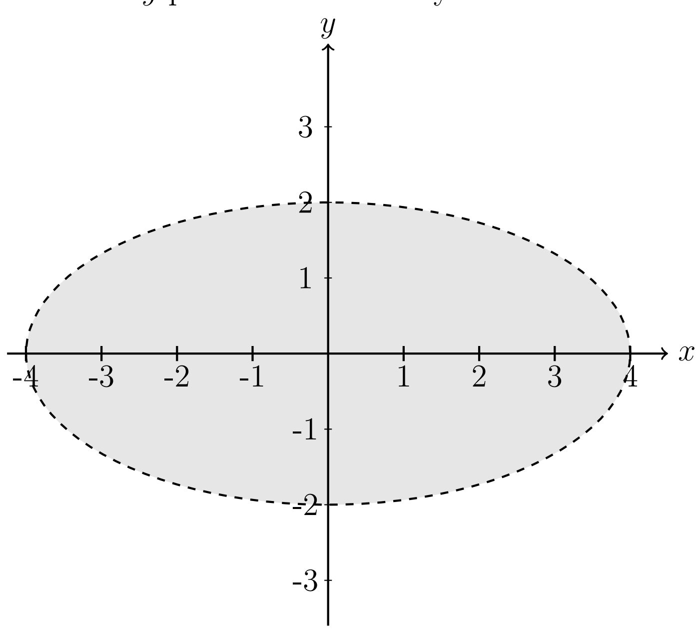
Thus we have that

$$\kappa(t) = \frac{1}{(e^t + e^{-t})^3}\sqrt{2(e^{2t} - 2 + e^{-2t}) + 4 + 4} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}.$$

Problem 3

Consider the function defined by the expression $f(x, y) = \log\big(16 - x^2 - 4y^2\big)$, where log is the natural logarithm.

a. Describe the region D where this function is defined (its domain) in the form of a set <u>and</u> draw it in the xy plane below. Clearly indicate what belongs to D and what does not.



$$D = \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, x^2 + 4y^2 < 16\Big\}$$

b. If you move upwards from the point $(2, -1)$ parallel to the y-axis, do the function values increase or decrease? Explain your reasoning.

c. What is a tangent vector to the level curve $L_0(f) = \big\{(x, y)\,\big|\, f(x, y) = 0\big\}$ at its point of intersection with the line $x = y$ with $y > 0$?

Solution:

b. We compute $\partial_y f(x,y) = \frac{-8y}{16-x^2-4y^2}$ and see that $\partial_y f(2,-1) = \frac{8}{8} = 1$. It follows that the function values grow.

c. Notice that $f(x,y) = 0$ if and only if $16 - x^2 - 4y^2 = 1$ or $x^2 + 4y^2 = 15$. Implicit differentiation yields

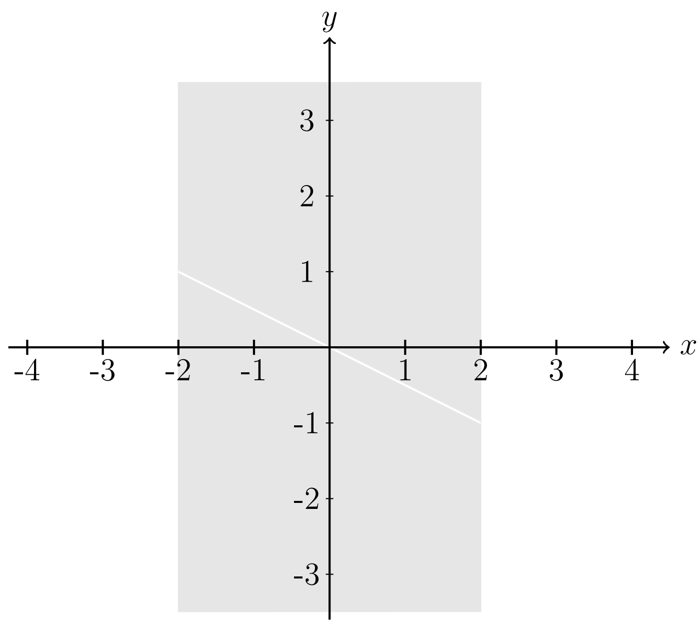
$$2x + 8y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{2x}{8y}$$

When $x = y$ the slope is $-\frac{1}{4}$ so that a tangent vector is given by $\langle -4, 1 \rangle$.

