Chapter 8

Euclidean Space and Metric Spaces

8.1 Structures on Euclidean Space

8.1.1 Vector and Metric Spaces

The set $\mathbb{K}^n$ of n-tuples $x = (x_1, x_2 \ldots, x_n)$ can be made into a *vector space* by introducing the standard operations of addition and scalar multiplication through

$$x + y = (x_1, x_2 \ldots, x_n) + (y_1, y_2 \ldots, y_n) := (x_1 + y_1, x_2 + y_2 \ldots, x_n + y_n)\,,$$
$$\lambda x = \lambda(x_1, x_2 \ldots, x_n) := (\lambda x_1, \lambda x_2 \ldots, \lambda x_n)\,,\ \lambda \in \mathbb{K}\,,\ x, y \in \mathbb{K}^n\,.$$

As for the topology of $\mathbb{K}^n$ we introduce the distance function

$$d(x, y) := \big[\sum_{k=1}^{n} |x_k - y_k|^2\big]^{1/2}$$

which satisfies the following properties
(m1) $d(x, y) \geq 0\,\forall\, x, y \in \mathbb{K}^n$ and $d(x, y) = 0 \Leftrightarrow x = y$.
(m2) $d(x, y) = d(y, x)\,\forall\, x, y \in \mathbb{K}^n$.
(m3) $d(x, z) \leq d(x, y) + d(y, z)\,\forall\, x, y, z \in \mathbb{K}^n$ (*triangle-inequality*)

Definition 8.1.1. A pair (M, d) is called *metric space* iff
(i) M is a set.
(ii) $d : M \times M \to [0, \infty)$ satisfies (m1)-(m3).

Examples 8.1.2. (a) $\big(\mathbb{K}^n, \big[\sum_{k=1}^n |x_k - y_k|^2\big]^{1/2}\big)$ is a metric space.
(b) $(\mathbb{R}, d_0)$ is a metric space for $d_0(x,y) := \begin{cases} 0\,, & x = y \\ 1\,, & x \neq y \end{cases} x, y \in \mathbb{R}$
(c) If $M = \mathbb{K}^n$ and d_p is defined by

$$d_p(x,y) := \begin{cases} \big[\sum_{k=1}^n |x_k - y_k|^p\big]^{1/p}\,, & 1 \leq p < \infty\,, \\ max_{k=1,\dots,n} |x_k - y_k|\,, & p = \infty\,, \end{cases}$$

then (M, d_p) is a metric spaces for $p \in [1, \infty]$.

8.1.2 Norms and Scalar Products

Observe that $d_2(x,y)$ only depends on the $x - y$. In particular, by defining $|x|_2 := \sqrt{\sum_{k=1}^n x_k^2}$, we recover $d(x,y) = |x - y|_2$.

Definition 8.1.3. A pair $(V, |\cdot|_V)$ is a *normed vector space* if
(i) V is a $\mathbb{K}$-vector space.
(ii) $|\cdot|_V : V \to [0, \infty)$ is a norm on V, that is, it satisfies

(n1) $|x|_V \geq 0\ \forall\, x \in V$, $|x| = 0 \iff x = 0$.

(n2) $|\alpha x|_V = |\alpha||x|_V\ \forall\, x \in V\ \forall\, \alpha \in \mathbb{K}$.

(n3) $|x + y| \leq |x| + |y|\ \forall\, x, y \in V$. (triangle inequality)

Remarks 8.1.4. (a) If $(V, |\cdot|_V)$ is a normed vector space, then (V, d_V) is a metric space for

$$d_v(x,y) := |x - y|_V\ \forall\, x, y \in V\,.$$

(b) For each $p \in [1, \infty]$

$$|x|_p := \big(\sum_{k=1}^n |x_k - y_k|^p\big)^{1/p}$$

is a norm on $V = \mathbb{R}^n\,,\ \mathbb{C}^n$.
(c) If $V = \mathrm{C}([a,b]) \ni f$ and $\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$ then $(V, \|\cdot\|_\infty)$ is a normed vector space.
(d) If $V = \mathrm{C}^1([a,b]) \ni f$ and $\|f\|_{1,\infty} := \|f\|_\infty + \|f'\|_\infty$ then $(V, \|\cdot\|_{1,\infty})$ is a normed vector space.

Definition 8.1.5. Let V be a vector space. Then

$$\langle\cdot,\cdot\rangle = V \times V \to \mathbb{K}\,,\ (x,y) \mapsto \langle x,y\rangle$$

is called *inner product* if
(i1) $\langle x,y\rangle = \overline{\langle y,x\rangle}\ \forall\, x,y \in V$.
(i2) $\langle x+\alpha y,z\rangle = \langle x,z\rangle + \alpha\langle y,z\rangle$ and $\langle x,y+\alpha z\rangle = \langle x,y\rangle + \bar{\alpha}\langle x,z\rangle$ $\forall\, x,y,x \in V\ \forall\,\alpha \in \mathbb{K}$.
(i3) $\langle x,x\rangle \geq 0\ \forall\, x \in V$ and $\langle x,x\rangle = 0$ iff $x = 0$. The pair $\big(V,\langle\cdot,\cdot\rangle\big)$ is called *inner product space.*

Example 8.1.6. If $V = \mathrm{C}([a,b],\mathbb{K})$ and $\langle f,g\rangle := \int_a^b f\bar{g}\,dx$ for $f,g \in V$, then $\big(V,\langle\cdot,\cdot\rangle\big)$ is an inner product space.

