Math 205a FALL TERM 2002

Assignment 1

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1. Form the negation of the following statements:
   (i) $\forall x \in A \, \exists y \in B \, : \, (x,y) \in G$.
   (ii) $\forall x \in A \, \exists! \, y \in B \, : \, (x,y) \in G$.
   (iii) $\exists y \in B \, : \, (x,y) \notin G \, \forall x \in A$.
   (iv) $\forall n \in \mathbb{N} \, \exists m \in \mathbb{N} \, : \, m \geq n$ and m is prime.

2. Let A, B be non-empty sets. Then $G \subset A \times B$ determines a map $f_G : A \to B$, $x \mapsto y$ iff
   $$(x,y) \in G \, , \; (x,\tilde{y}) \in G \; \Rightarrow \; y = \tilde{y} \, .$$
   If this is the case, then
   $$\mathrm{dom}(f) := \{x \in A \,|\, \exists y \in B \text{ with } (x,y) \in G\} \, , \text{ the domain of } f \, ,$$
   $$\mathrm{im}(f) := \{y \in B \,|\, \exists x \in A \text{ with } (x,y) \in G\} \, , \text{ the range of } f \, .$$
   For $x \in \mathrm{dom}(f)$ we write $y = f_G(x)$ if $(x,y) \in G$. Use quantifiers to describe the following:
   (i) The map is one-to-one.
   (ii) The map is onto.
   (iii) The map is bijective.
   (iv) $\widetilde{G} = \{(y,x) \in B \times A \,|\, (x,y) \in G\}$ defines a set. What is $f_{\widetilde{G}}$ when it exists?
3. Let $\mathbb{N}_m := \{1, 2, \ldots, m-1, m\}$. How many maps $f : \mathbb{N}_m \to \mathbb{N}_n$ are there for $m, n \in \mathbb{N}$? How many are the bijections among them? [Consider the cases $m < n$, $m = n$, $m > n$ separately.]
   Is the set $map(\mathbb{N},\mathbb{N}) := \{f : \mathbb{N} \to \mathbb{N} \,|\, f \text{ is a map}\}$ countable? What about $map_b(\mathbb{N},\mathbb{N}) := \{f : \mathbb{N} \to \mathbb{N} \,|\, f \text{ is a bijection}\}$?
   Justify your answers.
4. In class we proved that $\mathbb{N} \times \mathbb{N}$ is countable. Can you construct a bijective map $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ explicitly? Also prove the more general statement that $M \times N$ is countable if M and N are and conclude that $\mathbb{N}^m$ is countable for any $m \in \mathbb{N}$.

The Homework is due Friday, October 11