Assignment 13

1. Determine the radius of convergence of the following power series
\[ \sum_{n=0}^{\infty} \frac{\sqrt{n}}{(n+1)^3} x^n, \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n^n} x^n. \]

2. Assume that the power series
\[ \sum_{n=0}^{\infty} a_n x^n \text{ and } \sum_{n=0}^{\infty} b_n x^n \]
have positive radii of convergence. Suppose that there exists a sequence \((y_j)_{j \in \mathbb{N}}\) with \(y_j \to 0 (j \to \infty)\) and \(y_j \neq 0\) such that
\[ \sum_{n=0}^{\infty} a_n y_j^n = \sum_{n=0}^{\infty} b_n y_j^n. \]
Prove that \(a_n = b_n, n \in \mathbb{N}\).

3. Assume \(D \subseteq \mathbb{R}\) and let \(f : D \to \mathbb{R}\) be analytic. Show that, for every \(x_0 \in D\), constants \(M, r, \delta > 0\) can be found such that
\[ |f^{(k)}(x)| \leq Mk! r^k, x \in (x_0 - \delta, x_0 + \delta). \]

4. For \(\alpha \in \mathbb{R}\) define the (general) binomial coefficient
\[ \binom{\alpha}{n} := \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!}, n \in \mathbb{N}, \binom{\alpha}{0} := 1. \]
Show that the radius of convergence of the binomial series
\[ \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \]
is 1 and determine what function it represents.
[Hint: Use the relations
\[ \binom{\alpha}{n} = \left( \frac{\alpha-1}{n} \right) + \left( \frac{\alpha-1}{n-1} \right) \text{ and } \alpha \binom{\alpha-1}{n} = (n+1) \binom{\alpha}{n+1} \]
to derive an ordinary differential equation satisfied by the power series.]

5. Let \(\rho\) be the radius of convergence of the power series \(\sum_{k=0}^{\infty} a_n x^n\).
Show that
\[ \liminf_{n \to \infty} \frac{a_n}{a_{n+1}} \leq \rho \leq \limsup_{n \to \infty} \frac{a_n}{a_{n+1}} \]

Homework due by Thursday, February 16 2006