Theorem 8.1.7. *(Cauchy-Schwarz)*
Let $\big(V,\langle\cdot,\cdot\rangle\big)$ *be an inner product space. Then*

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} =: |x|_V|y|_V\ \forall\, x,y \in V\,.$$

Proof. (i) V is a $\mathbb{R}$-vector space: If either $x = 0$ or $y = 0$ the inequality is obvious. Assume therefore that $x \neq 0$ and $y \neq 0$. Then we can define

$$\tilde{x} = \frac{x}{|x|_V} \text{ and } \tilde{y} = \frac{y}{|y|_V}$$

and obtain $|\tilde{x}|_V = |\tilde{y}|_V = 1$. Observing that

$$0 \leq \langle\tilde{x}+\tilde{y},\tilde{x}+\tilde{y}\rangle = \langle\tilde{x},\tilde{x}\rangle + 2\langle\tilde{x},\tilde{y}\rangle + \langle\tilde{y},\tilde{y}\rangle = 2\langle\tilde{x},\tilde{y}\rangle + 2$$
$$0 \leq \langle\tilde{x}-\tilde{y},\tilde{x}-\tilde{y}\rangle = \langle\tilde{x},\tilde{x}\rangle - 2\langle\tilde{x},\tilde{y}\rangle + \langle\tilde{y},\tilde{y}\rangle = -2\langle\tilde{x},\tilde{y}\rangle + 2$$

the claim follows by combining the two inequalities.
(ii)V is a $\mathbb{C}$-vector space: We can again assume that $x \neq 0$ and $y \neq 0$ and define $\tilde{x}$ and $\tilde{y}$ as above to obtain

$$0 \leq \langle\tilde{x}+\tilde{y},\tilde{x}+\tilde{y}\rangle = \langle\tilde{x},\tilde{x}\rangle + \langle\tilde{x},\tilde{y}\rangle + \langle\tilde{y},\tilde{x}\rangle + \langle\tilde{y},\tilde{y}\rangle$$
$$0 \leq \langle\tilde{x}-\tilde{y},\tilde{x}-\tilde{y}\rangle = \langle\tilde{x},\tilde{x}\rangle - \langle\tilde{x},\tilde{y}\rangle - \langle\tilde{y},\tilde{x}\rangle + \langle\tilde{y},\tilde{y}\rangle$$

which implies $|\Re\langle\tilde{x},\tilde{y}\rangle| \leq 1$ since $\langle\tilde{x},\tilde{y}\rangle + \langle\tilde{y},\tilde{x}\rangle = 2\Re\langle\tilde{x},\tilde{y}\rangle$. It always is that

$$\langle\tilde{x},\tilde{y}\rangle = r\,e^{i\theta} \text{ for some } r \geq 0 \text{ and } \theta \in [0,2\pi)$$

and, from this, it follows that

$$|\langle\tilde{x},\tilde{y}\rangle| = |\langle e^{-i\theta}\tilde{x},\tilde{y}\rangle| = |\Re\langle e^{-i\theta}\tilde{x},\tilde{y}\rangle| \leq 1$$

since $\langle e^{-i\theta}\tilde{x},\tilde{y}\rangle$ is real and $|e^{-i\theta}\tilde{x}| = 1$. $\surd$

Remarks 8.1.8. (a) If V is an $\mathbb{R}$-vector space and $\langle \cdot, \cdot \rangle$ is an inner product on it, we obtain

$$\langle x, y \rangle = \frac{1}{4}\big(|x+y|_V^2 - |x-y|_V^2\big)\,,\ x, y \in V$$

for $|\cdot|_V$ defined by $|x|_V = \sqrt{\langle x, x\rangle}$.
(b) If V is an $\mathbb{C}$-vector space and $\langle \cdot, \cdot \rangle$ is an inner product on it, we obtain

$$\langle x, y \rangle = \frac{1}{4}\big(|x+y|_V^2 - |x-y|_V^2 + i|x+iy|^2 - i|x-iy|^2\big)\,,\ x, y \in V$$

(c) Let $\big(V, \langle \cdot | \cdot \rangle\big)$ be an inner product space. Then the following *parallelogram identity*

$$|x+y|_V^2 + |x-y|_V^2 = 2\big(|x|_V^2 + |y|_V^2\big)\,,\ x, y \in V$$

holds.
(d) On a normed vactor space $(V, |\cdot|_V)$ there exists an inner product $\langle \cdot, \cdot \rangle$ such that $|\, x|_V = \sqrt{\langle x, x\rangle}$ **iff** the parallelogram identity holds true.

Theorem 8.1.9. *Let $\big(V, \langle \cdot, \cdot \rangle\big)$ be an inner product space. Then*

$$|x|_V = \sqrt{\langle x, x\rangle}\,,\ x \in V$$

defines a norm on V.

Proof. We need to verify the validity of conditions (n1)-(n3).
(i) (n1) follows from (i3).
(ii) As for (n2) we have

$$|\alpha x|_V = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x, x\rangle} = \sqrt{|\alpha|^2\langle x, x\rangle} = |\alpha||x|_V\ \forall\, x \in V\ \forall\, \alpha \in \mathbb{K}\,.$$

(iii) Finally the triangular inequality follows from Cauchy-Schwarz. In fact

$$\begin{aligned}|x+y|_V^2 = \langle x+y, x+y\rangle &= |x|_V^2 + 2\Re\langle x, y\rangle + |y|_V^2 \\ &\le |x|_V^2 + 2|x|_V|y|_V + |y|_V^2 = \big(|x|_V + |y|_V\big)^2\ \forall\, x, y \in V\,.\end{aligned}$$

Here we used that $\Re z \le |z|$ for any $z \in \mathbb{C}$. $\surd$

8.2 Topology of Metric Spaces

8.2.1 Open Sets

We now generalize concepts of open and closed further by giving up the linear structure of vector space. We shall use the concept of distance in order to define these concepts maintaining the basic intuition that open should amount to every point having still some space around.

