Assignment 12

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1. Determine the convergence of the following series:
$$\sum_{k=1}^{\infty} \frac{e^{i\frac{\pi}{2}k}}{k}\,,\ \sum_{k=1}^{\infty}(-1)^k\big(\sqrt{k+1}-\sqrt{k}\big)\,,\ \sum_{k=1}^{\infty}\frac{k!}{k^k}\,.$$
2. Give an example of $(a_{mn})_{m,n\in\mathbb{N}}$ for which
$$\sum_{m=1}^{\infty}\big(\sum_{n=1}^{\infty}a_{mn}\big)\neq\sum_{n=1}^{\infty}\big(\sum_{m=1}^{\infty}a_{mn}\big)\,.$$
3. Find a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{R}([0,1],\mathbb{R})$ which converges pointwise to zero but for which
$$\int_0^1 f_n(x)\,dx\not\to 0\,(n\to\infty)\,.$$
Also find a sequence which does not converge to zero pointwise but for which
$$\int_0^1 f_n(x)\,dx\to 0\,(n\to\infty)\,.$$
4. For $\alpha\in(0,1)$ let
$$\begin{aligned}&\mathrm{C}^\alpha([0,1],\mathbb{K})\\ &\quad:=\big\{f\in\mathrm{C}([0,1],\mathbb{K})\,\big|\,[f]_\alpha=\sup_{x\neq y\in[0,1]}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty\big\}\end{aligned}$$
be the space of Hölder continuous real- or complex-valued functions defined on $[0,1]$. Show that it is complete w.r.t. $\|\cdot\|_\alpha=\|\cdot\|_\infty+[\cdot]_\alpha$. In other words, prove that any sequence satisfying
$$\forall\varepsilon>0\ \exists M\in\mathbb{N}\text{ s.t. }\|f_n-f_m\|_\alpha\le\varepsilon,\ m,n\ge M\,.$$
converges to some limit $f\in\mathrm{C}^\alpha([0,1],\mathbb{K})$.

5. You ask a question.

The Homework is due Friday, February 7