Midterm Examination

Print your name: ____________________________
Print your ID #: ______________________________

You have 50 minutes to solve the problems. Good luck!
1. Prove that the closure \( \bar{O} \) of a convex set \( O \subset \mathbb{R}^n \) is convex.

**Solution:**
Take any \( x, y \in \bar{O} \). Then there are sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) in the set \( O \) such that
\[
x_n \to x \text{ and } y_n \to y \text{ as } n \to \infty.
\]
Since \( O \) is convex, \((1 - t)x_n + ty_n \in O\) for any \( t \in [0, 1] \). Therefore, for any \( t \in [0, 1] \), \((1 - t)x + ty \in \bar{O} \) since
\[
(1 - t)x_n + ty_n \to (1 - t)x + ty \text{ as } n \to \infty.
\]

2. Prove that a Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \( \mathbb{R}^n \) which possesses a convergent subsequence is already convergent. Recall that a sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence if
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \|x_n - x_m\| \leq \varepsilon \forall n, m \geq N.
\]

**Solution:**
Fix \( \varepsilon > 0 \) and let \( x_\infty \) be the limit of the convergent subsequence. By assumption there exists \( N \in \mathbb{N} \) such that
\[
\|x_n - x_m\| \leq \varepsilon/2, \ m, n \geq N
\]
There also is \( N \leq M \in \mathbb{N} \) such that
\[
\|x_M - x_\infty\| \leq \varepsilon/2
\]
since \( x_\infty \) is the limit of a subsequence. We therefore conclude that
\[
\|x_n - x_\infty\| = \|x_n - x_M + x_M - x_\infty\| \leq \|x_n - x_M\| + \|x_M - x_\infty\| \leq \varepsilon, \ n \geq N
\]
and the claimed convergence is established.

3. For a sequence \((x_n)_{n \in \mathbb{N}}\) in \( \mathbb{R}^m \) define \( X = \{x_n \mid n \in \mathbb{N}\} \).
   Is every accumulation point of \((x_n)_{n \in \mathbb{N}}\) a limit point of \( X \)?
   Is every limit point of \( X \) an accumulation point of \((x_n)_{n \in \mathbb{N}}\)?
   If your answer is yes, prove it. If your answer is no, give a counterexample.

**Solution:**
The answer to the first question is \(\textbf{no} \). Just take the sequence \((x_n)_{n \in \mathbb{N}}\) given by \( x_n = x_\infty, \ n \in \mathbb{N} \) for some fixed \( x_\infty \in \mathbb{R}^m \). In this case
$X = \{x_\infty\}$ is a set without limit points but $x_\infty$ is certainly an accumulation point of the sequence. The converse is true. In fact, if $x_\infty$ is a limit point of $X$, then, by definition, there is a sequence $(x_{n_k})_{k \in \mathbb{N}}$ in $X \setminus \{x_\infty\}$ which converges to $x_\infty$ making it an accumulation point of $(x_n)_{n \in \mathbb{N}}$.

4. Let $f \in C(\mathbb{R}^n, \mathbb{R})$. Is it possible that $f(\mathbb{R}^n) = \{0,1\}$? Motivate your answer.

**Solution:**
The answer is no. In fact continuous functions map connected sets to connected sets. Since $\mathbb{R}^n$ is convex and therefore connected, the range of the continuous function $f$ cannot be the disconnected set $\{0,1\}$.

5. For $U \subset \mathbb{R}^n$ let $f : U \to \mathbb{R}$ be differentiable at $x \in U$ with $f(x) = 0$ and $g : U \to \mathbb{R}$ be continuous at $x$. Prove that $fg$ is differentiable at $x$ and compute $\nabla (fg)(x)$.

**Solution:**
Notice that

$$f(x + h) = \nabla f(x) \cdot h + o(\|h\|) \quad \text{as} \quad h \to 0$$

since $f$ is differentiable at $x$ and $f(x) = 0$. Also

$$\lim_{h \to 0} (g(x + h) - g(x)) = 0$$

since $g$ is assumed to be continuous at $x$. Therefore

$$f(x + h)g(x + h) - g(x)\nabla f(x) \cdot h = \left[f(x + h) - \nabla f(x) \cdot h\right]g(x + h) + \left[g(x + h) - g(x)\right] \nabla f(x) \cdot h = o(\|h\|) \quad \text{as} \quad h \to 0$$

which gives differentiability and $\nabla (fg)(x) = g(x) \nabla f(x)$. 

3