Definition 8.2.1. (Open Ball)
Let (M, d) be a metric space and $r \in (0, \infty)$ Then the *open ball about* $x \in M$ *with radius* r is defined by

$$\mathrm{B}(x, r) := \{y \in M \,|\, d(x, y) < r\} .$$

Definition 8.2.2. (Open Sets)
(i) $O \subset M$ is called *open* or, in short $O \overset{o}{\subset} M$, iff

$$\forall\, x \in O\, \exists\, r > 0 \text{ s.t. } x \in \mathrm{B}(x, r) \subset O .$$

(ii) Any set $\mathcal{U} \subset M$ containing a ball $\mathrm{B}(x, r)$ about x is called *neighborhood of* x. The collection of all neighborhoods of a given point x is denoted by $\mathcal{U}(x)$.

Remark 8.2.3. The collection $\tau_M := \{O \subset M \,|\, O \text{ is open }\}$ is a topology on M.

Theorem 8.2.4. *(Induced/Relative Metric)*
Let (M, d) be a metric space and $N \subset M$. Then (N, d_N) is a metric space with

$$d_N : N \times N \to [0, \infty) \,,\; (x, y) \mapsto d(x, y) .$$

Then $O \overset{o}{\subset} N$ iff $O = N \cap \tilde{O}$ for some $\tilde{O} \overset{o}{\subset} M$.

Corollary 8.2.5. *If $N \overset{o}{\subset} M$, then $O \overset{o}{\subset} N$ iff $O \overset{o}{\subset} M$.*

Proof. "$\Longrightarrow$": If $O \overset{o}{\subset} N$ and $x \in O$, then we find $r_x > 0$ such that

$$\mathrm{B}_N(x, r_x) = \{y \in N \,|\, d(x, y) < r_x\} \subset O .$$

Defining $\tilde{O}_x = \mathrm{B}_M(x, r_x)$ we obtain an open set in M. Setting

$$\tilde{O} = \bigcup_{x \in O} \tilde{O}_x \overset{o}{\subset} M$$

we arrive at $\tilde{O} \cap N = O$.
"$\Longleftarrow$": If $O = N \cap \tilde{O}$ for some $\tilde{O} \overset{o}{\subset} M$, then for any $x \in O$ we find $r_x > 0$ such that $\mathrm{B}_M(x, r_x) \subset \tilde{O}$. In this case

$$\mathrm{B}_N(x, r_x) = \mathrm{B}_M(x, r_x) \cap N \subset \tilde{O} \cap N = O$$

which shows that $O \overset{o}{\subset} N$. $\surd$

8.2.2 Limits and Closed Sets

Definitions 8.2.6. Let (M,d) be a metric space and $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$. Then we define
(i) $x_n \to x_\infty \, (n \to \infty) \iff \forall\, \varepsilon > 0 \,\exists\, N \in \mathbb{N}$ s.t. $d(x_n, x_\infty) \leq \varepsilon \,\forall\, n \geq N$.
(ii) A point x is called *limit point* of the sequence $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ if there is a subsequence $(n_j)_{j\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$x_{n_j} \to x \, (j \to \infty)\,.$$

(iii) A point x is called *limit point* of the the set $A \subset M$ iff

$$\dot{\mathrm{B}}(x,r) \cap A \neq \emptyset \,\forall\, r > 0$$

for $\dot{\mathrm{B}}(x,r) := \{y \in M \,|\, d(x,y) < r\,,\; y \neq x\}$.
(iv) A set $A \subset M$ is closed iff $\mathrm{LP}(A) \subset A$.
(v) The *closure* $\bar{A}$ of a set $A \subset M$ is given by $\bar{A} = A \cup \mathrm{LP}(A)$. In this case $\bar{A}$ is closed.
(vi) A set $A \subset B \subset M$ is *dense* in B iff $B \subset \bar{A}$.

Theorem 8.2.7. *Let (M,d) be a metric space. Then*

$$A = \bar{A} \iff A^c \overset{o}{\subset} M\,.$$

Proof. "$\Longleftarrow$": Let $x \in A^c \overset{o}{\subset} M$. Then we find $r > 0$ such that $\mathrm{B}(x,r) \subset A^c$, which means $x \notin \mathrm{LP}(A)$. Rephrasing we obtain

$$\mathrm{LP}(A) \subset (A^c)^c = A$$

which gives $A = \bar{A}$.
"$\Longrightarrow$": Let $A = \bar{A}$ and $x \in A^c$. Then x cannot be a limit point of A, that is, there is $r > 0$ with $\mathrm{B}(x,r) \cap A = \emptyset$, which amounts to $\mathrm{B}(x,r) \subset A^c$. $\surd$

8.2.3 Completeness

Let (M,d) be a metric space and $x \in M^{\mathbb{N}}$. The sequence x is called *Cauchy* iff

$$\forall\, \varepsilon > 0 \,\exists\, N \in \mathbb{N} \text{ s.t. } d(x_n, x_m) \leq \varepsilon \,\forall\, m,n \geq N\,.$$

Any convergent sequence is Cauchy.

