Assignment 13

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1. Determine the radius of convergence of the following power series
$$\sum_{n=0}^{\infty} \frac{\sqrt{n\,2^n}}{(n+1)^5} x^n \,, \ \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n^n} x^n \,.$$
2. Assume that the power series
$$\sum_{n=0}^{\infty} a_n x^n \text{ and } \sum_{n=0}^{\infty} b_n x^n$$
have positive radii of convergence. Suppose that there exists a sequence $(y_j)_{j\in\mathbb{N}}$ with $y_j \to 0$ $(j \to \infty)$ and $y_j \neq 0$ such that
$$\sum_{n=0}^{\infty} a_n y_j^n = \sum_{n=0}^{\infty} b_n y_j^n \,.$$
Prove that $a_n = b_n$, $n \in \mathbb{N}$.

3. Assume $D \overset{o}{\subset} \mathbb{R}$ and let $f : D \to \mathbb{R}$ be analytic. Show that, for every $x_0 \in D$, constants $M, r, \delta > 0$ can be found such that
$$|f^{(k)}(x)| \leq M k!\, r^k \,, \ x \in (x_0 - \delta, x_0 + \delta) \,.$$
4. For $\alpha \in \mathbb{R}$ define the *(general) binomial coefficient*
$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \,, \ n \in \mathbb{N} \,, \ \binom{\alpha}{0} := 1 \,.$$
Show that the radius of convergence of the binomial series
$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$
is 1 and determine what function it represents.
[Hint: Use the relations
$$\binom{\alpha}{n} = \binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} \text{ and } \alpha \binom{\alpha-1}{n} = (n+1)\binom{\alpha}{n+1}$$
to derive an ordinary differential equation satisfied by the power series.]
5. Let ρ be the radius of convergence of the power series $\sum_{k=0}^{\infty} a_n x^n$. Show that
$$\liminf_{n\to\infty} \Big|\frac{a_n}{a_{n+1}}\Big| \leq \rho \leq \limsup_{n\to\infty} \Big|\frac{a_n}{a_{n+1}}\Big|$$

Homework due by Thursday, February 16 2006