Harnessing the Power of Calculus

Half derivatives and Image Processing

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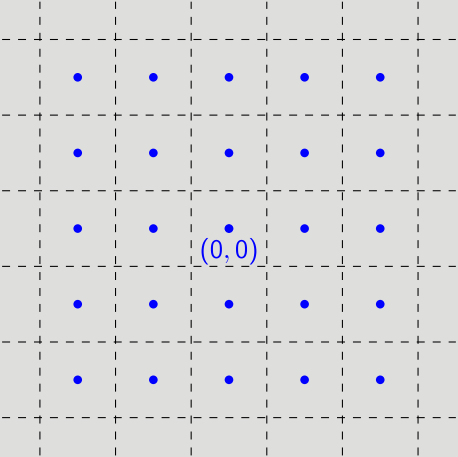
Run Experiment 1

Derivation of ...

Consider an infinite grid

Consider an infinite grid



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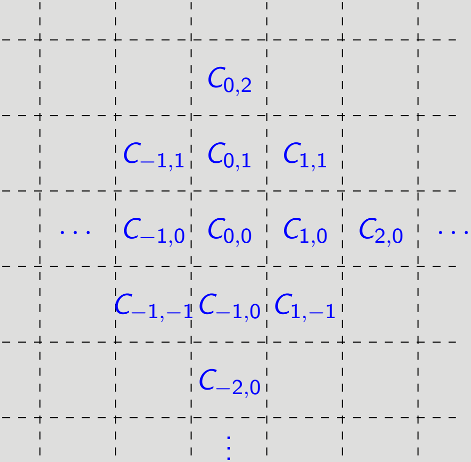

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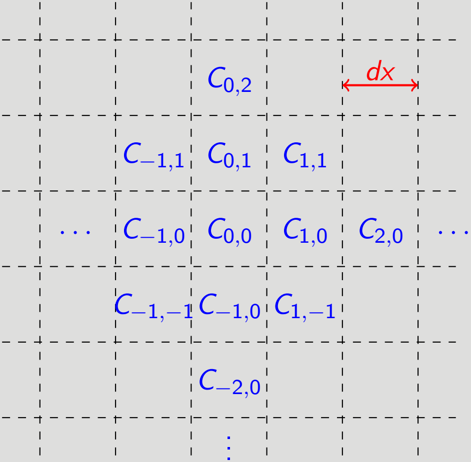

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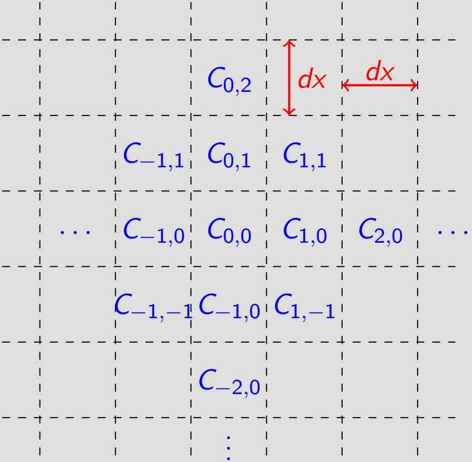

Derivation (continued)

Consider the following process:

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A particle found in cell $C_{j,k}$

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A particle found in cell $C_{j,k}$
centered at the point $(j, k)dx$

Derivation (continued)

Consider the following process:

A particle found in cell $C_{j,k}$
centered at the point $(j, k)dx$
at time $n\,dt$

Consider the following process:

A particle found in cell $C_{j,k}$
centered at the point $(j,k)dx$
at time $n\,dt$ (step $n \in \mathbb{N}$)

Derivation (continued)

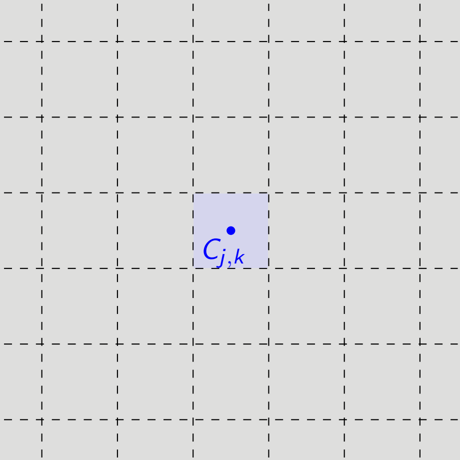
Consider the following process:

A particle found in cell $C_{j,k}$
centered at the point $(j,k)dx$
at time $n\,dt$ (step $n \in \mathbb{N}$) for $dt > 0$

Derivation (continued)

with probability 1

step n



Derivation (continued)

Consider the following process:

A particle found in cell $C_{j,k}$
centered at the point (jdx, kdx)
at time ndt (step n) for $n \in \mathbb{N}$ and $dt > 0$,

in step $n+1$ will move one cell

Derivation (continued)

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A particle found in cell $C_{j,k}$
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up or **down**

Derivation (continued)

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up or **down**, **left** or **right**

Derivation (continued)

Consider the following process:

A particle found in cell $C_{j,k}$
centered at the point (jdx, kdx)
at time ndt (step n) for $n \in \mathbb{N}$ and $dt > 0$,

in step $n+1$ will move one cell
up or **down**, **left** or **right**
with probability $\mathbf{1/4}$ each