Definition 8.2.8. A metric space (M,d) is called *complete* iff every Cauchy sequence has a limit in M.

Theorem 8.2.9. *Let a sequence $(x_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^n$ be given. Then*

$$|x_m - x_\infty|_2 \to 0\,(n\to\infty) \iff x_m^k \to x_\infty^k\,(n\to\infty)\,.$$

Corollary 8.2.10. *The normed vector space $\mathbb{R}^n$ is complete.*

Proof. (of theorem 8.2.9) "$\Longrightarrow$": The simple inequality

$$|x_m^k - x_\infty^k| \leq \big(\sum_{j=1}^n |x_m^j - x_\infty^j|^2\big)^{\frac{1}{2}} \to 0\,(m\to\infty)$$

is valid for any $k \in \{1,\dots,n\}$ and implies the stated convergence for the components.
"$\Longleftarrow$": The assumed componentwise convergence implies the existence, for any given $\varepsilon > 0$, of $N_k(\varepsilon) \in \mathbb{N}$ such that

$$|x_m^k - x_\infty^k| \leq \frac{\varepsilon}{\sqrt{n}}\,\forall\, m \geq N_k(\varepsilon)\,.$$

For $m \geq N := \max_{k=1,\dots,n} N_k$, we then get

$$\big(\sum_{j=1}^n |x_m^j - x_\infty^j|^2\big)^{\frac{1}{2}} \leq \big(\sum_{j=1}^n \frac{\varepsilon^2}{n}\big)^{\frac{1}{2}} = \varepsilon$$

which gives the desired convergence. $\surd$

Examples 8.2.11. (a) The normed vector space $\big(\mathrm{C}([a,b]), \|\cdot\|_\infty\big)$ is complete.
(b) The normed space $\big(\mathrm{C}([a,b]), \|\cdot\|_1\big)$ is not. Recall that

$$\|f\|_1 = \int_a^b |f(x)|\,dx\,,\ f \in \mathrm{C}([a,b])\,.$$

(c) Let (M,d) be an incomplete metric space. Then, just as we orignally did with $\mathbb{Q}$, it can be completed. The procedure is completely analogous. First define

$$\mathrm{C\,S}(M) := \{x \in M^{\mathbb{N}}\,|\, x \text{ is a Cauchy sequence }\}$$

and the relation $\sim$ on it by

$$x \sim y \;:\Longleftrightarrow\; \lim_{n\to\infty} d(x_n, y_n) = 0\,.$$

It can be shown that $\sim$ is an equivalence relation and the *completion* $\overline{M}$ of M can be defined by

$$\overline{M} = \mathrm{C\,S}(M)/\sim\,.$$

How would you introduce a concept of distance on $\overline{M}$ to show that

$$M \hookrightarrow \overline{M}$$

and that $\overline{M}$ is indeed complete?

8.2.4 Compactness

Definition 8.2.12. (Compactness)
A subset K of a metric space (M, d) is called *compact* iff

$$\forall\, x \in K^{\mathbb{N}}\, \exists\, (n_k)_{k\in\mathbb{N}}\,,\; x \in K \text{ s.t. } x_{n_k} \to x\; (k \to \infty)\,.$$

A set is therefore compact if every sequence within the set has a convergent subsequence with limit in the set itself.

Remark 8.2.13. Can M itself be compact? What is the relation to completeness in this case? Compactness always implies completeness since Cauchy sequences can only have one limit. Completeness does not necessarily imply compactness as the example $M = \mathbb{R}^n$ shows.

Lemma 8.2.14. *Any compact metric space M has a countable dense subset.*

Proof. Choose any $x_1 \in M$. Then pick recursively x_k for $k = 2, 3, \ldots$ with

$$\min_{j=1,\ldots,k-1} d(x_j, x_k) \geq \frac{1}{2} \sup_{x\in M} \min_{j=1,\ldots,k-1} d(x_j, x) =: R_k\,.$$

Clearly this is only possible if the supremum is finite. Suppose it is not for some k; then, for that k, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in M with

$$d(x_{k-1}, y_n) \geq n\,,\; n \in \mathbb{N}\,.$$

The sequence $(y_n)_{n\in\mathbb{N}}$ would have to have a convergent subsequence $(y_{n_j})_{j\in\mathbb{N}}$ which has a limit $y\infty \in M$ (by compactness of M). Thus it would follow that

$$0 \leq d(x_{k-1}, y_{n_j}) \leq d(x_{k-1}, y_\infty) + d(y, y_{n_j})\, \forall\, j \in \mathbb{N}$$

which is impossible since the right hand side is bounded whereas the left one is unbounded. We thus obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the above properties. We claim that it is dense in M. To see that, first observe that R_k has to converge to 0. Indeed, if that were not the case, we would have

$$d(x_j, x_k) \geq \varepsilon > 0\, \forall\, j, k \in \mathbb{N}\,.$$

Thus the sequence $(x_n)_{n\in\mathbb{N}}$ would not contain any convergent subsequence contradiciting compactness. Then, given any $x \in M$ and $\varepsilon > 0$, we find $x_k \in M$ with

$$d(x, x_k) \le \varepsilon \text{ for some } k \le n$$

if n is chosen so large that $2R_n \le \varepsilon$. $\surd$

Lemma 8.2.15. *Let (M, d) be a compact metric space. Then each of its open covers possesses a countable subcover.*

