Assignment 3

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1. Let $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. For any $m \in \mathbb{N}$ define
$$x_m = k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cfrac{1}{\ddots\; k_{m-1} + \frac{1}{k_m}}}}.$$
Prove that $(x_m)_{m\in\mathbb{N}} \in CS(\mathbb{Q})$ and that any $x \in [0,\infty)$ is obtained as a limit of such a sequence.

2. Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $x_n \le x_{n+1} \le c < \infty \, \forall n \in \mathbb{N}$ and show that there exists $x_\infty \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x_\infty$.
[Hint: Show that it is a Cauchy sequence.]

3. Let $(x_n)_{n\in\mathbb{N}}\,, (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ converge to the common limit x_∞. Prove that any sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with
$$x_n \le z_n \le y_n \, \forall\, n \ge m$$
for some $m \in \mathbb{N}$ also converges to the same limit x_∞.

4. Construct sequences $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that
(i) $LP(x) = \mathbb{N}$.
(ii) $LP(x) = \{y\}$ for some $y \in \mathbb{R}$ but x is not convergent.
(iii) $LP(x) = [0,2]$.

5. Prove that $\mathbb{R}$ is uncountable and has the same cardinality as $2^{\mathbb{N}}$.

Homework due by Thursday, October 20 2005.