Problem 4

Consider the function given by the expression $f(x,y) = \frac{\sqrt{4-x^2}}{x+2y}$.

a. Sketch the domain of f clearly indicating what belongs to the set and what does not.



The domain is the vertical strip between the lines $x = -2$ and $x = 2$ (included) minus the segment with equation $y = -\frac{x}{2}$ and $x \in [-2, 2]$.

b. Determine the limit $\lim_{(x,y)\to(2,-1)} f(x,y)$. If it exists, provide a justification; if it does not, explain why.

c. What is $\frac{\partial f}{\partial y}(0,1)$? Show your calculations.

Solution:

b. We take the limit parallel to the x-axis by setting $y = -1$ to obtain

$$\lim_{x\to 2} \frac{\sqrt{(2-x)(2+x)}}{x-2} = -\lim_{x\to 2} \frac{\sqrt{2+x}}{\sqrt{2-x}} = -\infty.$$

It follows that the limit does not exist.

c. We compute

$$\partial_y f(x,y) = -\frac{2\sqrt{4-x^2}}{(x+2y)^2} \text{ and } \partial_y f(0,1) = -\frac{4}{4} = -1.$$

Bonus Problem

Find a parametrization of the curve of intersection of the paraboloid with equation $z = x^2 + y^2$ and the cylinder with equation $z = 1 - y^2$.

Solution: Points in the intersection satisfy both equations and thus also $1 - y^2 = x^2 + y^2$ or, equivalently, $x^2 + 2y^2 = 1$, which is an ellipse parametrized by $\left(\cos(t), \frac{1}{\sqrt{2}}\sin(t)\right)$ for $t \in [0, 2\pi)$. Using again that $z = 1 - y^2$, we obtain the desired parametrization

$$\left(\cos(t), \frac{1}{\sqrt{2}}\sin(t), 1 - \frac{1}{2}\sin^2(t)\right),\ t \in [0, 2\pi).$$

Week 8

Section 14.5 The Chain Rule

Video Lecture. The Chain Rule I with Examples.
The case of a function $f : \mathbb{R}^n \to \mathbb{R}$ of several variables composed with a curve $x : \mathbb{R} \to \mathbb{R}^n$ is considered in this video.

Video Lecture. The Chain Rule II with Examples.
The case of a function $f : \mathbb{R}^n \to \mathbb{R}$ of several variables composed with a function $x : \mathbb{R}^m \to \mathbb{R}^n$ is considered in this video.

At around minute 9:00 a partial derivative is incorrectly computed. The correct expression is actually give by $\frac{\partial y}{\partial x_2} = x_1^4 + 2x_2x_3^3$. The rest of the calculation needs adjustments as well and should read

$$\begin{aligned}\frac{\partial y}{\partial s} &= \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial y}{\partial x_3}\frac{\partial x_3}{\partial s} = 4x_1^3x_2re^t + (x_1^4 + 2x_2x_3^3)2rse^{-t} + 3x_2^2x_3^2r^2\sin(t)\\ &= 128 + 64 + 0 = 192,\end{aligned}$$

since $(x_1, x_2, x_3) = (2, 2, 0)$ at $(r, s, t) = (2, 1, 0)$.

Video Lecture. Implicit Differentiation with Examples.

Section 14.6 Directional Derivatives and the Gradient Vector

Video Lecture. The Gradient Vector with Examples.

Video Lecture. The Gradient Vector and Level Sets with Examples.

Reading assignment *Significance of the gradient vector*.

Section 14.7 Maxima and Minima

Video Lecture. Maxima and Minima with Examples.

Towards the end of this video I make reference to eigenvalues of matrices. If you never had any linear algebra, just disregard my comment that is only meant for those of you who know what eigenvalues are. The criterion to distinguish extrema presented does not require this knowledge.

Homework Assignment 7 **Due May 23 by 11:59pm**

Week 9

Section 14.8 Lagrange Multipliers

Video Lecture. Lagrange Multipliers.