Proof. Let any open cover $\{B_\alpha \,|\, \alpha \in \Lambda\}$ be given. Let $\{x_n \,|\, n \in \mathbb{N}\}$ be a dense subset which exists thanks to lemma 8.2.14 and observe that, for any $n \in \mathbb{N}$, we can find $\alpha_n \in \Lambda$ with $x_n \in B_{\alpha_n}$. Since latter set is open, we then find $m \in \mathbb{N}$ such that $B(x_n, \frac{1}{m}) \subset B_{\alpha_n}$. Next define

$$I = \{(n, m) \,|\, \exists\, \alpha_{n,m} \in \Lambda \text{ s.t. } \mathrm{B}(x_n, \frac{1}{m}) \subset B_{\alpha_{n,m}}\}$$

and observe that $\mathrm{pr}_1(I) = \mathbb{N}$. Finally we claim that the countable subcollection $\{B_{\alpha_{n,m}} \,|\, (n, m) \in I\}$ is a subcover. Indeed, if we pick any $x \in K$, then we first find $\alpha \in \Lambda$ such that $x \in B_\alpha$ and, subsequently, a $m \in \mathbb{N}$ with $\mathrm{B}(x_n, \frac{1}{m}) \subset B_\alpha$. Next we find $n \in \mathbb{N}$ with $d(x, x_n) < \frac{1}{2m}$. Finally, $\mathrm{B}(x_n, \frac{1}{2m}) \subset B_\alpha$ implies $(n, 2m) \in I$ and therefore

$$x \in \mathrm{B}(x_n, \frac{1}{2m}) \subset B_{\alpha_{n,2m}}$$

which concludes the proof. $\surd$

Theorem 8.2.16. *A subset $K \subset M$ of a metric space M is compact iff it has the Heine-Borel property. Recall that a set K has the Heine-Borel property iff each of its open covers admits a finite subcover.*

Proof. "$\Longleftarrow$": We shall argue by contraposition. Let a sequence $(x_n)_{n\in\mathbb{N}}$ in K be given with no accumulation point in K. We can assume w.l.o.g that all points x_n are distinct. Define the sets

$$B_k = K \setminus \{x_k, x_{k+1}, \dots\}\,,\; k \in \mathbb{N}\,.$$

It follows that $\bigcup_{k\in\mathbb{N}} B_k = K$. Since B_k^c has no limit points by assumption, it must be closed. Thus $\{B_k \,|\, k \in \mathbb{N}\}$ is an open cover for K which clearly cannot admit any finite subcover.
"$\Longrightarrow$": Here we argue by contradition. By lemma 8.2.15 we can assume that

a countable open cover $\{B_k \,|\, k \in \mathbb{N}\}$ can be found which does not admit a finite subcover. The modified cover

$$B_1 =: A_1, B_1 \cup B_2 =: A_2, B_1 \cup B_2 \cup B_3 =: A_3 \ \ldots$$

will not cover, either. We therefore can construct a sequence $(x_n)_{n\in\mathbb{N}}$ in K with $x \in A_k^c$. Then we can find a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$ with limit $x_\infty \in M$. We clearly also have that $\{x_k, x_{k+1}, \dots\} \subset B_k^c$. Since B_k^c is closed, it follows that $x_\infty \in B_k^c$ for each $k \in \mathbb{N}$. This implies

$$x_\infty \notin \bigcup_{k\in\mathbb{N}} B_k = K$$

which is impossible. $\surd$

As a corollary to the proof just finished we obtain the first two remarks below.

Remarks 8.2.17. (a) A compact metric space (M, d) is bounded. Why?
(b) If (M, d) is a compact metric space, then, given any $\varepsilon > 0$, points $x_1, \dots, x_n \in M$ can be found with

$$M = \bigcup_{k=1}^{n} \mathrm{B}(x_k, \varepsilon)\,.$$

(c) Let $N \subset M$ be a subset of a complete metric space M. Then

$$(N, d_N) \text{ is complete } \iff N \text{ is closed in } M\,.$$

Theorem 8.2.18.

$$K \subset \mathbb{R}^n \textit{ is compact } \iff K \textit{ is closed and bounded.}$$

Proof. "$\Longrightarrow$": Clear.
"$\Longleftarrow$": Let a sequence $(x_j)_{j\in\mathbb{N}}$ in K be given. Then, by boundedness of K we find a constant $c \in (0, \infty)$ such that

$$|x_j^k| \le |x_j|_2 \le c < \infty \,\forall\, j \in \mathbb{N} \,\forall\, k \in \{1, \dots, n\}\,.$$

Then all sequences $(x_j^k)_{j\in\mathbb{N}}$ are bounded in $\mathbb{R}$. We therefore find a subsequence of $(j)_{j\in\mathbb{N}}$ for which

$$x_{j_m^1}^1 \to x_\infty^1 \; (m \to \infty) \text{ for some } x_\infty^1 \in \mathbb{R}\,.$$

And then a further subsequence $(j^2_m)_{m\in\mathbb{N}}$ of $(j^1_m)_{m\in\mathbb{N}}$ for which

$$x^k_{j^2_m} \to x^k_\infty \; (m \to \infty) \text{ for some } x^2_\infty \in \mathbb{R} \text{ and } k = 1, 2\,.$$

Continuing to choose subsequences in the same fashion we eventually obtain one $(j^n_m)_{m\in\mathbb{N}}$ for which

$$x^k_{j^n_m} \to x^k_\infty \; (m \to \infty) \text{ for } k = 1, \dots, n\,.$$

The gives that $x_{j^n_m} \to x_\infty = (x^1_\infty, \dots, x^n_\infty)\,(m \to \infty)$. and, by theorem 8.2.9 the claim. $\surd$