Derivation (continued)

with probability $1/4$

step $n+1$

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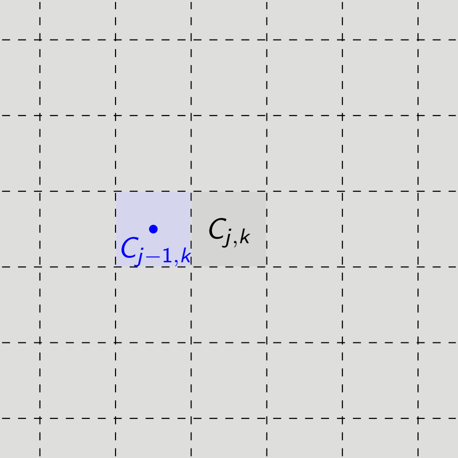

Derivation (continued)

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$$= \frac{1}{4}\Big[P\big(X^n \in C_{j-1,k}\big) + P\big(X^n \in C_{j+1,k}\big) + P\big(X^n \in C_{j,k-1}\big) + P\big(X^n \in C_{j,k+1}\big)\Big]$$

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$$= \frac{1}{4}\Big[P\big(X^n \in C_{j-1,k}\big) + P\big(X^n \in C_{j+1,k}\big) + P\big(X^n \in C_{j,k-1}\big) + P\big(X^n \in C_{j,k+1}\big)\Big]$$

which amounts to

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which amounts to

$$p^{n+1}_{j,k} - p^n_{j,k} = \frac{1}{4}\big[p^n_{j-1,k} + p^n_{j+1,k} + p^n_{j,k-1} + p^n_{j,k+1} - 4p^n_{j,k}\big]$$

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which amounts to

$$\frac{p^{n+1}_{j,k} - p^n_{j,k}}{dt} = \frac{1}{4dt}\,\frac{p^n_{j-1,k} + p^n_{j+1,k} + p^n_{j,k-1} + p^n_{j,k+1} - 4p^n_{j,k}}{1}$$

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which amounts to

$$\frac{p^{n+1}_{j,k} - p^n_{j,k}}{dt} = \frac{dx^2}{4dt}\,\frac{p^n_{j-1,k} + p^n_{j+1,k} + p^n_{j,k-1} + p^n_{j,k+1} - 4p^n_{j,k}}{dx^2}$$

Derivation of the Heat Equation

Notice that

$$\frac{p^{n+1}_{j,k} - p^{n}_{j,k}}{dt} \simeq \frac{p(ndt+dt, jdx, kdx) - p(ndt, jdx, kdx)}{dt}$$

$$= \frac{p(t+dt, x_1, x_2) - p(t, x_1, x_2)}{dt}$$

for $ndt = t$ and $(jdx, kdx) = (x_1, x_2) = x$

$$\simeq \frac{d}{dt} p(t,x) = p_t(t,x)$$

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Similarly

$$\frac{p^n_{j-1,k} - 2p^n_{j,k} + p^n_{j+1,k}}{dx^2} \simeq \frac{p(t, x_1-dx, x_2) - 2p(t, x_1, x_2) + p(t, x_1+dx, x_2)}{dx^2}$$
$$\simeq \frac{\partial^2}{\partial {x_1}^2} p(t, x) = p_{x_1 x_1}(t, x)$$

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Letting $dx, dt \to 0$

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$$p(t,x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}},\ t > 0,\ x \in \mathbb{R}^2.$$

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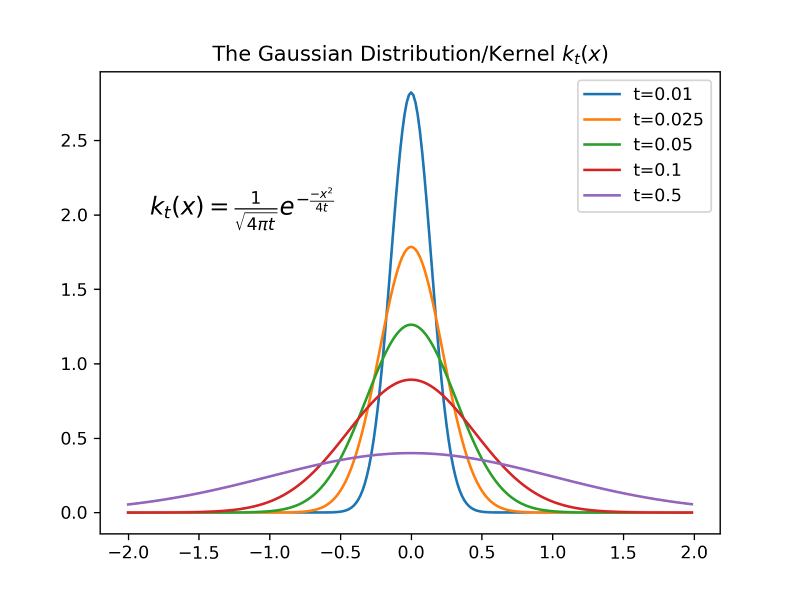
$$P\big(X^t \in A\big) = \int_A p(t,x)\, dx.$$

