1. Prove that a normed vector space \((V, |\cdot|_V)\) is an inner product space iff
\[
|x + y|_V^2 + |x - y|_V^2 = 2(|x|_V^2 + |y|_V^2), \ x, y \in V.
\]
[Hint: Use the polarization identity.]

2. Give an example of a norm which cannot be derived from an inner product and an example of a metric which is not induced by a norm.

3. Define
\[
l_\infty(\mathbb{K}) := \{x \in \mathbb{K}^\mathbb{N} \mid |x|_\infty := \sup_{n \in \mathbb{N}} |x_n| < \infty\}
\]
and
\[
c_0(\mathbb{K}) := \{x \in l_\infty(\mathbb{K}) \mid \lim_{n \to \infty} x_n = 0\}.
\]
Show that \((l_\infty(\mathbb{K}), |\cdot|_\infty)\) and \((c_0(\mathbb{K}), |\cdot|_\infty)\) are complete normed vector spaces.

4. Two norms \(|\cdot|_1\) and \(|\cdot|_2\) on a vector space \(V\) are said to be equivalent iff there exists a constant \(c \geq 1\) such that
\[
\frac{1}{c} |x|_1 \leq |x|_2 \leq c |x|_1, \ x \in V.
\]
Prove that equivalent norms induce the same topology.

5. Let \((M, d)\) be a metric space and show that
\[
\tau := \{O \subset M \mid O \text{ is open}\}
\]
defines a topology on \(M\). Compute \(\tau\) for \(M = \mathbb{R}\) and \(d\) defined by
\[
d(x, y) = \begin{cases} 
0, & x = y \\
1, & x \neq y.
\end{cases}
\]
For \(M = \mathbb{R}\) define
\[
d(x, y) = \frac{2}{\pi} \int_{\min(x,y)}^{\max(x,y)} \frac{d\xi}{1 + \xi^2}, \ x, y \in \mathbb{R}.
\]
Show that \((M, d)\) is a metric space and compute \(B(0, 1), B(1, \frac{\pi}{4})\).

Homework due by Thursday, March 9 2006