Assignment 15

1. Let $\alpha \in (0, 1)$ and assume that $A \subset C^\alpha([0, 1], \mathbb{K})$ be a bounded subset, that is, assume that
\[
\|f\|_\alpha = \|f\|_\infty + [f]_\alpha \leq M, \ f \in A
\]
for some $M > 0$. Show that $A$ is uniformly equicontinuous.

2. Let $f, g \in C_c(\mathbb{R}, \mathbb{K})$ and show that
\[
\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)
\]
\[
= \{x + y \mid x \in \text{supp}(f), \ y \in \text{supp}(g)\}.
\]

3. Let $0 \leq f \in C_c(\mathbb{R}, \mathbb{R})$ with $\int_{-\infty}^{\infty} f(x) \, dx = 1$. Show that $(f_n)_{n \in \mathbb{N}}$ defined through
\[
f_n(x) = n f(nx), \ x \in \mathbb{R}, \ n \in \mathbb{N},
\]
is an approximate identity.

4. Let $a < b \in \mathbb{R}$ and $\varepsilon > 0$ be given. Show that the constant function with value 1 on $[a, b]$ can be extended to a $C^\infty$-function of the line which vanishes outside $[a - \varepsilon, b + \varepsilon]$.

5. Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole real line which does not possess a uniformly convergent subsequence.