The Heat Kernel



Functions

Simple Functions:

Functions

Simple Functions:

$$f(x) = 1$$

Functions

Simple Functions:

$$f(x) = 1, \ \cos(x)$$

Functions

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$$f(x) = 1, \ \cos(x), \ \sin(x)$$

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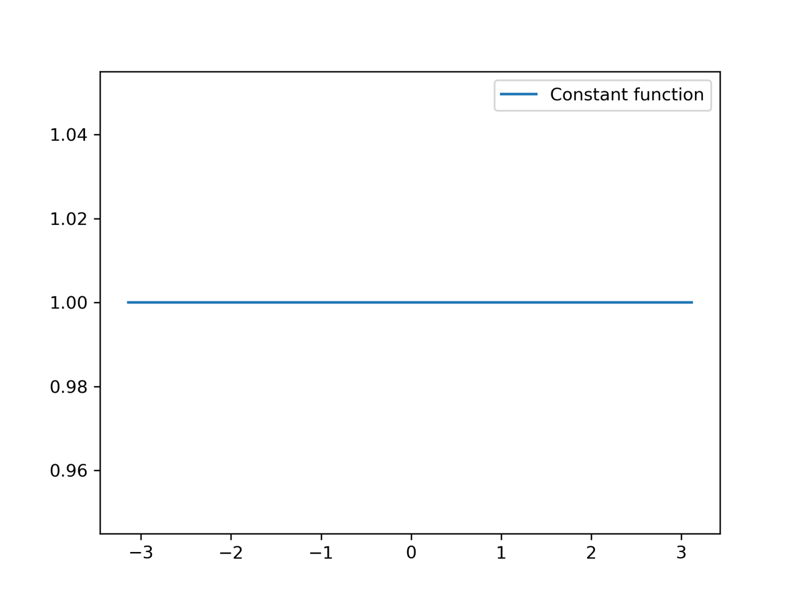
for the simple functions $\varphi_k = e^{ik\cdot}$.

Example of Functions/Expansions



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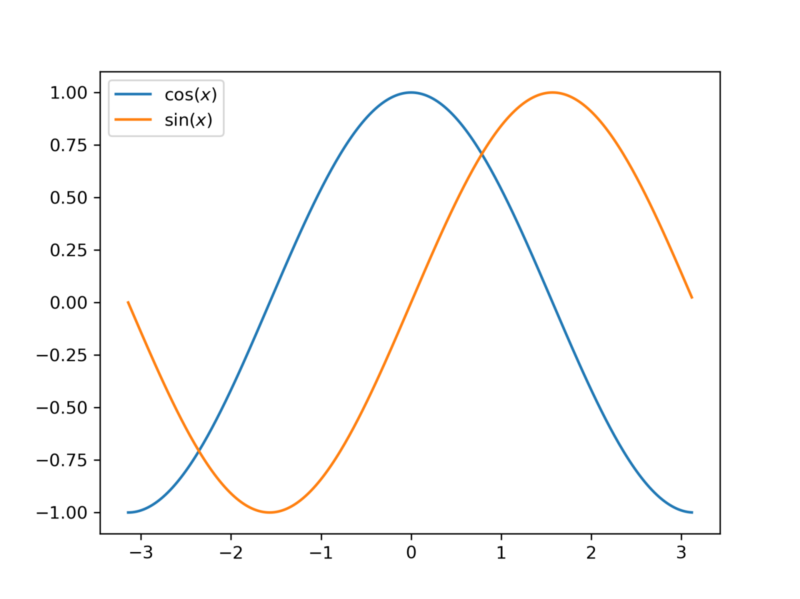

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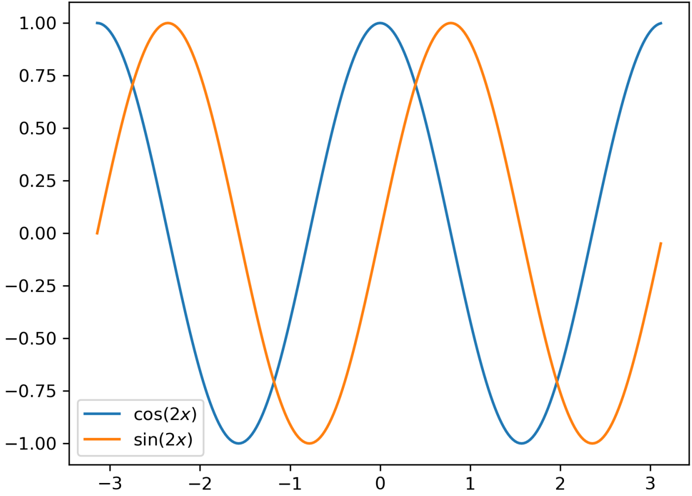