Remark 8.2.19. In general compactness is <u>not</u> equivalent to closure and boundedness. We give an example in the space $M = \mathrm{C}([0,1])$ which becomes a metric space if the distance is defined by

$$d(f, g) := \sup_{x\in[0,1]} |f(x) - g(x)| = \|f - g\|_\infty\,.$$

For $n \geq 3$ define

$$f_n(x) := \begin{cases} 0\,, & x \leq \frac{1}{2} - \frac{1}{n}\,, \\ \frac{n}{2}(x - \frac{1}{2} + \frac{1}{n})\,, & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]\,, \\ 1\,, & x \geq \frac{1}{2} + \frac{1}{n}\,. \end{cases}$$

Then $\{f_n \,|\, n \geq 3\}$ is closed and bounded but it is not compact! Why?

8.3 Continuous Functions on Metric Spaces

The distance function on a metric space allows to measure distances and therefore define concepts like convergence as we have seen in previous sections. Convergence in its turn can be used to define continuity, for instance. This is precisely what we shall do in this section. It turns out that much of the intuition we have developed in one dimension carries over to this more abstract setting of a metric space.

8.3.1 Characterizing Continuity

The intuitive non-rigorous definition of continuity of a function is that small perturbations in argument should lead to small perturbations in value. This leads to the usual $\varepsilon-\delta$ definition. But, as we have already seen for real real-valued functions, other equivalent definitions are possible, e.g. by means of sequences or open sets. The same turns out to be valid for metric spaces.

Theorem 8.3.1. *Let (M, d_M) and (N, d_n) be metric spaces and let $f : M \to N$. Then the following are equivalent:*

(i) $\forall\, x_0 \in M \;\forall\, \varepsilon > 0 \;\exists\, \delta = \delta(x_0, \varepsilon) > 0$ *s.t.* $d_N\big(f(x), f(x_0)\big) \le \varepsilon$ $\forall\, x \in M$ *with* $d_M(x, x_0) \le \delta\,.$

(ii) $d_N\big(f(x_n), f(x_\infty)\big) \to 0 \;\forall\, (x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ *with* $d_M(x_n, x_\infty) \to 0$ $(n \to \infty)$ *for some* $x \in M\,.$

(iii) $f^{-1}(O) \overset{o}{\subset} M \;\forall\, O \overset{o}{\subset} N\,.$

Proof. "(i)$\Longrightarrow$(ii)": By assumption, for any given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$d_N\big(f(x), f(x_\infty)\big) \le \varepsilon \text{ whenever } d_M(x, x_\infty) \le \delta\,.$$

Assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in M$. Then we find $N \in \mathbb{N}$ with

$$d(x_n, x_\infty) \le \delta \;\forall\, n \ge N\,,$$

which, then, clearly implies

$$d_N\big(f(x), f(x_\infty)\big) \le \varepsilon \;\forall\, n \ge N\,,$$

and concludes the first step of the proof.
"(ii)$\Longrightarrow$(i)": Assume the contraposition of (i). Then we find $x_\infty \in M$, $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$d_N\big(f(x_n), f(x_\infty)\big) \ge \varepsilon \text{ but } d_M(x_n, x_\infty) \le \frac{1}{n}\,.$$

This gives us a sequence in M which is mapped to a non-convergent sequence, which is the contraposition of (ii).
"(i)$\Longrightarrow$(iii)": Let $O \overset{o}{\subset} N$; we need to show that $f^{-1}(O) \overset{o}{\subset} M$. To that end, let $x_0 \inf^{-1}(O)$. It is then possible to find $\varepsilon > 0$ for which

$$\mathrm{B}_N\big(f(x_0), \varepsilon\big) \subset O\,.$$

But then, by assumption, there exists $\delta > 0$ such that

$$f\big(\mathrm{B}_M(x_0, \delta)\big) \subset \mathrm{B}_N\big(f(x_0), \varepsilon\big)$$

which is just a reformulation of the continuity condition. Since x_0 was arbitrary, this implies the claim by definition of open set in a metric space.

"(iii)$\Longrightarrow$(i)": For any $x_0 \in M$ and any $\varepsilon > 0$ the ball $\mathrm{B}_N\big(f(x_0), \varepsilon\big)$ is open in N. By assumption, then so is

$$f^{-1}\Big(\mathrm{B}_N\big(f(x_0), \varepsilon\big)\Big) \subset M\,.$$

This readily implies the existence of $\delta > 0$ such that

$$\mathrm{B}_M(x_0, \delta) \subset f^{-1}\Big(\mathrm{B}_N\big(f(x_0), \varepsilon\big)\Big)$$

which is nothing but the continuity condition as observed earlier. $\surd$

Remarks 8.3.2. Let (M, d_M), (N, d_N) and (P, d_P) be metric spaces.
(a) Assume that $f \in \mathrm{C}(M, N)$ and $g \in \mathrm{C}(N, P)$. Then $f \circ g \in \mathrm{C}(M, P)$.
(b) If $N = \mathbb{K}^n$ and $f, g \in \mathrm{C}(M, N)$, then $f + g, \lambda f \in \mathrm{C}(M, \mathbb{K}^n)$ for any $\lambda \in \mathbb{K}$. If $n = 1$, then also $fg \in \mathrm{C}(M, \mathbb{K})$.
(c) $f \in \mathrm{C}(M, \mathbb{K}^n) \Longleftrightarrow f_k \in \mathrm{C}(M, \mathbb{K})\; \forall\, k = 1, \ldots, n$.
(d) $X_k : \mathbb{K}^n \to \mathbb{K}\,,\; x \to x_k$ is continuous for $k = 1, \ldots, n$.
(e) $p = \sum_{|\alpha| \leq m} p_\alpha X^\alpha \in \mathrm{C}(\mathbb{K}^n, \mathbb{K})$ for any $m \in \mathbb{N}$ and any choice of $p_\alpha \in \mathbb{K}$. Here we use the notation $X^\alpha = \Pi_{k=1}^n X_k^{\alpha_k}$.

