MATH 205 FALL TERM 2005

Assignment 7

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1. Let a function $f \in \mathrm{BUC}\big((a,b),\mathbb{R}\big)$ be given for $a < b \in \mathbb{R}$. Show that
$$\lim_{x\to a+} f(x)\,,\ \lim_{x\to b-} f(x)$$
exist. Thus, f can be uniquely extended to a uniformly continuous function of the closed interval $[a,b]$.

2. Analyze the continuity and differentiability of the function
$$f(x) := \begin{cases} 0\,, & x \le 0\,,\\ \frac{1}{x}e^{-\frac{1}{x^2}}\,, & x > 0\,. \end{cases}$$
What can you say about higher order derivatives at $x = 0$?

3. Let $f : D \to \mathbb{R}$ be differentiable at $x_0 \in D \overset{o}{\subset} \mathbb{R}$. Prove or disprove:
(i) f is strictly increasing at x_0 implies that $f'(x_0) > 0$.
(ii) $f'(x_0) = 0$ implies that f has a local minimum or maximum at x_0.
(iii)f is Lipschitz continuous at x_0 and has local minimum there implies that f is differentiable at x_0.

4. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$. Define
$$d^+f(x_0) = \lim_{x\to x_0+0} \frac{f(x)-f(x_0)}{x-x_0}\,,\ x_0 \in [a,b)\,,$$
$$d^-f(x_0) = \lim_{x\to x_0-0} \frac{f(x)-f(x_0)}{x-x_0}\,,\ x_0 \in (a,b]\,,$$
the right and left derivative of f at x_0, respectively, if the limit exists.
(i) Show that f is differentiable at $x_0 \in (a,b)$ if and only if $d^\pm f(x_0)$ exist and coincide.
(ii) Assume that f takes on a minimum or a maximum at a or b and is one-side differentiable there. What can you say about $d^+f(a)$ and $d^-f(b)$?
5. Let $f \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, $g \in \mathrm{C}(\mathbb{R},\mathbb{R})$ and assume that $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Prove that $fg : \mathbb{R} \to \mathbb{R}\,,\ x \mapsto f(x)g(x)$ is differentiable at x_0 and compute $(fg)'(x_0)$. What can you say if g is only assumed to be bounded?

Homework due by Thursday, November 18 2005