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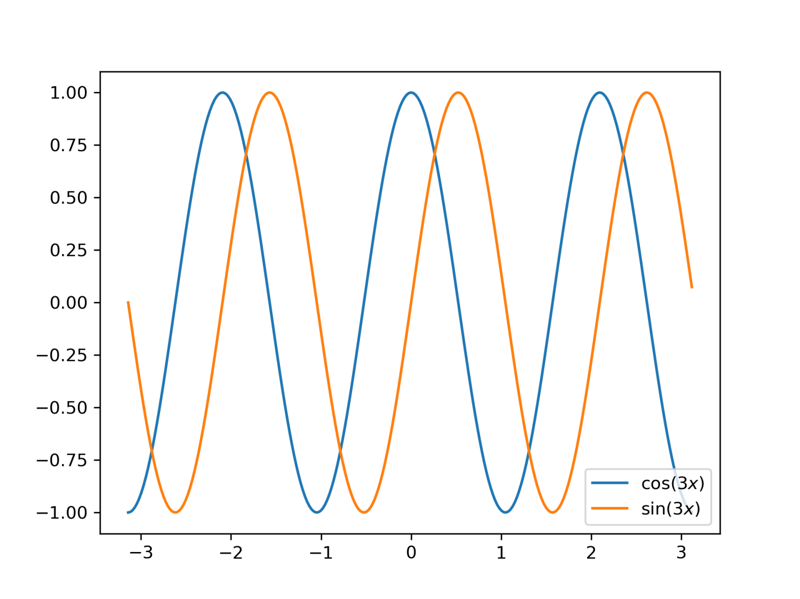

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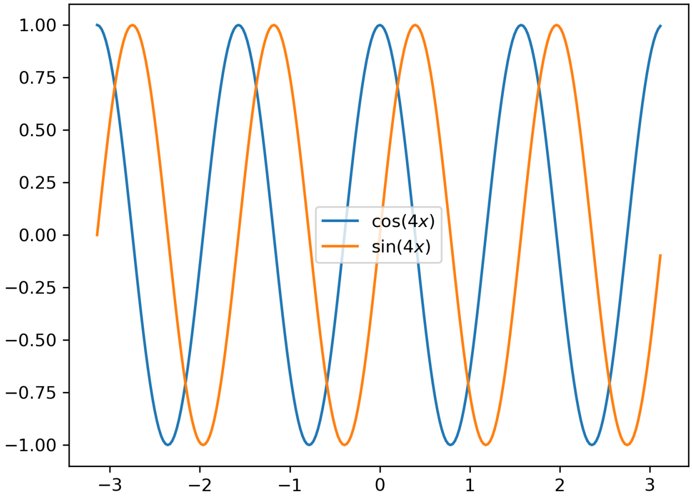

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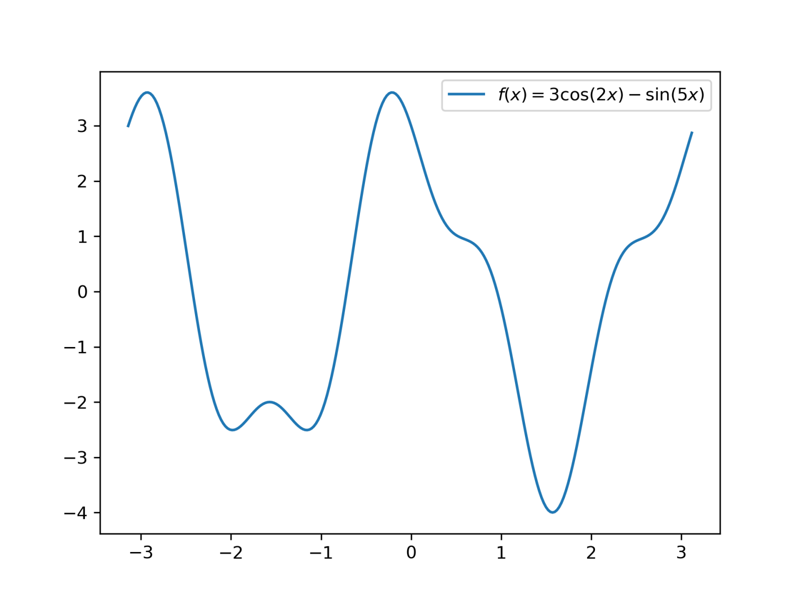

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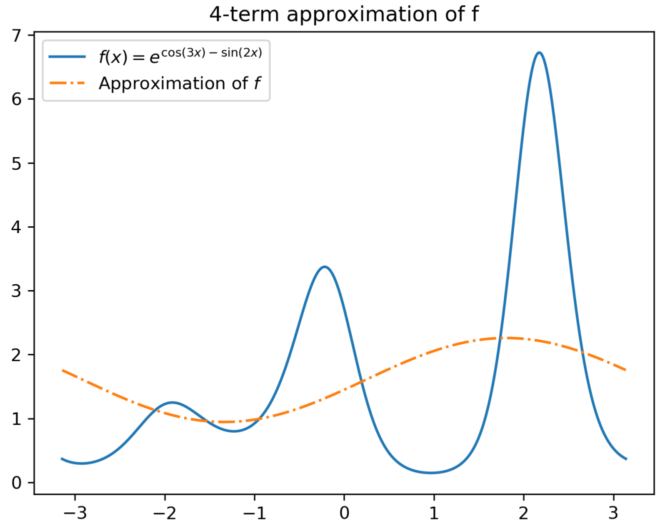