8.3.2 Continuous Functions on Compact Domains

From consideration in the one dimensional case we expect continuous functions on compact domains to be particularly "nice".

Theorem 8.3.3. *Let (M, d_M) and (N, d_n) be metric spaces and assume that $f \in \mathrm{C}(M, N)$. If M is compact, then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$, for each $x_0 \in M$, there is $\delta = \delta(x_0) > 0$ sucht that

$$f\big(\mathrm{B}_M(x_0, 2\delta)\big) \subset \mathrm{B}_N\big(f(x_0), \frac{\varepsilon}{2}\big)\,.$$

All these balls give an open cover of M

$$M = \bigcup_{x \in M} \mathrm{B}\big(x, \delta(x)\big)$$

which consequently has a finite subcover $\{\mathrm{B}\big(x_j, \delta(x_j)\big) \,|\, j = 1, \ldots, n\}$. Let $\delta = \min_{k=1,\ldots,n} \delta(x_j)$ and observe that $d_M(x, y) \leq \delta$ implies

$$d_M(x_{j_x}, y) \leq d_M(x, x_{j_x}) + d_M(x_{j_x}, y) \leq 2\delta(x_{j_x})$$

if j_x is chosen such $x \in \mathrm{B}(x_{j_x}, \delta_{j_x})$. Thus $x, y \in \mathrm{B}(x_{j_x}, 2\delta_{j_x})$ and therefore

$$d_N\big(f(x), f(y)\big) \leq d_N\big(f(x), f(x_{j_x})\big) + d_N\big(f(x_{j_x}), f(y)\big) \leq \varepsilon$$

which shows uniform continuity of f. $\surd$

Theorem 8.3.4. *Let (M, d_M) be a compact metric space and $f \in \mathrm{C}(M, \mathbb{R})$. Then*

$$\sup_{x\in M} f(x) \text{ and } \inf_{x\in M} f(x)$$

exist and there are $\overline{x}, \underline{x} \in M$ such that

$$\sup_{x\in M} f(x) = f(\overline{x}) \text{ and } \inf_{x\in M} f(x) = f(\underline{x})\,.$$

Proof. We prove the claim for the supremum only, since the same proof can be applied to find the infimum of f coincides with the negative of the supremum of $-f$. By definition we find a sequence of arguments $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ such that

$$f(x_n) \to \sup_{x\in M} f(x)\,(n\to\infty)\,.$$

Since M is compact, a subsequence $(x_{n_k})_{k\in\mathbb{N}})$ can be found that converges to some limit $\overline{x} \in M$. Then continuity of f yields

$$f(x_{n_k}) \to f(\overline{x}) = \sup_{x\in M} f(x)\,(k\to\infty)$$

which concludes the proof. $\surd$

Theorem 8.3.5. *Let (M, d_M) and (N, d_n) be metric spaces and assume that $f \in \mathrm{C}(M, N)$. Then*

$$K \subset M \text{is compact} \implies f(K) \subset N \text{ is compact.}$$

Proof. Let $\{O_\alpha \,|\, \alpha \in \Lambda\}$ be an open cover of $f(K)$. Then, since f is continuous, the preimages $\{f^{-1}(O_\alpha) \,|\, \alpha \in \Lambda\}$ of the sets in the cover are open and clearly cover K. Since K is compact, a finite subcover

$$\{f^{-1}(O_{\alpha_j}) \,|\, j = 1, \dots, n\}$$

can be found which leads to

$$\bigcup_{j=1,\dots,n} O_{\alpha_j} \supset f(K)$$

and the claim is proved. $\surd$

8.3.3 Connectedness

We know look into the generalization of the intermediate theorem. You will remember that it states the images of intervals under continuous functions are again intervals. The concept of interval is clearly confined to a one dimensional setting and we therefore need a more general concept which can be used for metric spaces.

Definition 8.3.6. (Connectedness)
Let (M, d) be a metric space. M is said to be *connected* iff

$$M = A \stackrel{\cdot}{\cup} B\,,\ A, B \stackrel{o}{\subset} M \Longrightarrow\ A = \emptyset \text{ or } B = \emptyset\,.$$

In other words, a connected set cannot be decomposed into two disjoint open subsets.

Remarks 8.3.7. (a) M is connected iff

$$A \subset M \text{ open and closed} \Longrightarrow\ A = M \text{ or } A = \emptyset\,.$$

(b) If $M = \mathbb{R}$, then $A \subset \mathbb{R}$ is connected iff it is an interval.

Definition 8.3.8. Let (M, d) be a metric space and $I \subset \mathbb{R}$ be an interval. Then $\gamma : I \to M$ is called *curve in* M if it is continuous.

Example 8.3.9. Let $M = \mathbb{R}^3$. Then $\gamma(t) := \big(\cos(t), \sin(t), t\big)\,,\ t \in [0, 2\pi]$, is a curve in $\mathbb{R}^3$. Can you picture its path?

