Final Examination

Print your name: ____________________ ____________________

Print your ID #: ________________________________

You have 2 hours to solve the problems. Good luck!

1. Let $\alpha \geq 0$ and define $f_\alpha \in C([0,1],\mathbb{R})$ by

$$f_\alpha(x) = \frac{1}{1 + \alpha x}, \ x \in (0,1).$$

For $A > 0$ consider the set $S_A = \{f_\alpha \mid \alpha \in [0,A]\}$. Is it equicontinuous? What if $A = \infty$? Motivate your answers.

2. Show that

$$f : \mathbb{R} \to \mathbb{R}, \ x \mapsto \frac{\cos(x) - 1}{x^2}$$

is analytic at $x = 0$.

3. Consider

$$f_n(x) = e^{-nx}, \ x \in (0,\infty), \ n \in \mathbb{N}.$$ 

What is $\sum_{n=0}^{\infty} f_n$? Is the convergence uniform? Motivate your answers.

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4. Determine the function which is represented by the following power series

\[ \sum_{n=1}^{\infty} n x^n \]

in the interval \((-1, 1)\). Justify your answer.

5. Let \((b_n)_{n \in \mathbb{N}}\) be a positive decreasing sequence. Show that \(\sum_{n=0}^{\infty} a_n\) converges absolutely if

\[ |a_n| \leq b_n - b_{n+1}, \quad n \in \mathbb{N}. \]

6. Consider the following spaces of functions

\[
\begin{align*}
BC(\mathbb{R}, \mathbb{R}) &= \{ f \in C(\mathbb{R}, \mathbb{R}) \mid \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| < \infty \}, \\
C_0(\mathbb{R}, \mathbb{R}) &= \{ f \in C(\mathbb{R}, \mathbb{R}) \mid \lim_{|x| \to \infty} f(x) = 0 \}.
\end{align*}
\]

Show that \(C_0(\mathbb{R}, \mathbb{R})\) is a closed subset of \((BC(\mathbb{R}, \mathbb{R}), \| \cdot \|_\infty)\).

7. Let \(M_1 = (M, d_1)\) and \(M_2 = (M, d_2)\) be metric spaces and assume that \(d_2 \leq c d_1\) for some positive \(c > 0\). Prove that

\[ O \subset M_2 \Rightarrow O \subset M_1. \]

8. Fix \(N \geq 1\) and show that the set

\[ C_N = \left\{ \sum_{n=1}^{N} a_n \cos(n\pi x) \mid \sup_{1 \leq n \leq N} |a_n| \leq 1 \right\} \subset C(\mathbb{R}, \mathbb{R}) \]

is uniformly bounded and uniformly equicontinuous.

9. Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in a metric space \((M, d)\) and assume it possesses a convergent subsequence. Show that the whole sequence converges.