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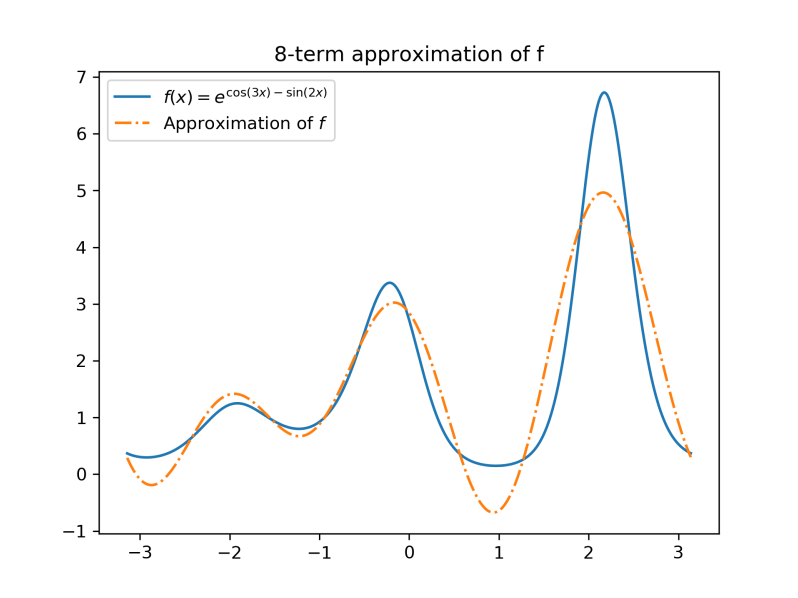

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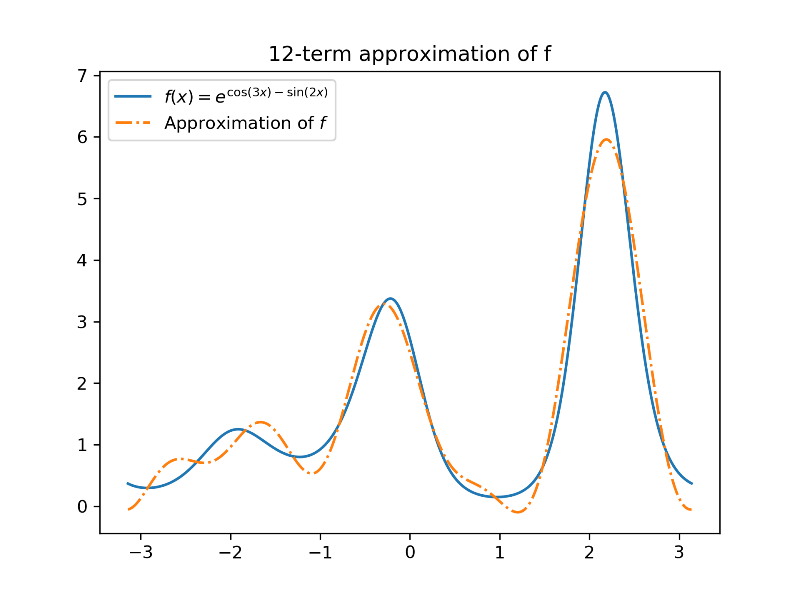

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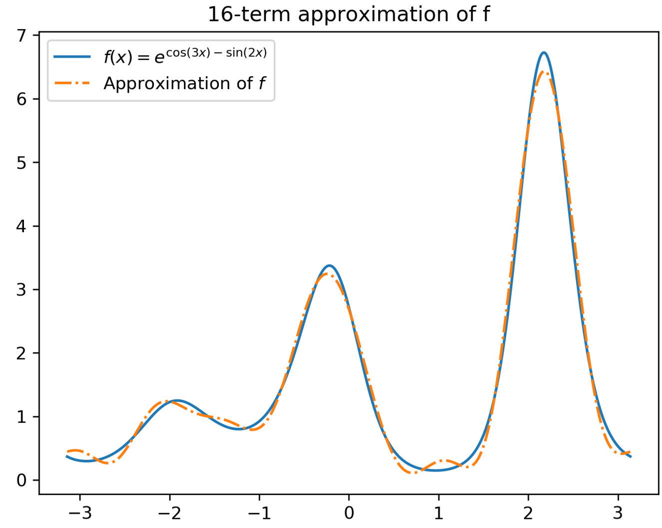

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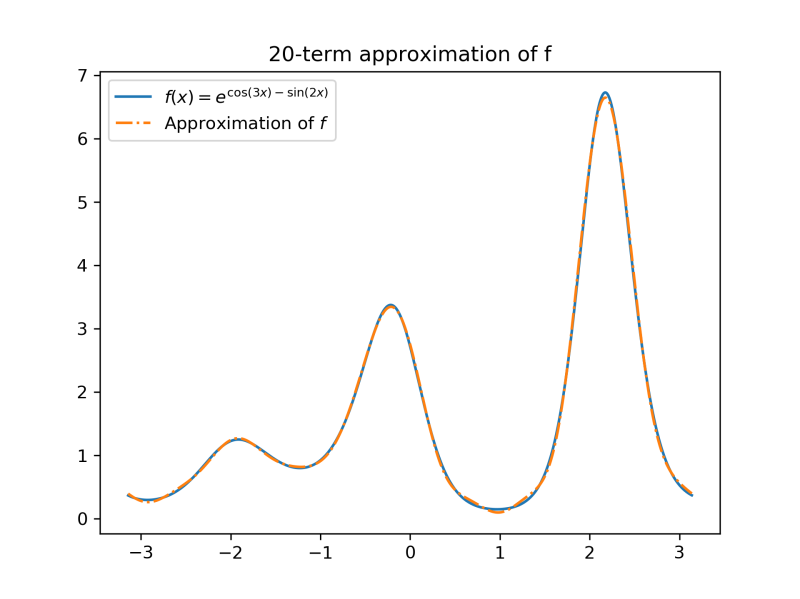