Definition 8.3.10. (Pathwise Connected)
A subset N of a metric space (M, d) is called *pathwise connected* iff

$$\forall\, x, y \in N\ \exists\, \gamma : I = [a, b] \to N \text{ s.t. } \gamma(a) = x\,,\ \gamma(b) = y\,.$$

Remark 8.3.11. Let (M, d) be a pathwise connected metric space. Then, if $g \in \mathrm{C}(M, \mathbb{R})$ and $x, y \in M$,

$$g(x) = a, g(y) = b \Longrightarrow\ [a, b] \subset g(M)\,.$$

Proof. Take any curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then $g \circ \gamma \in \mathrm{C}([0, 1], \mathbb{R})$ and, by theorem 4.2.5, we obtain that $g \circ \gamma([0, 1])$ is an interval and thus contains $[a, b]$. $\surd$

Theorem 8.3.12. *A pathwise connected metric space is connected.*

Proof. We argue by contradiction. Assume that M is not connected but is pathwise connected. Then nontrivial subsets $A, B \overset{o}{\subset} M$ can be found such that $M = A \dot{\cup} B$. Pick $x \in A$ and $y \in B$; then there is a curve $\gamma : [0,1] \to M$ connecting them, i.e., s.t.

$$\alpha(0) = x\,,\ \alpha(1) = y\,.$$

By continuity of γ the sets

$$\gamma^{-1}(A) \text{ and } \gamma^{-1}(B)$$

are open subsets of $[0,1]$ (w.r.t. the relative topology). Since $\gamma(t)$ is either in A or B for each $t \in [0,1]$ we see that

$$[0,1] = \gamma^{-1}(A) \,\dot{\cup}\, \gamma^{-1}(B)$$

which is impossible since $[0,1]$ is connected. $\surd$

Remark 8.3.13. The converse is not true in general.

Theorem 8.3.14. *Let* (M, d_M) *and* (N, d_n) *be metric spaces and assume that* $f \in \mathrm{C}(M,N)$ *is surjective. Then*
(i) If M *is connected, so is* N*.*
(ii) If M *is pathwise connected, then so is* N*.*

Proof. (i) Assume that $N = A \,\dot{\cup}\, B$ for some $A, B \overset{o}{\subset} N$. Then

$$M = f^{-1}(A) \,\dot{\cup}\, f^{-1}(B) = M$$

since f is surjective. Since $f^{-1}(A), f^{-1}(B) \overset{o}{\subset} M$ by continuity of f, it follows that M cannot be connected.
(ii) Let $x, y \in N = f(M)$; then we find $u, v \in M$ such that $f(u) = x$ and $f(v) = y$. Since M is pathwise connected, a curve $\gamma \in \mathrm{C}([0,1], M)$ can be found connecting u and v. But then $f \circ \gamma$ is a curve connecting x and y in N which is precisely what we need to conclude the proof. $\surd$

8.3.4 Contraction Mapping Theorem

Next we prove a simple theorem which has a very large number of applications. We shall assume throughout that (M, d) is a metric space and that $f \in \mathrm{C}(M, M)$.

Definition 8.3.15. (Contraction)
The self-map f is called *contractive* (or a *contraction*) iff it satisfies

$$d\big(f(x), f(y)\big) \leq r\, d(x, y)\; \forall\, x, y \in M$$

for some $r \in (0, 1)$.

We shall make use of the following notation

$$f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}}$$

for the n-fold composition of f with itself.

Theorem 8.3.16. *(Contractive Mapping Theorem)*
Let (M, d) be a complete metric space and $f \in \mathrm{C}(M, M)$ be a contraction. Then f possesses a unique fixed-point $x_0 \in M$, that is, a point for which $f(x_0) = x_0$.
Moreover, for each $x \in M$ it holds that

$$f^{\circ n}(x) \to x_0\; (n \to \infty) \text{ and } d\big(x_0, f^{\circ n}(x)\big) \leq c\, r^n$$

for a constant which depend on x.

Proof. (i) Uniqueness: Let x_0 and x_1 be two fixed-points of f, then

$$f(x_0, x_1) = d\big(f(x_0(, f(x_1)\big) \leq r\, d(x_0, x_1)$$

which is only possible if $x_0 = x_1$.
Existence: Let $x \in M$ and define $x_n = f^{\circ n}(x)$ for $n \in \mathbb{N}$. Then

$$d(x_{n+1}, x_n) \leq r\, d(x_n, x_{n-1}) \leq \cdots \leq r^n\, d(x_1, x)\,,\; n \in \mathbb{N}$$

and, consequently,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leq [r^{m-1} + \cdots + r^n] d(x_1, x)\,.$$

The convergence of the geometric series implies that, for any given $\varepsilon > 0$, a $N \in \mathbb{N}$ can be found such that

$$d(x_m, x_n) \leq \varepsilon\, \forall\, n, m \geq N$$

showing that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since M is complete, there exists a limit $x_0 \in M$ for this sequence. Since f is continuous, it follows that

$$f(x_0) = \big(\lim_{n\to\infty} f x_n\big) = \lim_{n\to\infty} f(x_n) = x_0\,.$$

Finally we see that

$$d(x_n, x) = \lim_{m\to\infty} d(x_n, x_m) \leq \sum_{k=n}^{\infty} r^k d(x_1, x) = \frac{r^n}{r-1} d(x, f(x))$$

and the proof is complete. $\surd$