There is mistake at the very end of this video. After computing x_1, which should read $x_1 = -\frac{1}{\mu}$, and x_2, we determine μ using the circle equation $h = 1$ to find the value of μ. Then, the equation of the plane $g = 1$ is used to compute

$$x_3 = 1 - x_1 + x_2 = 1 \pm \frac{2}{\sqrt{29}} \pm \frac{5}{\sqrt{29}} = 1 \pm \frac{7}{\sqrt{29}}$$

The corresponding extremal values of the function should be

$$f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3 = \frac{1}{\sqrt{29}}\big(\mp 2 \pm 10\big) + 3 \pm \frac{21}{\sqrt{29}} = 3 \pm \frac{29}{\sqrt{29}} = 3 \pm \sqrt{29}$$

Video Examples. Computing Maxima and Minima in the Presence of Side Conditions.

Section 15.1 Double Integrals Over Rectangles

Reading assignment *Review of the Definite Integral*.

Video Lecture. Double integrals of functions defined on a rectangle with examples.

Section 15.2 Double Integrals over General Domains

Video Lecture. Double integrals of functions defined on a general domain with examples.

Video Lecture. Properties of double integrals.

Homework Assignment 8 **Due May 30 by 11:59pm**

Week 10

Review–Final Examination

The final examination is comprehensive and includes all the material covered over the course the term with the exception of Sections 15.1 and 15.2.
A good way to review the material would be to use the review sections at the end of each chapter. Ponder and answer the **conceptual questions** that are raised there, take the **true/false quizzes** and **practice** with the following selection of **problems**:

- Chapter 10 Review: Problems 1, 2, 4, 7, 23, 24, 27, 39, 43.
- Chapter 12 Review: Problems 1, 3, 5, 7, 13, 15, 18, 20, 22, 24, 27, 37, 38.
- Chapter 13 Review: Problems 1, 2, 6, 8, 9, 10, 12, 13, 17, 17, 23
- Chapter 14 Review: Problems 1–4, 9, 10, 20, 22, 25, 26, 28, 31, 33, 36, 39, 40, 42–44, 47, 50, 52, 53, 56, 60, 61, 63, 65.

Solution to Some Problems

There was not enough time to go over all the problems I had selected in review and I am posting the solutions here.

Problem
What are the maximal and minimal volume of a rectangular box with surface area of 1300 square units and total edge length of 200 units?
Solution: We denote by x, y, and z the length, width, and height of the box. Then the surface area A, edge length L, and volume V are given by

$$A = 2xy + 2xz + 2yz, \; L = 4x + 4y + 4z, \; \text{and } V = xyz.$$

Thus the problem consists of finding the extrema of V with the side conditions that $S = 1300$ and $L = 200$. The Lagrange multiplier's condition for an extremum reads

$$\nabla V = \lambda \nabla A + \mu \nabla L,$$

which, together with the side conditions, yields the system

$$yz = 2\lambda(y + z) + 4\mu \tag{1}$$
$$xz = 2\lambda(x + z) + 4\mu \tag{2}$$
$$xy = 2\lambda(x + y) + 4\mu \tag{3}$$
$$x + y + z = 50 \tag{4}$$
$$xy + xz + yz = 650 \tag{5}$$

Subtracting (1) from (2), (1) from (3), and (2) from (3) we get the equations

$$z(x - y) = 2\lambda(x - y) \tag{6}$$
$$y(x - z) = 2\lambda(x - z) \tag{7}$$
$$x(y - z) = 2\lambda(y - z). \tag{8}$$

Equations (6)-(8) are satisfied when $x = y = z$ in which case the side conditions cannot be satisifed since
$$3x = 50 \text{ gives } x^2 = \frac{2500}{9} \neq \frac{650}{3}.$$
If, on the other hand $x \neq y$, $y \neq z$, and $x \neq z$, then it follows from (6)-(8) that $x = y = z = 2\lambda$, which is impossible as it contradicts the assumption.
The last option is that $x = y$ and $z \neq x$, that is, that two of the quantities are the same and the third is distinct from them (notice that the names x, y, and z are interchangeable). In this case we get that $z = 50 - 2x$ from (4) and then that
$$x^2 + x(50 - 2x) + x(50 - 2x) = 650 \text{ or } 3x^2 - 100x + 650 = 0$$
from (5).The latter equation has solutions
$$x_{1,2} = \frac{50}{3} \pm \frac{\sqrt{550}}{3} \text{ so that } z_{1,2} = 50 - 2x_{1,2} = \frac{50}{3} \mp 2\frac{\sqrt{550}}{3}.$$
The corresponding extremal volumes are
$$V_{1,2} = \Big(\frac{50}{3} \pm \frac{\sqrt{550}}{3}\Big)^2 \Big(\frac{50}{3} \mp 2\frac{\sqrt{550}}{3}\Big).$$