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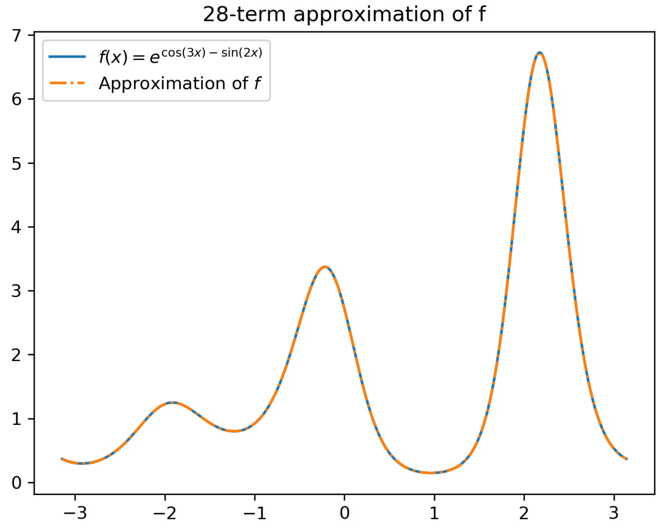

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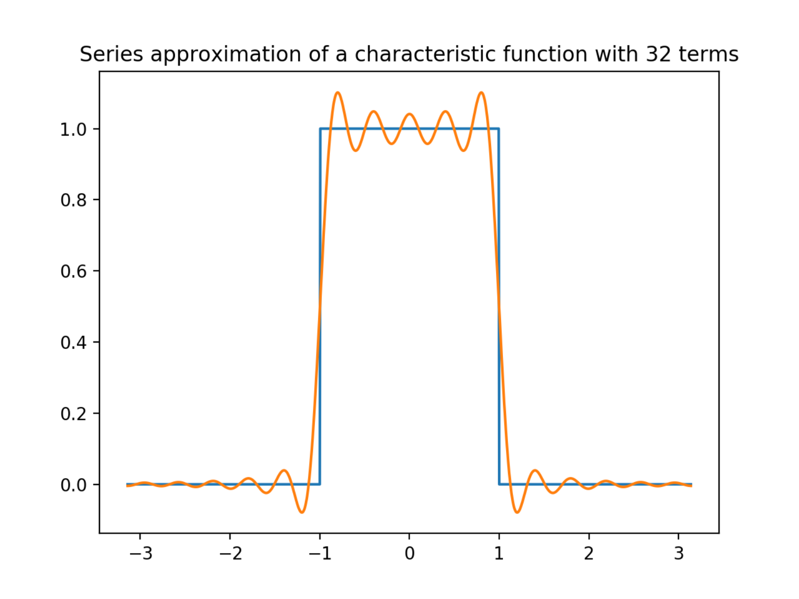

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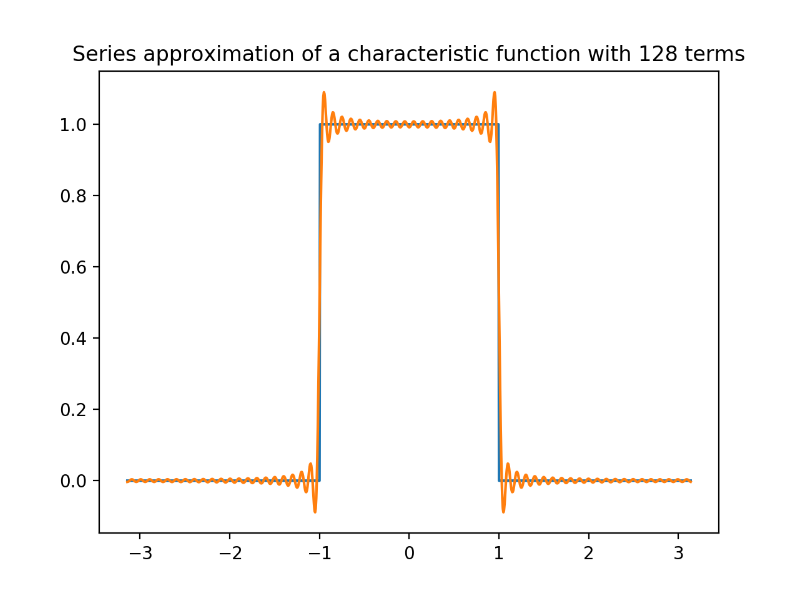

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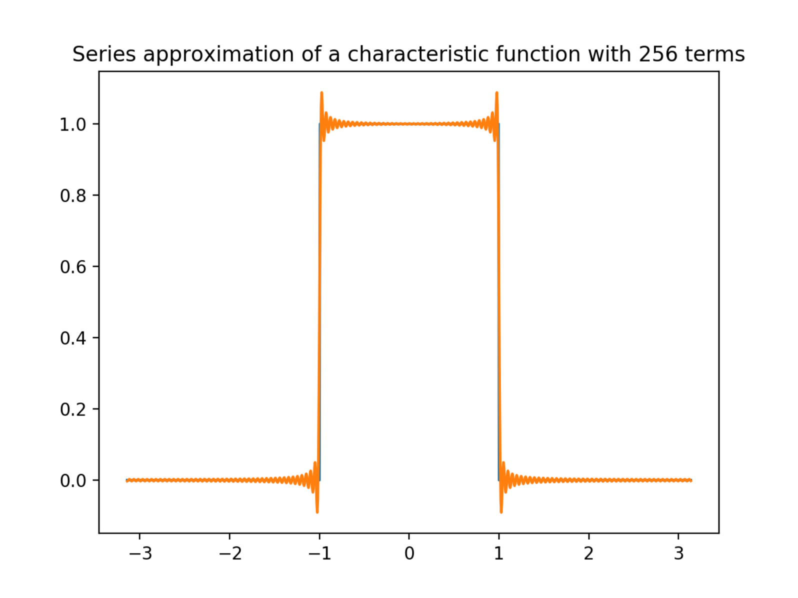

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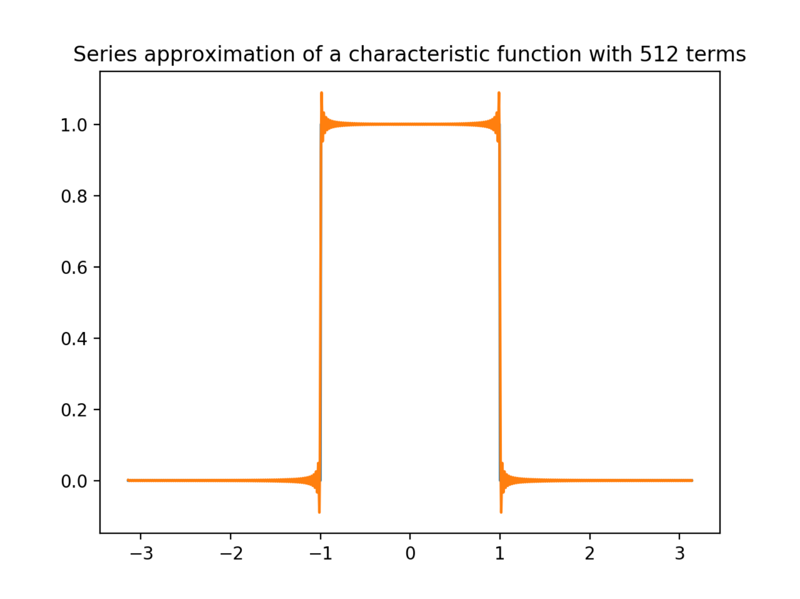

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Intuition

Vector space

Scalar product

Orthonormal basis

Basis expansion

Coefficients

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The Effect of Linear Diffusion

Click for video

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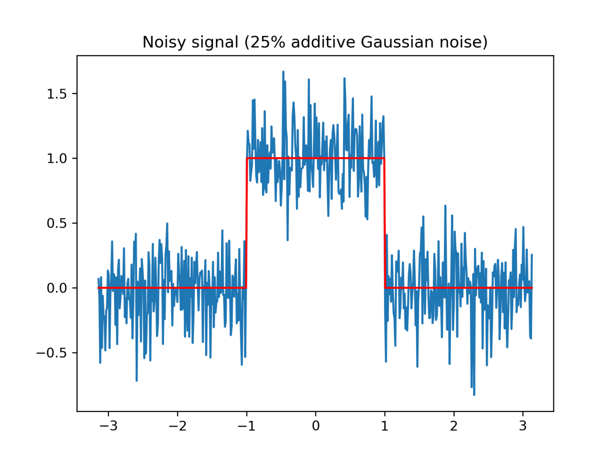

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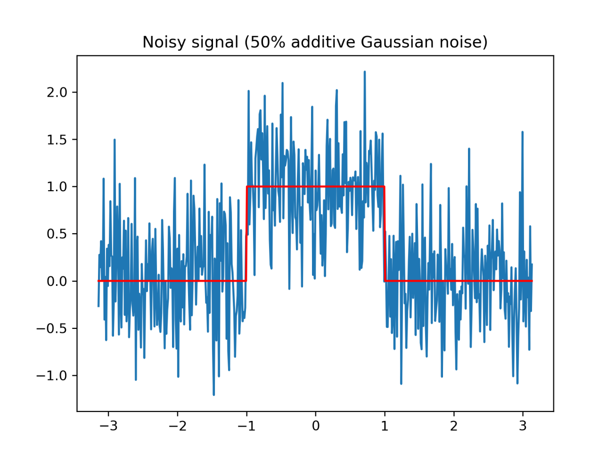