Problem

Find the extrema of $f(x, y) = e^{xy}$ on the curve with equation $g(x, y) = x^3 + y^3 = 16$.

Solution: We first observe that the curve can be parametrized by
$$\big(x, (16 - x^3)^{\frac{1}{3}}\big),\ x \in \mathbb{R}.$$
For x positive and large, y will be negative and large, while it will positive and large for x negative and large. More mathematically
$$\lim_{x\to\pm\infty} y(x) = \mp\infty,$$
and thus $\lim_{x\to\pm\infty} xy(x) = -\infty$. It follows that, along the curve, $f > 0$ attains no minimum but its values come arbitrarily close to zero. It follows also that the method of Lagrange multipliers will yield a maximum. To obtain it, we compute
$$\nabla f = \lambda\nabla g \text{ or } \langle ye^{xy}, xe^{xy}\rangle = \lambda\langle 3x^2, 3y^2\rangle,$$
which gives $3\lambda x^3 = 3\lambda y^3$, or simply $x = y$. Then, using $g(x, x) = 16$ we obtain $x = y = 2$.

Problem

Find the plane through $P = (3, 1, 4)$ and containing the line of intersection of the planes with equations $x + 2y + 3z = 1$ and $2x - y + z = 3$.

Solution: The first plane equation gives $x = 1 - 2y - 3z$ and, plugging into the second, we get that $2 - 4y - 6z - y + z = 3$ or $y = -\frac{1}{5} + z$. Then we also have that $x = 1 + \frac{2}{5} - 2z - 3z = \frac{7}{5} - 5z$. We conclude that the line of intersection can be parametrized by
$$\Big(\frac{7}{5} - 5z, -\frac{1}{5} + z, z\Big),\ z \in \mathbb{R}.$$
Next we take any two points on this line, say the points corresponding to $z = 0, \frac{1}{5}$ or $Q = (\frac{7}{5}, -\frac{1}{5}, 0)$ and $R = (\frac{2}{5}, 0, \frac{1}{5})$. Finally we use the P, Q and R to generate two vectors in the desired plane, say

$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to obtain a normal vector to the desired plane $\overrightarrow{PQ} \times \overrightarrow{PR}$. Using the latter and one of three points, we finally find the coefficients A, B, C, and D of its equation.

Alternatively, we try to use the given information to directly find A, B, C, and D. As P should lie in the plane we see that

$$3A + B + 4C + D = 0.$$

Next the plane needs to contain the line of intersection so that

$$A(\frac{7}{5} - 5z) + B(-\frac{1}{5} + z) + Cz + D = 0$$

must hold for every $z \in \mathbb{R}$. Choosing $z = \frac{7}{25}, \frac{1}{5}, 0$ we obtain three more equations

$$\frac{2}{25}B + \frac{7}{25}C + D = 0,\ \frac{2}{5}A + \frac{1}{5}C + D = 0, \text{ and } \frac{7}{5}A - \frac{1}{5}B + D = 0.$$

All that is left, is now to solve this system to find A, B, C, and D.

Problem
Find three parametrizations of $y = \sqrt{x}$, $x \geq 0$.
Solution: One is clearly $(x, \sqrt{x})$ for $x \geq 0$. Another is (x^2, x) for $x \geq 0$. Still another is (x^6, x^3) for $x \geq 0$.

Problem
If $z = y + f(x^2 - y^2)$, where f is differentiable , show that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$.
Solution: Using the chain rule of single variable calculus (partial derivatives are taken with respect to a variable while keeping the other(s) fixed), we get that

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)2x \text{ and } \frac{\partial z}{\partial y} = 1 - f'(x^2 - y^2)2y.$$

The rest is algebra.