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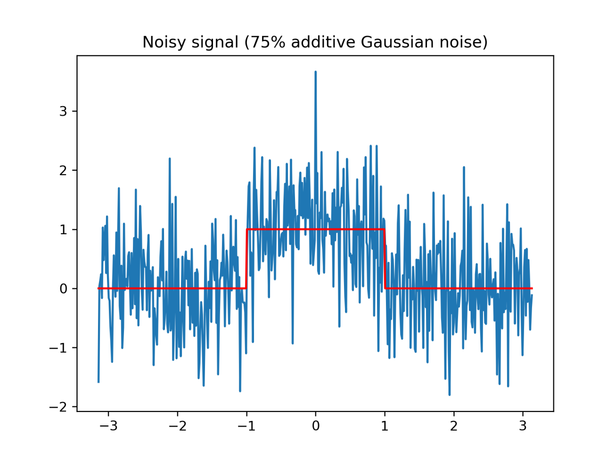

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The Effect of Linear Diffusion

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Back to the drawing board

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This is the so-called **Perona-Malik** model introduced in 1990.

The Effect of Perona-Malik

Click for video

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Finally we arrive at a noise reduction model with the desired properties.

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where $\varepsilon \in (0, 1)$ is chosen small but not too much. This model was introduced by **G & Lambers** in 2009.

The Effect of Fractional Diffusion

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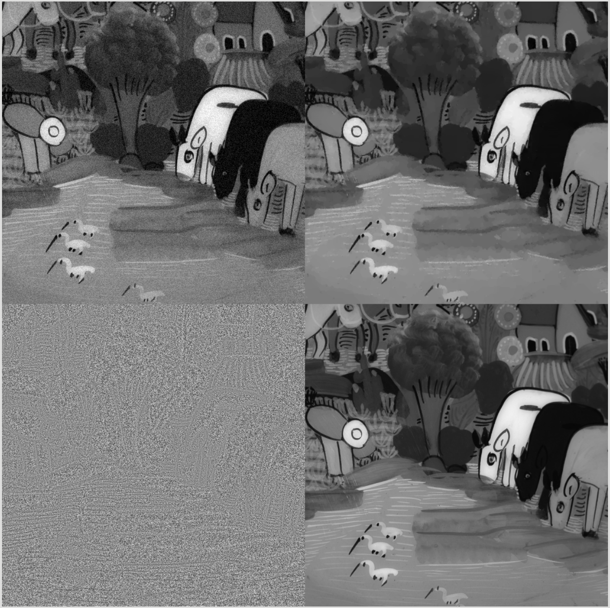
Experiments with Real Images



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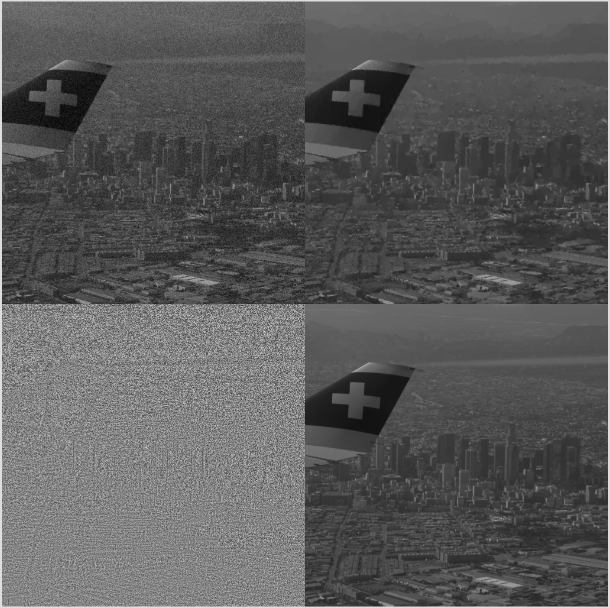
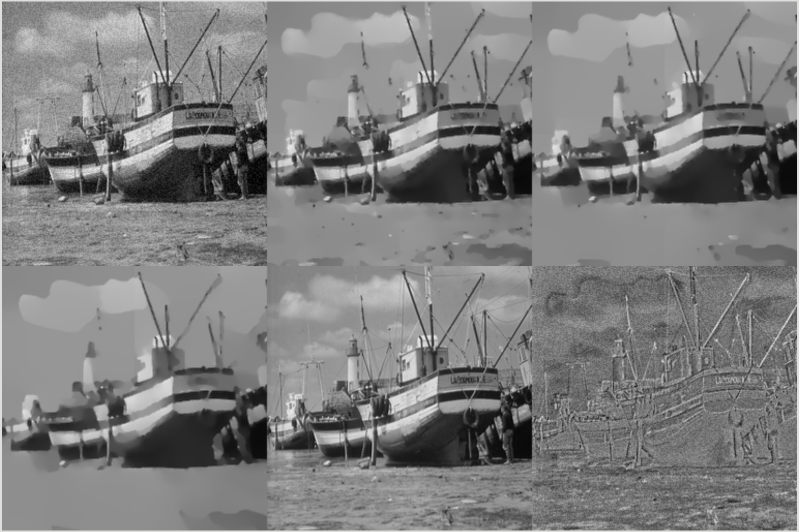


Experiments with Real Images



Experiments with Real Images

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Do you have any questions?

Thank you for your attention!

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